ARTICLES

Before The Conquest

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Introduction

In the late 15th century, through their explorers, Europeans "discovered" the New World. Although the discovery would cause drastic change, the New World was, of course, not new to its inhabitants. When the Europeans arrived, there were at least 9 million people in about 800 different cultures living in the Western Hemisphere. Because of the vast disruptions that eventually took place, what we know about them and their mathematical ideas is limited. Most of the cultures had no writing as we commonly use the term and so there are no writings by them in their own words. For the cultures that did not survive, we have primarily what can be learned from archeology and from the writings of the Europeans of the time, who had little understanding and little respect for these cultures so different from their own. For those that did survive, we also have their oral traditions.

We focus here on the mathematical ideas of two sizable groups, the Incas and the Maya. The regions the groups inhabited, their cultures, and their histories are quite distinct, as are their mathematical ideas. Fortunately, for both groups, there is sufficient information for us to gain some understanding of their rich and complex ideas. Here we present an abbreviated introduction to the content and context of the sophisticated data handling system of the Incas and the intricate calendric system of the Maya.

The Incas

The Incas comprised a complex state of about 5 million people that existed from about 1400 C.E. to 1560 C.E. in what is now Peru, and also parts of modern Ecuador, Bolivia, Chile, and Argentina. There were many different peoples in the region, but, starting about 1400, the Incas forcibly consolidated the others into a single bureaucratic entity. The consolidation was achieved by the overlay of a common state religion and a common language, relocation of groups of people, extensive systems of roads and irrigation, and a system of taxation involving, for example, agricultural products, labor, and cloth. The Incas also built a network of storehouses to hold and redistribute goods as well as to feed the army as it moved. The Inca bureaucracy can be characterized as methodical, highly organized, and intensive data users. Although the Incas did not have what we call writing, they did keep extensive records. These were encoded via a logical-numerical system on spatial arrays of colored, knotted cords called *quipus*.

A few selected people from each region that the Incas occupied were trained to serve in the Inca administration and, in particular, to be responsible for gathering, and then encoding and decoding a wide variety of information on the quipus. Believing the quipus to be works of the Devil, the Spanish destroyed thousands of them. Only about 500 remain. These were recovered from graves, probably buried with those who made them. Only rarely can we read the quipus in the sense that specific *meaning* can be assigned. However, we can reconstruct something of their logical-numerical system and, as a result, see the interrelationships of some of the data they contain.

A photograph of a quipu is in FIGURE 1. FIGURE 2 is a schematic. In general, a quipu has a *main cord* from which other cords are suspended. Most of the suspended cords are attached such that they fall in one direction (*pendant cords*); some few fall in the opposite direction (*top cords*). *Subsidiary cords* are often suspended from the pendant or top cords. And there can be subsidiaries of subsidiaries, and so on. (Notice that in FIGURE 2 the first pendant has two subsidiaries on the same level while the fourth pendant has two levels of subsidiaries.) Some pendant cords have as many as 18 subsidiaries on one level, and some have as many as 10 levels of subsidiaries. Sometimes a single cord (*dangle end cord*) is attached to the end of the main cord in a way that sets it apart from the pendant and top cords. All cord attachments are tight so that the spacing between the cords is fixed and serves to group or separate the cords. Overall, a quipu can be made up of as few as three cords or as many as 2000.

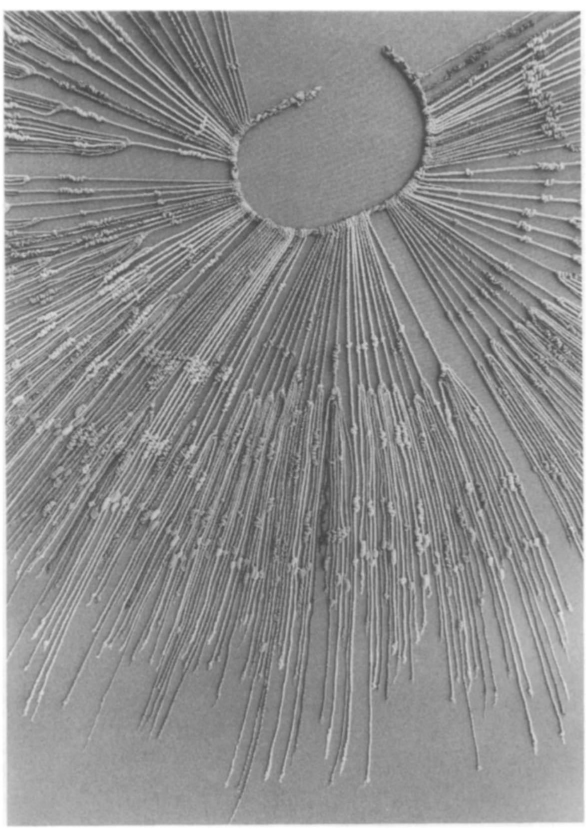

FIGURE 1

A quipu in the collection of the Museo Nacional de Anthropologia y Arquelogía, Lima, Peru. (Photo by Marcia and Robert Ascher.)

