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FREMLIN TENSOR PRODUCTS OF BANACH LATTICES

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Abstract. We prove that the Fremlin tensor product of two Banach lattices with the Radon-Nikodym property, with one of them atomic, has the Radon-Nikodym property.

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1. Introduction. In the study of the tensor product of Banach spaces (see [11]), it is natural to investigate which geometric properties of Banach spaces are preserved by the tensor product of these spaces. Diestel and Uhl (see [7]) posed the question whether the projective tensor product of two Banach spaces with the Radon-Nikodym property has the Radon-Nikodym property. Bourgain and Pisier (see [2]) answered this question in the negative. They constructed a Banach space X with the Radon-Nikodym property such that $X \overset{\wedge}{\otimes} X$ does not have the Radon-Nikodym property. In this paper, modifying ideas of Bu and Buskes (see [3]) and Bu, Diestel, Dowling, Oja (see [4]), we show that if X and Y are two Banach lattices with the Radon-Nikodym property, with one of them atomic, then the Fremlin tensor product of X and Y has the Radon-Nikodym property. The subtle techniques of measure theory, as used in the paper of Diestel, Fourie and Swart [6], was found to be useful in Step 3 of our proof of Theorem 5.

2. Preliminaries. Throughout this paper we will denote by X a Banach lattice, by X^* its topological dual and by B_X its closed unit ball. As usual we will denote by X^+ the positive cone of the Banach lattice X.

Definition 1. Let E and F be Banach lattices. A bounded linear operator $T : E \longrightarrow F$ is called *positive* if $T(E^+) \subseteq F^+$. Let $\mathcal{L}^+(E,F)$ be the collection of all positive operators from E into F. An operator $T \in \mathcal{L}(E,F)$ is called *regular* if there exist $T_1, T_2 \in \mathcal{L}^+(E,F)$ such that $T = T_1 - T_2$. Let $\mathcal{L}^r(E,F)$ denote the collection of regular operators from E into F. It is known that if F is Dedekind complete then $\mathcal{L}^r(E,F)$ is a Banach lattice with positive cone $\mathcal{L}^+(E,F)$ (see [14]) and norm

$$\|T\|_r = \inf\{\|S\| \ : \ S \in \mathcal{L}^+(E,F), \ |T(x)| \le S(|x|), \ x \in E^+\}$$

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and in this case, $\|T\|_r = \| \, |T| \, \|$.

For two Banach lattices X and Y, Fremlin in [8], and [9] introduced a lattice tensor product, called the *positive projective tensor product.* Let X, Y, Z be Banach lattices. A bilinear map $\phi : X \times Y \longrightarrow Z$ is called *positive* if $\phi(X^+, Y^+) \subseteq Z^+$; the projective cone on the tensor product $X \otimes Y$ is defined as:

$$X^+ \otimes Y^+ = \{\sum_{k=1}^{n} x_k \otimes y_k : \quad n \in \mathbb{N}, \ x_k \in X^+, \ y_k \in Y^+\}$$

The positive projective tensor norm on $X \otimes Y$ is defined as:

$$\|u\|_{|\pi|} = \sup\{|\sum_{i=1}^{n} \phi(x_i, y_i)| \ : \ u = \sum_{i=1}^{n} x_i \otimes y_i, \ \phi \text{ is a positive bilinear function on } X \times Y, \ \|\phi\| \leq 1 \ \}.$$

Let $X \overset{\wedge}{\otimes}_F Y$ be the completion of $X \otimes Y$ equipped with the norm $\| \cdot \|_{|\pi|}$. Then $X \overset{\wedge}{\otimes}_F Y$ is a Banach lattice, having as positive cone the closure in $X \overset{\wedge}{\otimes}_F Y$ of the projective cone $X^+ \otimes Y^+$. Fremlin ([9], Theorem 1E (vii)), gave a simpler equivalent form of the positive projective norm, as

$$\|u\|_{|\pi|} = \inf\{\sum_{k=1}^{\infty} \|x_k\| \cdot \|y_k\| : \quad x_k \in X^+, \ y_k \in Y^+, \ |u| \leq \sum_{k=1}^{\infty} x_k \otimes y_k\}.$$

