UNIVERSITY OF CATANIA

Department of Mathematics and Computer Sciences

Master degree in Mathematics

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Exercise 1. Write down polar coordinates in $\mathbb{R}^4$ and evaluate its Jacobian. Generalize to $\mathbb{R}^n$, $n \geq 5$.

Exercise 2. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set, $|E| < +\infty$. Prove

$$\|f\|_p \leq |E|^{1/p-1/q}\|f\|_q \qquad 1 \leq p \leq q \leq +\infty.$$

Exercise 3. For $f, g, h \in L^1(\mathbb{R}^n)$ prove that

1. $f * g = g * f$;
2. $f * (g * h) = (f * g) * h$;
3. $f * (g + h) = f * g + f * h$.

Exercise 4. Let $K(s,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be such that

$$K(\lambda s, \lambda t) = \lambda^{-1} K(s,t) \qquad \forall s, t \geq 0\,, \lambda > 0\,.$$

Assume that there exists $p \geq 1$ such that

$$\gamma = \int_0^{+\infty} K(1,t)\, t^{-1/p}\, dt < +\infty\,.$$

Show that

$$(Tf)(s) = \int_0^{+\infty} K(s,t) f(t)\, dt$$

satisfies

$$\|Tf\|_p \leq \gamma\, \|f\|_p \qquad \forall f \geq 0\,,\ f \in L^p\,.$$

Exercise 5. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{R\to 0} \fint_{B_R(0)} u(y)\, dy = u(0).$$

Exercise 6. Let $x_0 \in \mathbb{R}^n$ and $u \in C^2(\mathbb{R}^n)$. Prove that

$$\lim_{r\to 0} \frac{2n}{r^2} \left[u(x_0) - \fint_{\partial B_r(x_0)} u(x) d\sigma(x) \right] = -\Delta u(x_0).$$

Exercise 7. Evaluate $\lim\limits_{n\to+\infty}(1-n|x|)^+ \qquad \text{in } \mathcal{D}'(\mathbb{R})$.

Exercise 8. Evaluate $\dfrac{d}{dx}\left((-1)^{[x]}\right)$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 9. Let $\psi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$. Prove that

$$(\psi T)' = \psi' T + \psi T' \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

Exercise 10. Evaluate $\lim\limits_{j\to+\infty} f_j(x)$ where

$$f_j(x) = \begin{cases} \dfrac{1}{x} & \text{if } |x| > 1/j \\ j & \text{if } 0 < x < 1/j \\ -j & \text{if } -1/j < x < 0 \end{cases}$$

in $\mathcal{D}'(\mathbb{R})$.