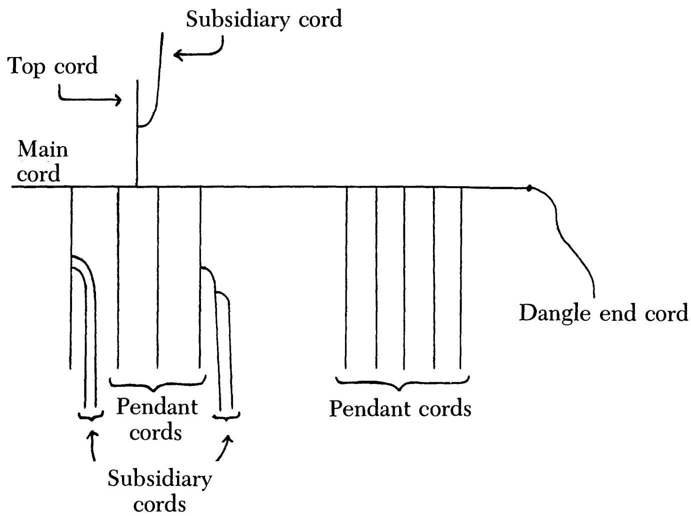

FIGURE 2
A schematic of a quipu.

Color is also a feature of the logical system. It is used primarily to associate or differentiate cords within a single quipu. Thus, color as well as space can create cord groupings. For example, eight pendants can be formed into two groups by having four white pendants followed by four green pendants, or by a four-color sequence repeated twice. In the latter case, each cord is not only associated with its group, but also with the like-colored cord in the other group. Similarly, subsidiaries are associated or differentiated by color as well as by level and relative position on the given level.

Spaced clusters of knots on the cords represent numbers. No matter what the cord placement, only three types of knots appear (single knots, long knots, and figure-8 knots). Depending on the knot types and relative cluster positions, each cord can be interpreted as one number or as multiple numbers. If it is one number, it is an integer in the base 10 positional system. Each knot cluster is read as a digit and each consecutive cluster, starting from the free end of the cord, is valued at one higher power of 10. The units position is always a long-knot cluster or a figure-8 knot, while all other positions are clusters of single knots. When, instead, the cord carries multiple numbers, long-knot clusters or figure-8 knots are interspersed with single-knot clusters thereby signaling the start of a new number. The color coding of the cords also helps in the interpretation of values by enabling the distinction between a numerical value of zero and an intentional omission or blank.

Knot types and knot positions, cord directions, cord levels, color, and spacing are all structural indicators that were combined together in sufficiently standardized ways to be read and interpreted by the community of quipumakers. That is, the quipus served for communication, not as ad hoc personal mnemonic devices. Top cords, for example, generally carry the sum of the pendant cords with which they are grouped on the main cord. Another aspect of the system that is crucial to its general applicability is that numbers were used as labels as well as magnitudes. Particularly with the advent of computers, this usage is now very prevalent in our own culture. For example, the composite number 202-387-5200 is a label identifying a geographic region, a locale within that region, and a specific telephone within that locale.

The quipus, then, are logically structured arrays of magnitudes and labels. Let us translate three of them into notation that is familiar but preserves their logical structure. Then we can delve into some of their internal data relationships.