DEFINITION 2. Let $(u_n)_{n\in\mathbb{N}}$ be a Schauder basis for a Banach space U and let $(u_n^*)_{n\in\mathbb{N}}$ be its biorthogonal sequence in U^*. The basis $(u_n)_{n\in\mathbb{N}}$ is said to be an *unconditional basis* for U if there exists a constant $\lambda \geq 1$ so that $\sum_{n=1}^{\infty} t_n \langle u_n^*, x\rangle \, u_n$ converges for every $(t_n)_{n\in\mathbb{N}} \in \ell_\infty$, $x \in U$ and

$$\left\|\sum_{n=1}^{\infty} t_n \langle u_n^*, x\rangle \, u_n\right\| \leq \lambda \left\|\sum_{n=1}^{\infty} \langle u_n^*, x\rangle \, u_n\right\| \quad \forall (t_n)_{n\in\mathbb{N}} \in B_{\ell_\infty}.$$

We recall that a Banach space U with an unconditional basis $(u_n)_{n\in\mathbb{N}}$ can be ordered in a natural way such that it becomes a Banach lattice with the new norm

$$|||u||| = \sup\{\|\sum_{n\in\mathbb{N}} t_n u_n^*(u) u_n\| \quad : \quad (t_n)_{n\in\mathbb{N}} \in \ell_\infty\}$$

where $\| \cdot \|$ denotes the original norm in U. Moreover

$$\| \cdot \| \leq ||| \cdot ||| \leq K \| \cdot \|,$$

where K is a constant depending only on the unconditional basis $(u_n)_{n\in\mathbb{N}}$. Throughout this paper U will denote a Banach lattice with a normalized unconditional basis $(u_n)_{n\in\mathbb{N}}$ with normalized biorthogonal functionals $(u_n^*)_{n\in\mathbb{N}}$ and throughout the sequel, U is endowed with the norm $||| \cdot |||$. Finally, following the terminology used by Heinrich Lotz, Tenny Peck and Horatio Porta (see [13]), we say that a Banach space X is *semi-embedded* into a Banach space Y, if there exists a bounded, linear injective operator $\sigma : X \longrightarrow Y$ such that $\sigma(B_X)$ is closed. The map σ is called a *semi-embedding*. In connection with this definition, we recall the following elegant result of J. Bourgain and H. Rosenthal (see [1]).

Theorem 3. *Suppose X is a separable Banach space that is semi-embeddable into a Banach space Y having with the Radon-Nikodym property. Then X also has the Radon-Nikodym property.*

3. The Radon-Nikodym property in Fremlin tensor products. Let X and U be Banach lattices, and let $(u_n)_n$ be a Schauder basis of U. It is easily verified that the space

$$U(X) = \{(x_n)_{n\in\mathbb{N}} \subseteq X : \sum_{n\in\mathbb{N}} \|x_n\|\, u_n \text{ is convergent in } U\}$$

endowed with the norm

$$\|(x_n)\|_{U(X)} = \|\sum_{n\in\mathbb{N}} \|x_n\|\,\|_U$$

and the order defined defined by

$$(x_n)_{n\in\mathbb{N}} \leq (y_n)_{n\in\mathbb{N}} \quad \Leftrightarrow \quad x_n \leq_X y_n \quad \forall n \in \mathbb{N},$$

is a Banach lattice.

