UNIVERSITY OF CATANIA – Department of Mathematics and Computer Sciences

Master degree in Mathematics - May 23^{rd} 2022

Exercise 1. Prove that

$$\int_{\Omega} |Du|^2 dx \le c \left(\int_{\Omega} |u|^2 dx \right)^{1/2} \left(\int_{\Omega} |D^2 u|^2 dx \right)^{1/2} \qquad \forall u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$$

where c is a constant that does not depend on u.

Exercise 2. Write down Euler–Lagrange equation for

$$\int_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + x^2 \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy$$

Exercise 3. Give an example of a domain where Harnack inequality for equation

$$u_{xx} + x^2 u_{yy} = 0 \qquad \text{in } \mathbb{R}^2.$$

is true.

Exercise 4. Find all real α such that

$$u(x) = |x|^{\alpha}$$

belongs to $W^{1,2}(B_1(0))$ where $B_1(0)$ is the unit ball in \mathbb{R}^n .