The quipu shown in FIGURE 1 contains solely quantitative data. Analysis of the pendant cord colors and spacing shows that there are six ordered sets of 18 values each. We will call the jth value in the ith set a_{ij} where $i = 1, \dots, 6$; $j = 1, \dots 18$. When the knots are interpreted as magnitudes, we find that, for all j,

$$a_{1j} = a_{2j} + a_{3j}$$

and, in turn,

$$a_{2j} = \sum_{i=4}^{6} a_{ij}.$$

Hence, the relationship

$$a_{1j} = \sum_{i=3}^{6} a_{ij}$$

also holds. Additionally, there are subsidiary cords on the pendants in five of the six cord groups. Thus, for each a_{ij}, for $i = 2, \dots, 6$; $j = 1, \dots, 18$, there are as many as 11 ordered subsidiary values. Call them

$$a_{ijk} \quad \text{with } k = 1, \dots, 11.$$

Here, too, consistent summation relationships exist:

$$a_{2jk} = \sum_{i=3}^{6} a_{ijk} \quad \text{for } k = 1, \dots, 11;\ j = 1, \dots 18.$$

A modern analogy of data with sums of sums and sets of sums, as is seen in this example, is an accounting scheme for a company, broken down to reflect that it is made up of several departments and producing a variety of products.

In our second example, the arrangement of values and their sums is analogous to a matrix that has, as a subset, the transpose of the sum of two other matrices. Specifically, this quipu's data can be thought of as two 3×3 matrices, each preceded by a single value, and a 3×5 matrix. Calling the elements of the matrices

$$a_{ijk} \quad i = 1, 2, 3;\ j = 1, 2, 3;\ k = 1, 2,$$

and

$$b_{ij} \quad i = 1, 2, 3;\ j = 1, 2, 3, 4, 5,$$

the relationship is

$$b_{i, 2j-1} = \sum_{k=1}^{2} a_{jik} \quad \text{for } i = 1, 2, 3;\ j = 1, 2, 3.$$

And, continuing the analogy to matrices, the single value preceding each of the 3×3's would be the sum of its first row; that is,

$$c_k = \sum_{j=1}^{3} a_{1jk} \quad \text{for } k = 1, 2.$$

Some of the data structures remind us of spreadsheets, matrices, and tree diagrams. Other quipus have other layouts, nonquantitative as well as quantitative data, other kinds of internal data relationships, or even relationships with data on other quipus. Many remain fascinating puzzles. One of these, our final example, is from a pair of quipus that were found together.

The specific numbers on these two quipus are different, but the quipus share several internal data relationships, including what we commonly call a difference table. While both appear to be expressions of the same algorithm, a concise unifying description escapes me. Translated into tabular form, they are compared in FIGURE 3. I have superimposed arrows and heavy lines on the tables to indicate the similarities that I see. Perhaps you can find additional similarities or, perhaps, you can find a generalization that unites the data sets.

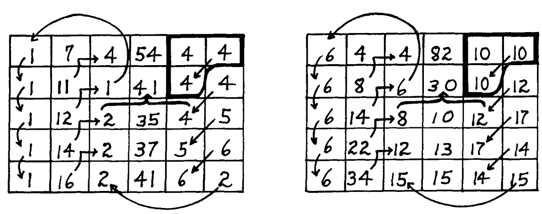

FIGURE 3

Data excerpted from a pair of quipus found together. Arrows and heavy lines highlight some of their similarities. In both, for example, all values are the same in the first column and in row 2, column 3. Also, in both, the third column contains the differences of consecutive values in the second column. (The quipus are described by C. Radicati di Primeglio in La "seriación" como posible clave para descifrar los quipus extranumerales, *Documenta: Revista de La Sociedad Peruana de Historia* 4 (1965), 112–215.)

Reference [1] contains more details, examples, and discussion of the context, contents, and interpretation of quipus. Although we lack the cultural associations needed to know what a specific quipu means, the quipus surely are records of human and material resources and calendric information. But they contain much more—possibly information as diverse as construction plans, dance patterns, and even aspects of Inca history. Overall, the logical-numerical system embedded in these spatial arrays of colored, knotted cords was sufficiently *general* to serve the needs of the Inca bureaucracy. Their use was terminated soon after the destruction of the Inca state in 1560 C.E.

The Maya

The Mayan peoples have a complex cultural tradition extending over a long period of time and encompassing different groups speaking about 25 different languages. They shared much in the way of culture but, spread through time and space, they had different centers and political organizations, some different ideas, and some different practices. Discussions of their history usually begin sometime before 1000 B.C.E. The period 200–1000 C.E. is referred to as the Classic period and is marked by ceremonial centers with monumental architecture, a system of writing, an elaborate astrological science, and numerous centers of social, religious, economic, and political activities interrelated by marriage and trade networks. During the Classic period, the

Maya inhabited what are now the eastern Mexican states of Chiapas, Tabasco, Campeche, Quintano Roo, and Yucatan; Belize; Guatemala; and the western portions of Honduras and El Salvador. On the eve of the Spanish conquest, there were spread out in this area, many independent yet culturally interrelated states, none as grandiose as earlier. Because they were dispersed and independent, they did not succumb to the Spanish as quickly and easily as did the Incas. Today, primarily in Chiapas and the highlands of Guatemala, some Maya traditions continue.

