UNIVERSITY OF CATANIA – A.A.2019-20

Department of Mathematics and Computer Sciences

Master degree in Mathematics

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Exercise 1. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{R\to 0} \fint_{B_R(0)} u(y)\, dy = u(0).$$

Exercise 2. let Ω be a domain in $\mathbb{R}^n$ and $d : \Omega \to \mathbb{R}$ be such that $d(x) = \mathrm{dist}(x, \partial\Omega)$.

Prove d is Lipschitz continuous and for any $\varepsilon > 0$ the set $\Omega_\varepsilon = \{x \in \Omega \ : \ d(x) > \varepsilon\}$ is open.

Exercise 3. Let $X \subseteq \mathbb{R}^n$ and $\varepsilon > 0$. Prove that

$$X + B_\varepsilon(0) = \{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, X) < \varepsilon\}$$

is open.

Exercise 4. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{1-|x|^2}\right) & \text{se } |x| < 1 \\ 0 & \text{se } |x| \geq 1 \end{cases}$$

belongs to $C_0^\infty(\mathbb{R})$.

Exercise 5. Prove that $C_0^\infty(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ w.r.t.

$$\|f\| = \int_{\mathbb{R}} |f(x)|\, dx$$

Exercise 6. Prove that for any $C^2(\mathbb{R}^n)$ function u we have

$$\lim_{r\to 0} \frac{2n}{r^2} \left[u(x_0) - \fint_{\partial B_r(x_0)} u(x) d\sigma(x) \right] = -\Delta u(x_0)$$

where x_0 is any point in $\mathbb{R}^n$.