Relation algebra (in a nutshell)
State transition dynamics
Recurrence and attractors
Research outlook
References

Recurrence and attractors in state transition dynamics A relational view

Giuseppe Scollo¹
in collaborazione con
Giuditta Franco², Vincenzo Manca²

¹ Dipartimento di Matematica e Informatica Università di Catania

> ²Dipartimento di Informatica Università di Verona

> > 3 luglio 2006





- Relation algebra (in a nutshell)
 - standard concepts and notation
 - concepts and notation for state transition dynamics
- State transition dynamics
 - basic concepts
 - a contrived example
- Recurrence and attractors
 - basic definitions and facts
 - existence of nonrecurrent flights
 - nonexistence of the unavoidable attractor
 - flights in absence of eternal recurrence
 - characterization of recurrence and attractors
- Research outlook
 - research questions and perspectives
- 5 References
 - state transition dynamics
 - background for further study
 - metabolic P systems





- Relation algebra (in a nutshell)
 - standard concepts and notation
 - concepts and notation for state transition dynamics
- 2 State transition dynamics
 - basic concepts
 - a contrived example
- Recurrence and attractors
 - basic definitions and facts
 - existence of nonrecurrent flights
 - nonexistence of the unavoidable attractor
 - flights in absence of eternal recurrence
 - characterization of recurrence and attractors
- 4 Research outlook
 - research questions and perspectives
- References
 - state transition dynamics
 - background for further study
 - metabolic P systems





Relation algebra (in a nutshell) State transition dynamics Recurrence and attractors Research outlook References

- Relation algebra (in a nutshell)
 - standard concepts and notation
 - concepts and notation for state transition dynamics
- 2 State transition dynamics
 - basic concepts
 - a contrived example
- 3 Recurrence and attractors
 - basic definitions and facts
 - existence of nonrecurrent flights
 - nonexistence of the unavoidable attractor
 - flights in absence of eternal recurrence
 - characterization of recurrence and attractors
- A Research outlook
 - research questions and perspectives
- References
 - state transition dynamics
 - background for further study
 - metabolic P systems





Relation algebra (in a nutshell)
State transition dynamics
Recurrence and attractors
Research outlook
References

- Relation algebra (in a nutshell)
 - standard concepts and notation
 - concepts and notation for state transition dynamics
- 2 State transition dynamics
 - basic concepts
 - a contrived example
- 3 Recurrence and attractors
 - basic definitions and facts
 - existence of nonrecurrent flights
 - nonexistence of the unavoidable attractor
 - flights in absence of eternal recurrence
 - characterization of recurrence and attractors
- Research outlook
 - research questions and perspectives
- 5 References
 - state transition dynamics
 - background for further study
 - metabolic P systems





- Relation algebra (in a nutshell)
 - standard concepts and notation
 - concepts and notation for state transition dynamics
- 2 State transition dynamics
 - basic concepts
 - a contrived example
- 3 Recurrence and attractors
 - basic definitions and facts
 - existence of nonrecurrent flights
 - nonexistence of the unavoidable attractor
 - flights in absence of eternal recurrence
 - characterization of recurrence and attractors
- Research outlook
 - research questions and perspectives
 - 5 References
 - state transition dynamics
 - background for further study
 - metabolic P systems





relation algebra: a complete, atomic boolean algebra enriched with

```
binary associative composition: p; (q; r) = (p; q); r an identity constant: 1'; r = r; 1'= r
```

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p ; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

provable laws (among many others):

```
r = r

(q; r) = r ; q

p; (q + r) = (p; q) + (p; r), 	 <math>(p + q); r = (p; r) + (q; r)
```

- a relation algebra is representable if
 it is isomorphic to a boolean algebra of binary relations,
 with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



4日 > 4周 > 4 至 > 4 至

• relation algebra: a complete, atomic boolean algebra enriched with

```
binary converse: r
binary associative composition: p; (q; r) = (p; q); an identity constant: 1'; r = r; 1' = r
```

that satisfy some further laws

```
Schröder equivalences: p; q \le r \Leftrightarrow p^*; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q^* \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

provable laws (among many others):