Christopher Columbus, in 1502, is said to have been the first European to encounter the Maya, and his brother, Bartholomew, was the first to record the name of the group. By that time, however, remains of the Classic Maya period were already covered over, and so another "discovery"—this time archeological—took place beginning in the mid-1800s. In addition to some ongoing traditions, what we know of the Maya, and in particular of their mathematical ideas, comes from archeological materials, including thousands of inscribed stone monuments (*stelae*) and four post-Classic books (codices), the only ones remaining of the *thousands* that were burned by the Spanish.

We will concentrate on the idea of *time* as it permeates the Mayan culture. Time is considered to be cyclic. Supernatural forces and beings are associated with and influence units of time. Events of the past, present, and future are related through the recurrence of named time units. There are, however, not just one, but several, overlapping cycles that all must be taken into consideration to give meaning to any particular time unit. Although their calendric concerns extend to the incorporation of astronomical phenomena, the Maya were preoccupied with the interrelationship of the arbitrary cycles they created and imposed on time. For this reason, the Maya are said to have "mathematized" time and, through it, their religion and cosmology.

There is, first of all, a 260-day *ritual almanac*. Each day within it is identified by a number in a cycle of 13 and a named deity in a cycle of 20. (Each of the 13 numbers also has an associated deity.) There is a *vague year* of 365 days (called "vague" because it does not keep in alignment with the seasons). It results from a cycle of 20 numbers *within* a cycle of 18 named deities plus five unnamed days. (The cycle of 20 is now referred to as a *month* but does not have a lunar correspondence.) One *calendar round* is 18,980 days (52 vague years, 78 almanac cycles) since that is the least common multiple of 365 and 260. A date within this, made up of an almanac date and a vague year date, reads, for example, 4 Ahau 8 Cumku where Ahau and Cumku are names of deities.

In the ceremonial centers of the Classic period, there were temples atop large, stepped pyramid frusta as high as 213 feet. Hundreds of stelae, some as tall as 32 feet, were erected around them to commemorate different events. To mark an event, what was needed was to accurately and *fully* identify it in time and, sometimes, to state how many days it was from another event. In addition to the calendar round date, another significant identifier was a *Long Count*: the number of days from the beginning of the then current *Great Cycle*. A Great Cycle is based on a 360-day period (a *tun*) consisting of 18 *uinals* of 20 days each; 20 tuns are a *katun*; 20 katuns are a *baktun*; and 13 baktuns are a Great Cycle. An example of a Long Count transcribed into our numerals is 9.0.19.2.4. From left to right this reads "9 baktuns, 0 katuns, 19 tuns, 2 uinals, and 4 days." To convert to our system, starting at the right, each position—with the exception of the third—is multiplied by one higher power of 20. In the third position, an 18 is used instead. Hence, the Long Count date of 9.0.19.2.4 is interpreted as:

$$9 \cdot 18 \cdot 20^3 + 0 \cdot 18 \cdot 20^2 + 19 \cdot 18 \cdot 20 + 2 \cdot 20 + 4 = 1{,}302{,}884 \text{ days}$$

from the beginning of the Great Cycle that started on the calendar round date of 4 Ahau 8 Cumku. The exact correlation of this date with the Gregorian calendar is not known. However, by one commonly accepted correlation, the beginning of the Great Cycle was in 3114 B.C.E. and the date given by the Long Count above is, thus, in 454 C.E.

This Long Count date appeared on a stela that also was dated in the calendar round as 2 Kan 2 Yax. But, as with most stelae, it had dates placing it within still more cycles. There was a 9-day cycle of *Lords of the Night*, each associated with one of the nine levels of the underworld. Hence, a specific Lord of the Night also dates the day being marked. And, in addition, the day is placed within a *lunar cycle*. Lunar years and half-years are made up of 29- and 30-day lunar months. The stela contains the moon number within the lunar half-year, the age of the moon, and whether it is a 29- or 30-day month.

