UNIVERSITY OF CATANIA – A.A.2019-20

Department of Mathematics and Computer Sciences

Master degree in Mathematics

March 23^{rd} 2020

Exercise 1. Let p, p' be real numbers such that $1 \leq p, p' < +\infty$ e $1/p + 1/p' = 1$.
Prove that

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \qquad \forall a, b \geq 0$$

and then show Hölder inequality as a Corollary.

Exercise 2. Write down polar-coordinates in $\mathbb{R}^4$ and evaluate its Jacobian. Then, generalize to n dimensions where $n \geq 5$.

Exercise 3. Let E be a Lebesgue measurable set with finite measure. Prove that

$$\|f\|_p \leq |E|^{1/p-1/q} \|f\|_q \qquad 1 \leq p \leq q \leq +\infty$$

Exercise 4. Let E be a Lebesgue measurable set with finite measure. Prove that

$$\lim_{p \to +\infty} \|f\|_p = \|f\|_\infty$$

for any $f \in L^\infty(E)$.

Exercise 5. Let f, g and h be summable functions in $\mathbb{R}^n$. Prove that

1. $f * g = g * f$;
2. $f * (g * h) = (f * g) * h$;
3. $f * (g + h) = f * g + f * h$.

Exercise 6. Let $K(s,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be such that

$$K(\lambda s, \lambda t) = \lambda^{-1} K(s,t) \qquad \forall s, t \geq 0\,, \lambda > 0\,.$$

Assume that there exists $p \geq 1$ such that

$$\gamma = \int_0^{+\infty} K(1,t)\, t^{-1/p}\, dt < +\infty\,.$$

Show that

$$(Tf)(s) = \int_0^{+\infty} K(s,t)f(t)\,dt$$

satisfies

$$\|Tf\|_p \leq \gamma\,\|f\|_p \qquad \forall f \geq 0\,,\ f \in L^p\,.$$