Lemma 4. (Fremlin's theorem [9]) *Let X and Y be Banach lattices. Then for each Banach lattice Z and for each continuous bilinear map $\phi : X \times Y \longrightarrow Z$ there exists a unique continuous linear map $T : X\widehat{\otimes}_F Y \longrightarrow Z$ such that*

(i) $\|T\| = \|\phi\|$

(ii) $T(x \otimes y) = \phi(x, y)$

(iii) *ϕ is a positive if and only if T is a positive.*

Theorem 5. *Let X be a separable Banach lattice and U be a Banach space with an unconditional basis. If U and X have the Radon-Nikodym property then the Fremlin tensor product $U\widehat{\otimes}_F X$ of U and X can be semi-embedded in $U(X)$*

Proof. To start, consider the bilinear operator

$$\begin{aligned} \widetilde{\Psi} : U \times X &\longrightarrow U(X) \\ (u, x) &\longmapsto (u_n^*(u)x)_{n\in\mathbb{N}}. \end{aligned}$$

Note that $\widetilde{\Psi}$ is bilinear bounded and positive. If $u \in U^+$ and $x \in X^+$, then $\widetilde{\Psi}(u,x)_i = u_i^*(u)x \geq 0$ for each $i \in \mathbb{N}$; therefore, $\Psi(u,x) \geq 0$ in $U(X)$; moreover,

$$\begin{aligned}\| \sum_{n\in\mathbb{N}} \|\widetilde{\Psi}(u,x)\|_X \; u_n \; \|_U &= \| \sum_{n\in\mathbb{N}} \|u_n^*(u)x\|_X u_n \; \|_U \\ &= \|x\|_X \| \sum_{n\in\mathbb{N}} |u_n^*(u)| u_n\|_U \\ &\leq \|x\|_X \|u\|_U.\end{aligned}$$

So $\widetilde{\Psi}$ is bounded with $\|\widetilde{\Psi}\| \leq 1$.

By Fremlin‘s theorem there exists a unique continuous positive linear $\Psi : U\widehat{\otimes}_F X \longrightarrow U(X)$ such that

(i) $\|\Psi\| \leq 1$ and

(ii) $\widetilde{\Psi}(u,x) \;=\; \Psi(u \otimes x)$ for each $u \in U, x \in X$.

Step 1. Ψ is injective.
First, consider Ψ on $U \otimes_F X$. If $v = \sum_{k=1}^p v_k \otimes x_k \in (U \otimes_F X)^+$ (with $v_k, x_k \geq 0$ for each k) so that $\Psi(v) = 0$, then

$$0 = \Psi(v) = \sum_{k=1}^p \widetilde{\Psi}(v_k, x_k) = (\sum_{k=1}^p u_n^*(v_k)x_k)_{n\in\mathbb{N}};$$

thus

$$\sum_{k=1}^p u_n^*(v_k)x_k = 0, \quad \forall n \in \mathbb{N};$$

but every $u_n^*(v_k)x_k$ is in X^+, so

$$u_n^*(v_k)x_k = 0 \quad \forall n \in \mathbb{N},\; k = 1,...,p.$$

This means: either $x_k = 0$ or $u_n^*(v_k) = 0$ for each $n \in \mathbb{N}$; hence, either $x_k = 0$ or $u_k = 0$. Either way we have $v = \sum_{k=1}^p v_k \otimes x_k = 0$.
If there is a $z > 0$ in $U\widehat{\otimes}_F X$ so that $\Psi(z) = 0$, then we can choose a sequence $(z_n)_{n\in\mathbb{N}}$, of positive elements of $U \otimes_F X$ such that $z_n \leq z$ for every $n \in \mathbb{N}$, convergent to z (see [10]). Therefore,

$$0 \leq \Psi(z_n) \leq \Psi(z) = 0 \quad \forall n \in \mathbb{N};$$

so

$$z_n = 0, \quad \forall n \in \mathbb{N},$$

and so

$$z = \|\cdot\|_{|\pi|} - \lim_n z_n = 0,$$

a contradiction. This means Ψ is injective on the positive cone of $U\widehat{\otimes}_F X$ and so injective on $U\widehat{\otimes}_F X$.