```
r^- = r

(q; r)^- = r^-; q^-

p; (q+r) = (p; q) + (p; r), 	 (p+q); r = (p; r) + (q; r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



4日 > 4周 > 4 至 > 4 至

 relation algebra: a complete, atomic boolean algebra enriched with unary converse: r~

```
binary associative composition: p; (q; r) = (p; q); r an identity constant: 1'; r = r; 1' = r
```

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p^*; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q^* \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

provable laws (among many others):

```
r = r

(q; r)^{\sim} = r^{\sim}; q^{\sim}

p; (q+r) = (p; q) + (p; r), (p+q); r = (p; r) + (q; r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



4日 > 4周 > 4 至 > 4 至

 relation algebra: a complete, atomic boolean algebra enriched with unary converse: r

```
binary associative composition: p; (q; r) = (p; q); r
```

at satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p^*; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q^* \le p^-
Tarski rule: r \ne 0 \Rightarrow 1 : r : 1 = 1
```

```
 (q; r) = r ; q 
 (p; q) = r ; q 
 p; (q+r) = (p; q) + (p; r), 
 (p+q); r = (p; r) + (q; r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r
```

```
binary associative composition: p; (q; r) = (p; q); r an identity constant: 1'; r = r; 1' = r
```

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p ; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

```
r = r

(q; r)^{\sim} = r^{\sim}; q^{\sim}

p; (q + r) = (p; q) + (p; r), 	 (p + q); r = (p; r) + (q; r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r
```

an **identity** constant: 1'; r = r; 1' = r

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p^*; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q^* \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

```
(q;r) = r

(q;r) = r; q

p; (q+r) = (p;q) + (p;r), 	 <math>(p+q); r = (p;r) + (q;r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r
```

```
binary associative composition: p; (q; r) = (p; q); r an identity constant: 1'; r = r; 1' = r
```

that satisfy some further laws:

Schröder equivalences:
$$p$$
; $q \le r \Leftrightarrow p^{\check{}}$; $r^{-1} \le q^{-1} \Leftrightarrow r^{-1}$; $q^{\check{}} \le p^{-1}$

```
(q;r)^{-} = r^{-}; q^{-}

(q;r)^{-} = r^{-}; q^{-}

p; (q+r) = (p;q) + (p;r), (p+q); r = (p;r) + (q;r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

unary **converse**: r

binary associative **composition**: p; (q; r) = (p; q); r an **identity** constant: 1'; r = r; 1' = r

that satisfy some further laws:

Schröder equivalences: p; $q \le r \Leftrightarrow p$; $r^{-1} \le q^{-1} \Leftrightarrow r^{-1}$; q $\le p^{-1}$

Tarski rule: $r\neq 0 \Rightarrow 1$; r; 1 = 1

provable laws (among many others):

(q;r) = r ; qp; (q+r) = (p;q) + (p;r), (p+q); r = (p;r) + (q;r)

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r an identity constant: 1': r = r; 1' = r
```

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p ; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

```
r^{"} = r

(q; r)^{"} = r^{"}; q^{"}

p; (q + r) = (p; q) + (p; r), (p + q); r = (p; r) + (q; r)
```

- a relation algebra is representable if it is isomorphic to a boolean algebra of binary relations, with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r an identity constant: 1': r = r; 1' = r
```

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p ; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

```
r^{w} = r

(q; r)^{\tilde{}} = r^{\tilde{}}; q^{\tilde{}}

p; (q+r) = (p; q) + (p; r), (p+q); r = (p; r) + (q; r)
```

- a relation algebra is representable if
 it is isomorphic to a boolean algebra of binary relations,
 with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)





• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r an identity constant: 1': r = r; 1' = r
```

that satisfy some further laws:

```
Schröder equivalences: p; q \le r \Leftrightarrow p ; r^{-1} \le q^{-1} \Leftrightarrow r^{-1}; q \le p^{-1}
Tarski rule: r \ne 0 \Rightarrow 1; r; 1 = 1
```