Just as there are nine levels below the earth, there are 13 levels in the heavens above. There are four cardinal directions and each of the quadrants they define is associated with a different color. Uniting time and space, the days of the 260-day ritual almanac move in a counterclockwise direction through the four quadrants. Hence, not only are time and space related, but the ritual almanac has within it a four-color cycle. In some cases, where dates also identify days within a 819-day cycle associated with the rain god, the use of four colors effectively makes that cycle $819 \cdot 4 = 3276$ days.

Many of the Maya computations are projections into the past or into the future that require dovetailing the cycles. For instance, one inscription, commemorating the enthronement of a ruler, gives the calendar round dates of his birth and his enthronement, as well as of the enthronement of an earlier, somehow related ruler or deity. The number of days between these events is also included in Long Count form. For example, the time elapsed since the enthronement of the deity is 7.18.2.9.2.12.1 days. Hence, given one calendar round date, a calendar round date some $1\frac{1}{4}$ million years earlier was calculated or, given two calendar round dates, their Long Count difference (number of days between them) was calculated.

To more fully savor the calculation, you might try to do such a problem. First, recall that each of the 260 days in the ritual almanac is identified by a number in a cycle of 1 to 13 and a named deity in a cycle of 20. For simplicity, let us call the deities $D_1, \dots, D_{20}$. In the 365-day vague year, five days are unnamed while, for the rest, each day is identified by a number in a cycle of 1 to 20 within each of 18 months named for deities. Call these deities $d_1, \dots, d_{18}$ and assume that the five unnamed days follow $20d_{18}$. The calendar round date is the almanac date followed by the vague year date. What, then, is the calendar round date that is 2.3.5.10 days after $8D_{10}13d_{10}$? And, what is the Long Count Difference between $12D_8 6d_2$ and the next $5D_4 12d_{17}$? (Answers are on page 235.)

The Dresden Codex, attributed to the eleventh century in Yucatan is the most mathematical of the codices. It is constructed as a long strip of paper made from tree bark, folded into pages, coated with white plaster, and painted. Perhaps as aids to computation, the codex includes several tables of multiples; for example, there are tables of multiples of 5.1.0 (that is, of 1820 that equals $7 \cdot 260$ and $5 \cdot 364$) up to 1.0.4.8.0. But, even more important, other tables in the codex combine backward and forward calendar projections with evidence of keen astronomical knowledge. One set of tables correlates lunar cycles with ritual almanac dates. These tables cover 405 lunations and are interpreted as prediction tables for possible eclipses. Another set of tables in this codex correlates Venus visibility events with ritual almanac and vague year dates. Covering 65 Venus cycles, which is 146 almanac cycles and 104 vague years, it includes corrections reflecting the fact that the mean synodic year of Venus is not an integral number of days. (The mean synodic year of Venus is 583.92 days.) The corrections are such that the error between real and tabulated times of the positions of Venus would be off by just two hours in about 500 years!

We know that much is unknown about the knowledge and mathematical ideas of the Maya. Dates and numbers, written in a variety of symbolic forms, have been recognized and deciphered. But the Maya writings contain much more. The writing system is complex because it contains about 1000 symbols and both phonetic and nonphonetic elements. A great deal of recent activity in decipherment raises the hope that more will become known about Maya ideas and the Maya culture in general. (For more details on number representation, the tables in the Dresden Codex, and possible algorithms for the date difference calculations, see [2] and [5]. Reference [3] discusses the importance of the Maya scribes including evidence that they were both women and men. Also, [4] is an excellent comprehensive overview of the Maya.)

Conclusion

The Inca and Maya are two substantial examples of cultures whose mathematical ideas were both sophisticated and independent of those of Western culture. We can never know about all of the mathematical ideas they had and, what is more, we cannot know what they might have developed had they continued to thrive.

REFERENCES

1. M. Ascher and R. Ascher, *Code of the Quipu: A Study in Media, Mathematics and Culture*, University of Michigan Press, Ann Arbor, MI, 1981.
2. M. Closs, The mathematical notation of the ancient Maya, *Native American Mathematics*, ed. by M. Closs, University of Texas Press, Austin, TX, 1986, pp. 291–369.
3. ______, My mother and my father were both scribes, *Research Reports on Ancient Maya Writing*, Center for Maya Research, Washington, DC, in press.
4. J. S. Henderson, *The World of the Ancient Maya*, Cornell University Press, Ithaca, NY, 1981.
5. F. G. Lounsbury, Maya numeration, computation, and calendrical astronomy, *Dictionary of Scientific Biography*, Vol. 15, suppl. 1, Scribners, New York, 1978, pp. 759–818.