We want to show that Ψ is a semi-embedding, i.e., we want to show that for a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq B_{U\widehat{\otimes}_F X}$ and $(y_i)_{i\in\mathbb{N}} \in U(X)$ such that $\lim_n \Psi(z_n) = (y_i)_i$ in $U(X)$, there exists $z \in B_{U\widehat{\otimes}_F X}$ such that $\Psi(z) = (y_i)_i$.

Step 2. Now, fix $T \in \mathcal{L}^r(U, X^*)$ and consider the series $\sum_{i\in\mathbb{N}} \langle y_i, T(u_i)\rangle$. First, we show that the series is absolutely convergent: We suppose that $(z_n)_n \subseteq B_{U\otimes X}$, so we can write each z_n as

$$z_n = \sum_{k=1}^{p(n)} v_{k,n} \otimes x_{k,n}.$$

Then,

$$\begin{aligned}(y_i)_i &= \lim_n \Psi(z_n)\\ &= \lim_n \sum_{k=1}^{p(n)} \widetilde{\Psi}(v_{k,n}, x_{k,n})\\ &= \lim_n (\sum_{k=1}^{p(n)} (u_i^*(v_{k,n}) x_{k,n})_i\\ &= (\lim_n \sum_{k=1}^{p(n)} (u_i^*(v_{k,n}) x_{k,n})_i.\end{aligned}$$

Hence,

$$\lim_n \sum_{k=1}^{p(n)} u_i^*(v_{k,n}) x_{k,n} = y_i \quad i \in \mathbb{N}.$$

Then for fixed $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ so that for $i = 1, 2, ..., m$ we have

$$\| \sum_{k=1}^{p(n_0)} u_i^*(v_{k,n_0}) x_{k,n_0}) - y_i\|_X \le \frac{\epsilon}{m}.$$

Therefore,

$$
\begin{aligned}
&\sum_{i=1}^{m} |\langle y_i, T(u_i)\rangle| \\
&\leq \sum_{i=1}^{m} |\langle y_i - \sum_{k=1}^{p(n_0)} u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle| + \sum_{i=1}^{m} |\langle \sum_{k=1}^{p(n_0)} u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle| \\
&\leq \epsilon\|T\| + \sum_{i=1}^{m} | \sum_{k=1}^{p(n_0)} \langle u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle| \\
&(\text{which, if } \theta_i = \text{sign} \sum_{k=1}^{p(n_0)} \langle u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle) \\
&= \epsilon\|T\| + |\sum_{i=1}^{m} \theta_i \sum_{k=1}^{p(n_0)} \langle u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle| \\
&= \epsilon\|T\| + |\sum_{i_1}^{m} \theta_i T(u_i) \otimes u_i^*(\sum_{k=1}^{p(n_0)} v_{k,n_0} \otimes x_{k,n_0})| \\
&= \epsilon\|T\| + |\sum_{i=1}^{m} \theta_i T(u_i) \otimes u_i^*(z_{n_0})| \\
&\leq \epsilon\|T\| + \|\sum_{i=1}^{m} \theta_i T(u_i) \otimes u_i^*\|_{\mathcal{L}(U,X^*)} \|z_{n_0}\| \\
&\leq \epsilon\|T\| + \|T\|, \quad \text{for each } m \in \mathbb{N}. \qquad (*)
\end{aligned}
$$