$$r = r$$

 $(q; r) = r ; q$
 $p; (q + r) = (p; q) + (p; r), (p + q); r = (p; r) + (q; r)$

- a relation algebra is representable if
 it is isomorphic to a boolean algebra of binary relations,
 with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r an identity constant: 1': r = r; 1' = r
```

that satisfy some further laws:

Schröder equivalences:
$$p$$
; $q \le r \Leftrightarrow p$; $r^{-1} \le q^{-1} \Leftrightarrow r^{-1}$; $q \le p^{-1}$
Tarski rule: $r \ne 0 \Rightarrow 1$; r ; $1 = 1$

$$r^{\sim} = r$$

 $(q; r)^{\sim} = r^{\sim}; q^{\sim}$
 $p; (q + r) = (p; q) + (p; r), (p + q); r = (p; r) + (q; r)$

- a relation algebra is representable if
 it is isomorphic to a boolean algebra of binary relations,
 with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r an identity constant: 1'; r = r; 1' = r
```

that satisfy some further laws:

Schröder equivalences:
$$p$$
; $q \le r \Leftrightarrow p$; $r^{-1} \le q^{-1} \Leftrightarrow r^{-1}$; $q \le p^{-1}$
Tarski rule: $r \ne 0 \Rightarrow 1$; r ; $1 = 1$

$$r^{\sim} = r$$

 $(q; r)^{\sim} = r^{\sim}; q^{\sim}$
 $p; (q + r) = (p; q) + (p; r), (p + q); r = (p; r) + (q; r)$

- a relation algebra is representable if
 it is isomorphic to a boolean algebra of binary relations,
 with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



• relation algebra: a complete, atomic boolean algebra enriched with

```
unary converse: r binary associative composition: p; (q; r) = (p; q); r an identity constant: 1'; r = r; 1' = r
```

that satisfy some further laws:

Schröder equivalences:
$$p$$
; $q \le r \Leftrightarrow p$; $r^{-1} \le q^{-1} \Leftrightarrow r^{-1}$; $q \le p^{-1}$
Tarski rule: $r \ne 0 \Rightarrow 1$; r ; $1 = 1$

$$r = r$$

 $(q; r) = r ; q$
 $p; (q + r) = (p; q) + (p; r), $(p + q); r = (p; r) + (q; r)$$

- a relation algebra is representable if
 it is isomorphic to a boolean algebra of binary relations,
 with set-theoretic interpretation of converse, composition and identity
- while every boolean algebra is representable (Stone), not every relation algebra so is (Lyndon)



References

- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 \Rightarrow y = x
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if q is a binary relation and x, y are monotypes in a relation algebra, **define**:
 - $x <_{\alpha}^{(n)} v$ if $\operatorname{img}(x : \alpha \ge n) < v$
 - the eventually below under q-iteration relation on monotypes





- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; 1)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 \Rightarrow y = x
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if q is a binary relation and x, y are monotypes in a relation algebra, **define**:
 - \bullet $x \leq_{\sigma}^{(n)} y$ if $img(x; \sigma^{\geq n}) \leq y$





- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q; dom $q = 1' \cdot (q : 1)$, image of q; image q = 1
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1; y; 1 = 1 \Rightarrow y = x$
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if q is a binary relation and x, y are monotypes in a relation algebra, **define**:
 - $x \leq_q^{w'} y$ if $\operatorname{img}(x; q^{\leq w}) \leq y$
 - the eventually below under q-iteration relation on monotypes:





- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q : 1)$, image of q: image $q = 1' \cdot (1 : 1)$
- atomic monotypes: characterized by the quasiequations x < 1'. 1: x : 1 = 1. and $y < x \land 1 : y : 1 = 1 \Rightarrow y = 3$
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if *q* is a binary relation and *x*, *y* are monotypes in a relation algebra, **define**:
 - $x \leq_q^{-1} y$ if $\operatorname{img}(x; q-1) \leq y$
 - the **eventually below under** q-iteration relation on monotypes:





- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; q)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 \Rightarrow y = x
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if *q* is a binary relation and *x*, *y* are monotypes in a relation algebra, **define**:







- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; q)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 \Rightarrow y = x
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if *q* is a binary relation and *x*, *y* are monotypes in a relation algebra, **define**:







- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; q)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 \Rightarrow y = x
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if *q* is a binary relation and *x*, *y* are monotypes in a relation algebra, **define**:





- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

note: $q^+ = q^{\geq 1}$ and $q^* = q^{\geq 0}$.