To see why it is so, recall: if $z \in (U \stackrel{\wedge}{\otimes}_F X)^+$ then for each $\epsilon > 0$ there exists $(x_j)_{j=1}^n \subseteq X^+$, $(v_j)_{j=1}^n \subseteq U^+$ so that $z \leq \sum_{j=1}^n v_j \otimes x_j$ and

$$
\sum_{j=1}^{n} \|v_j\| \ \|x_j\| \leq \|z\|_{U \otimes_F X} + \epsilon.
$$

So

$$
\begin{aligned}
|\sum_{i=1}^{m} \theta_i T(u_i) \otimes u_i^*(z)| &\leq \sum_{i=1}^{m} T(u_i) \otimes u_i^*(z) \\
&\leq \sum_{i=1}^{m} T(u_i) \otimes u_i^*(\sum_{j=1}^{n} v_j \otimes x_j) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} T(u_i)(x_j) u_i^*(v_j)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} T(\sum_{i=1}^{m} u_i^*(v_j)u_i)(x_j) \\
&\le \sum_{j=1}^{n} \|T(\sum_{i=1}^{m} u_i^*(v_j)u_i)\|_{X^*} \ \|x_j\|_X \\
&\le \|T\| \ \sum_{j=1}^{n} \| \sum_{i=1}^{m} u_i^*(v_j)u_i\|_U \ \|x_j\|_X \\
&\le \|T\| \ \sum_{j=1}^{n} \|v_j\|_U \|x_j\|_X \\
&\le \|T\| \ (\|z\|_{U\hat{\otimes}_F X} + \epsilon).
\end{aligned}
$$

Now (*) follows and with it we see that

$$\sum_{i\in\mathbb{N}} |\langle y_i, T(u_i)\rangle| \le \|T\|.$$

Now it makes sense to define $\phi : \mathcal{L}^r(U, X^*) \longrightarrow \mathbb{K}$ by

$$\phi(T) := \sum_{i\in\mathbb{N}} \langle y_i, T(u_i)\rangle \text{ for each } T \in \mathcal{L}^r(U, X^*).$$

From the note above, ϕ is well defined, with

(a) $\phi \in \mathcal{L}^r(U, X^*)^*$ and

(b) $\|\phi\| \le 1$.

Step 3. We show that there exists $z \in B_{U\hat{\otimes}_F X}$ so that $\Psi(z) = (y_i)_i$.
Let $K = \beta((B_U \ , \ \|\cdot\|) \times (B_{X^{**}} \ , \ weak^*))$ where $\beta(S)$ is the Čech-Stone compactification of S. K is a compact Hausdorff space and, because $(B_U \ , \ \|\cdot\|)$ is a Polish space, $(B_U \ , \ \|\cdot\|) \times (B_{X^{**}} \ , \ weak^*)$ is universally measurable with respect to all Radon measures on K (see [5]). Define

$$J : \mathcal{L}^r(U, X^*) \longrightarrow C(K)$$

by

$$J(T)(u, x^{**}) = x^{**}(T(u))$$

on $(B_U \times B_{X^{**}})$ and extend using the Čech-Stone nature of K. J is a bounded linear operator with $\|JT\|_{C(K)} = \|T\|$ on the positive cone $\mathcal{L}^r(U, X^*)$. Now consider

$$F_\phi : J(\mathcal{L}^r(U, X^*)) \longrightarrow \mathbb{K}$$

defined by

$$F_\phi(JT) = \langle T, \phi\rangle \quad \forall T \in \mathcal{L}^r(U, X^*).$$

Note $\|F_\phi\| = \|\phi\|$. So by the Hahn-Banach theorem and the Riesz representation theorem, there exists a regular Borel measure ν so that

$$F_\phi(JT) = \int_K JT(\omega)d\nu(\omega) \qquad \forall T \in \mathcal{L}^r(U, X^*) \tag{A}$$

and

$$|\nu|(K) = \|F_\phi\| = \|\phi\|.$$

Define $h_1 : (B_U\ ,\ \|\cdot\|) \times (B_{X^{**}}\ ,\ weak^*) \longrightarrow B_U$ by $h_1(u, x^{**}) = u$; h_1 is continuous into $(B_{U^{**}}\ ,\ weak^*)$ and so extends to a continuous function, still called h_1, from K to $(B_{U^{**}}\ ,\ weak^*)$. B_U's Polish character now allows us to look at