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; q)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 $\Rightarrow y = x$
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if *q* is a binary relation and *x*, *y* are monotypes in a relation algebra, **define**:
 - $x \leq_{a}^{(n)} y$ if $\operatorname{img}(x; q^{\geq n}) \leq y$
 - the **eventually below under** *q***-iteration** relation on monotypes: $q = \sum_{i=1}^{n} q_i e^{in}$





4日×4周×4厘×4厘×

- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; q)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 $\Rightarrow y = x$
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful higher-order binary relations on monotypes: if q is a binary relation and x, y are monotypes in a relation algebra, define:
 - $x \leq_a^{(n)} y$ if $img(x; q^{\geq n}) \leq y$
 - the **eventually below under** q-iteration relation on monotypes:





- iteration: $q^0 = 1', q^{i+1} = q; q^i$
- Kleene closure and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

note: $q^+ = q^{\geq 1}$ and $q^* = q^{\geq 0}$.

- monotypes: subrelations of the identity, viz. $x \le 1'$, such as domain of q: dom $q = 1' \cdot (q; 1)$, image of q: img $q = 1' \cdot (1; q)$
- atomic monotypes: characterized by the quasiequations $x \le 1'$, 1; x; 1 = 1, and $y \le x \land 1$; y; 1 = 1 \Rightarrow y = x
- **notation:** $x \le y$ means that x is an atomic monotype and $x \le y$
- useful *higher-order binary relations on monotypes*: if *q* is a binary relation and *x*, *y* are monotypes in a relation algebra, **define**:
 - $x \leq_q^{(n)} y$ if $\operatorname{img}(x; q^{\geq n}) \leq y$
 - the eventually below under q-iteration relation on monotypes: $\le q = \sum_{n \in \mathbb{N}} \le_q^{(n)}$



4日×4周×4厘×4厘×

• state transition (ST) dynamics: (S, q)

```
S : (discrete) state space q \subseteq S \times S : [total] transition dynamics [dom q = 1'_S
```

- straightforward extension of q to **quasistates**, represented by nonzero monotypes $x \le 1_S'$ —the atomic ones represent **individual states**: $x \le 1_S'$
- fixed point of q: a state x such that img(x; q) = x
- **orbit** (of *origin* x_0) : $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory: function $\xi: \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight: an x-flight in the converse q-dynamics



state transition (ST) dynamics: (S, q)

```
S : (discrete) state space q \subseteq S \times S : [total] transition dynamics [dom q = 1'_S
```

- straightforward extension of q to **quasistates**, represented by nonzero monotypes $x \le 1_S'$ —the atomic ones represent **individual states**: $x \le 1_S'$
- fixed point of q: a state x such that img(x; q) = x
- **orbit** (of *origin* x_0) : $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** $b: 0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory: function $\xi: \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight: an x-flight in the converse q dynamics



state transition (ST) dynamics: (S, q)

S: (discrete) state space

```
q \subseteq S \times S: [total] transition dynamics [dom q = 1'_{S}]
```

- straightforward extension of q to **quasistates**, represented by nonzero monotypes $x \le 1_S'$ —the atomic ones represent **individual states**: $x \le 1_S'$
- fixed point of q: a state x such that img(x; q) = x
- **orbit** (of *origin* x_0) : $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_{S}$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight: an x-flight in the converse q-dynamics



state transition (ST) dynamics: (S, q)

```
S: (discrete) state space q \subseteq S \times S: [total] transition dynamics [dom q = 1'_S]
```