$$k_1 = h_1 \cdot \chi_{B_U \times B_{X^{**}}};$$

k_1 is scalarly ν-measurable and U-valued; hence, strongly ν-measurable. Also, k_1 is bounded in norm by 1 so

$$\int_K \|k_1(\omega)\| d|\nu| \leq |\nu|(K)$$

that is, k_1 is Bochner ν-integrable. Now we know that (see [7], p. 172) for every $\epsilon > 0$ there exists a sequence $(v_n)_{n\in\mathbb{N}} \subseteq U$ and a sequence of Borel sets $(B_n)_{n\in\mathbb{N}} \subseteq K$ such that

$$k_1(\omega) = \sum_{n=1}^{\infty} v_n \chi_{B_n}(\omega) \qquad |\nu|\text{-a.e.}$$

with

$$\sum_{n=1}^{\infty} \|u_n\|_U\ |\nu|(B_n) \leq \int_K \|k_1(\omega)\|\ d|\nu| + \epsilon \leq |\nu|(K) + \epsilon.$$

Let

$$h_2 : K \longrightarrow X^{**}$$

be given by

$$h_2(u, x^{**}) := x^{**}.$$

Then h_2 is weak*-continuous and hence weak*-measurable. Moreover, for each $x^* \in X^*$,

$$\int_K |\langle x^*, h_2(\omega)\rangle d|\nu|(\omega) \leq \|x^*\| \int_K |h_2(\omega)| d|\nu|(\omega) \leq \|x^*\| |\nu|(K) < \infty.$$

So h_2 is Gelfand integrable (see [7], p. 42).

Now, if we consider, for each $i \in \mathbb{N}$ and $x^* \in (X^*)^+$, $T_i = u_i^* \otimes x^* \in \mathcal{L}^+(U, X^*)$, we have by (A) that

$$\begin{aligned}
\langle y_i, x^* \rangle &= \langle T_i, \phi \rangle \\
&= \int_K \langle T_i u, x^{**} \rangle d\nu(u, x^{**}) \\
&= \int_K \langle x^*, h_2(u, x^{**}) \rangle \langle k_1(u, x^{**}), u_i^* \rangle d\nu(u, x^{**}) \\
&= \int_K \langle x^*, h_2(u, x^{**}) \rangle \langle \sum_{n=1}^{\infty} v_n \chi_{B_n}(u, x^{**}), u_i^* \rangle d\nu(u, x^{**}) \\
&= \sum_{n=1}^{\infty} u_i^*(v_n) \int_{B_n} \langle x^*, h_2(u, x^{**}) \rangle d\nu(u, x^{**}) \\
&= \sum_{n=1}^{\infty} u_i^*(v_n) \langle x^*, a_n^{**} \rangle
\end{aligned}$$

where

$$a_n^{**} = Gelfand - \int_{B_n} h_2(u, x^{**}) d\nu(u, x^{**}).$$

Therefore,

$$\langle y_i, x^* \rangle = \sum_{n=1}^{\infty} u_i^*(v_n) \langle x^*, a_n^{**} \rangle. \tag{B}$$

For every $x^* \in (X^*)^+$ and $n \in \mathbb{N}$

$$\begin{aligned}
|\langle x^*, a_n^{**} \rangle| &= |\int_{B_n} \langle x^*, h_2(u, x^{**}) d\nu(u, x^{**})| \\
&\leq \int_{B_n} |\langle x^*, h_2(u, x^{**}) \rangle| d\nu(u, x^{**}) \\
&\leq \|x^*\| |\nu|(B_n).
\end{aligned}$$

Hence,

$$\|a_n^{**}\| \leq |\nu|(B_n).$$

Moreover

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \|u_i^*(v_n) a_n^{**}\| &= \sum_{n \in \mathbb{N}} |u_i^*(v_n)| \|a_n^{**}\| \\
&\leq \sum_{n \in \mathbb{N}} \|v_n\| |\nu|(B_n) \\
&\leq |\nu|(K) + \epsilon.
\end{aligned} \tag{C}$$

That means the series $\sum_{n \in \mathbb{N}} u_i^*(v_n) a_n^{**}$ is absolutely convergent in X^{**}.
Note, now, that since X is a Banach lattice with the Radon-Nikodym property, X is norm-one complemented in X^{**}. Therefore there exists a norm one linear

projection $P : X^{**} \longrightarrow X^{**}$, so that $P(X^{**}) = X$. Let $a_n = P(a_n^{**})$ and $z = \sum_{n\in\mathbb{N}} v_n \otimes a_n$. We have

$$\begin{aligned}
\|z\|_{U\hat{\otimes}_F X} &\leq \sum_{n\in\mathbb{N}} \|u_n\|\|a_n\| \\
&\leq \|P\| \sum_{n\in\mathbb{N}} \|u_n\|\|a_n^{**}\| \\
&\leq \sum_{n\in\mathbb{N}} \|u_n\||\nu|(B_n) \\
&\leq (|\nu|(K) + \epsilon) \\
&= (\|\phi\| + \epsilon)
\end{aligned}$$

so that $\|z\|_{U\hat{\otimes}_F X} \leq \|\phi\| \leq 1$. In particular $z \in B_{U\hat{\otimes}_F X}$. Here is the catch: from (B), (C) and from the definition of Ψ we have, for each $i \in \mathbb{N}$

$$\begin{aligned}
y_i &= P(y_i) \\
&= P(\sum_{n\in\mathbb{N}} u_i^*(v_n) a_n^{**}) \\
&= \sum_{n\in\mathbb{N}} u_i^*(v_n) P(a_n^{**}) \\
&= \sum_{n\in\mathbb{N}} u_i^*(v_n) a_n \\
&= \sum_{n\in\mathbb{N}} (\Psi(u_n \otimes a_n))_i \\
&= (\ \sum_{n\in\mathbb{N}} \Psi(u_n \otimes a_n)\ )_i \\
&= \Psi(\ \sum_{n\in\mathbb{N}} u_n \otimes z_n\ )_i = \Psi(z)_i.
\end{aligned}$$

Hence

$$\Psi(z) = (y_i)_i.$$

We are done. □

Corollary 6. *Let X be a separable Banach lattice and U be a Banach space with an unconditional basis. If U and X have the Radon-Nikodym property then the Fremlin tensor product $U\widehat{\otimes}_F X$ of U and X has the Radon-Nikodym property*

Proof. It is enough to note that from our hypothesis follow that $U(X)$ has the Radon-Nikodym property (that was proved by Bu, Diestel, Dowling, Oja (see [4]). Now the result is a direct consequence of a theorem of Bourgain and Rosenthal (see [1], Theorem 3). □

Now,using Heinrich-Mankiewicz [12] and N. Randrianantoanina [15] and the result of Uhl [7] (which asserts that a general Banach space has the Radon-Nikodym property if and only if does every of its separable closed subspace) We have the following

Corollary 7. *If U and X are two Banach lattices, one of them atomic, then the Fremlin tensor product of U and X, $U\widehat{\otimes}_F X$, has the Radon-Nikodym property if both U and X possess this property.*

Remark 1. Note that we cannot have stability of the Radon-Nikodym property in the Fremlin tensor product of general Banach lattices because Fremlin proved that (see [9]) $L_2[0,1]\widehat{\otimes}_F L_2[0,1]$ is not Dedekind complete and so it cannot have the Radon-Nikodym property (see [14]).

The author initially tried to show the stability of the Radon-Nikodym property for the Fremlin tensor product. Qingying Bu pointed out to the author that Fremlin proved $L_2[0,1]\widehat{\otimes}_F L_2[0,1]$ is not Dedekind complete, and I take this occasion to thank him.

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