- straightforward extension of q to **quasistates**, represented by nonzero monotypes $x \le 1_S'$ —the atomic ones represent **individual states**: $x \le 1_S'$
- fixed point of q: a state x such that img(x; q) = x
- **orbit** (of *origin* x_0) : $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \text{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory: function $\xi: \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight: an x-flight in the converse q-dynamics





- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- **orbit** (of *origin* x_0) : $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \text{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q-dynamics



- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- **orbit** (of *origin* x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q -dynamics



state transition (ST) dynamics: (S, q)

```
S: (discrete) state space
```

 $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]

- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight: an x-flight in the converse q -dynamics



- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q dynamics



state transition (ST) dynamics: (S, q)

```
S: (discrete) state space q \subseteq S \times S: [total] transition dynamics [dom q = 1'_S]
```

- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1_S'$ —the atomic ones represent individual states: $x \le 1_S'$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- **basin** b: $0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight: an x-flight in the converse q dynamics



- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- basin $b: 0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- x-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q-dynamics



- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : img(x_0; q^{k+n}) = img(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- basin $b: 0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- *x*-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q dynamics



- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1_S'$ —the atomic ones represent individual states: $x \le 1_S'$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- basin $b: 0 \neq b \leq 1'_S$ s.t. $img(b; q) \leq b$
- *x*-trajectory : function $\xi: \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q-dynamics



- state transition (ST) dynamics: (S, q)
 - S: (discrete) state space
 - $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]
- straightforward extension of q to quasistates, represented by nonzero monotypes $x \le 1'_S$ —the atomic ones represent individual states: $x \le 1'_S$
- fixed point of q: a state x such that img(x; q) = x
- orbit (of origin x_0): $(x_i \mid i \in \mathbb{N})$ such that $x_{i+1} = \operatorname{img}(x_i; q)$
- eventually periodic orbit: $\exists k \ge 0 \exists n > 0 : \text{img}(x_0; q^{k+n}) = \text{img}(x_0; q^k)$ (periodic orbit if k = 0)
- orbit $(x_i \mid i \in \mathbb{N})$ (eventually) included in $(y_i \mid i \in \mathbb{N})$: $(\exists j \geq 0) \ \forall i (\geq j) \ x_i \leq y_i$
- basin $b: 0 \neq b \leq 1'_{S}$ s.t. $img(b; q) \leq b$
- *x*-trajectory : function $\xi : \mathbb{N} \to 1_S'$ s.t. $\xi_0 = x \le 1_S'$ and $\xi_{n+1} \le \operatorname{img}(\xi_n; q)$
- x-flight : an injective x-trajectory
- x-antiflight : an x-flight in the converse q-dynamics



a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see

 [(Rights at al., 2000s)] (Rights at al., 2000s)] (Manage 2000s)]







a simple model of epidemic propagation

as well as of unstable catalythic reaction:

 $CG \rightarrow GG$

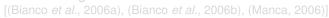
 $C \rightarrow C$

 $G \rightarrow \lambda$

 $G \rightarrow K$

 $G \rightarrow G$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow X$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states : (|C| + |K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see







a simple model of epidemic propagation

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$${m G}
ightarrow {m G}$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states: (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see





a simple model of epidemic propagation

as well as of unstable catalythic reaction:

$$CG \rightarrow GG$$

$$C \rightarrow C$$

$$G \rightarrow \lambda$$

$$G \rightarrow K$$

$$G \rightarrow G$$

- instability of agent G: it either dies or recovers (becoming immune: K)
- states: (|C|+|K|, |G|)
- transitions can be determined by equipping rules with a measure of relative strength: see

[(Bianco et al., 2006a), (Bianco et al., 2006b), (Manca, 2006)]





recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set, recurrence, attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a \leq b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \le \sum_{r - \le q} r$$

• a is a potential attracting set of b, a \diamond -attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r = \leq_q a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, $x \lozenge \mathbf{rec} b$, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and x; $q^* \le x$; q^*



recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set**, **recurrence**, **attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a \leq b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \le \sum_{r - \le q} r$$

a is a potential attracting set of b, a ◊-attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r-\leq_q a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, $x \lozenge \mathbf{rec} b$, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and x; $q^* \le x$; q^*



recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set**, **recurrence**, **attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a \leq b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \le \sum_{r \le q} r$$

• a is a potential attracting set of b, a \diamond -attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r \leq_q a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, $x \lozenge$ -rec b, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and x; $q^* \le x$; q^*



recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set**, **recurrence**, **attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a < b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \le \sum_{r \le q} r$$

a is a potential attracting set of b, a ◊-attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r-\leq_q a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, $x \lozenge \mathbf{rec} b$, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and x; $q^* \le x$; $q^* \le x$



recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set**, **recurrence**, **attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a \leq b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \le \sum_{r \le q} r$$

• a is a potential attracting set of b, a \lozenge -attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r - \leq_q a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, $x \lozenge \mathbf{rec} b$, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and x; $q^* \le x$; q^*



recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set**, **recurrence**, **attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a < b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \leq \sum_{r \leq_q a} r$$

• a is a potential attracting set of b, a \lozenge -attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r \leq_a a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is **recurrent** in basin b, $x \lozenge$ -rec b, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and $x : q^* \le x : q^*$



recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set**, **recurrence**, **attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a \leq b$:

• a is an unavoidable attracting set of b, a \square -attracts b, if

$$b \leq \sum_{r \leq_q a} r$$

a is a potential attracting set of b, a ◊-attracts b, if

$$b \leq \operatorname{dom}(q^*; \sum_{r - \leq_q a} r)$$

- the (unavoidable | potential) attractor of a basin b is a minimal (unavoidable | potential) attracting set of b, under boolean ordering
- x is recurrent in basin b, $x \lozenge$ -rec b, if $x \le b$ and $x \le \operatorname{img}(x; q^+)$
- x is eternally recurrent in basin b, $x \square$ -rec b, if $x \le b$ and x; $q^* \le x$; q^*



recurrence and attractors: basic facts

let b be a basin in the q-dynamics

- b □-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts $b \bowtie a \lozenge$
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists otherwise we write $a_{\square} = 0$
- a_{s} is the *unique* potential attractor of b, when it exists otherwise we write $a_{s} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

- Definition:
 - flight ξ is recurrent in b if ξ_n≤ img(r_κ; q*) for some n∈N
 - flight ξ is eternally recurrent in b if $\xi_N < \text{img}(r_n : q^{**})$





let b be a basin in the q-dynamics:

- b □-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts $b \bowtie a \lozenge$
- a_{\square} is the *unique* unavoidable attractor of b, when it exists otherwise we write $a_{\square} = 0$
- a_{s} is the *unique* potential attractor of b, when it exists otherwise we write $a_{s} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

- Definition:
 - flight ξ is **recurrent in** b if $\xi_n \leq \operatorname{img}(r_{\wedge}; q^*)$ for some $n \in \mathbb{N}$
 - flight \mathcal{E} is eternally recurrent in b if $\mathcal{E}_{\mathbb{N}} < \operatorname{img}(r_{-} : \sigma^{**})$





let b be a basin in the q-dynamics:

- b□-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts $b \bowtie a \lozenge$
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists otherwise we write $a_{\square} = 0$
- a_{0} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{0} = 0$
- recurrence sets in b: let

$$r_{\lozenge} = \sum_{x \, \lozenge \text{-rec } b} x \,, \quad r_{\square} = \sum_{x \, \square \text{-rec } b} x$$

- Definition:
 - flight ξ is **recurrent in** b if $\xi_n \leq \operatorname{img}(r_{\lambda}; q^*)$ for some $n \in \mathbb{N}$
 - flight \mathcal{E} is eternally recurrent in b if $\mathcal{E}_{\mathbb{N}} < \operatorname{img}(r_{-} : \sigma^{**})$





let b be a basin in the q-dynamics:

- b□-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_o is the unique potential attractor of b, when it exists, otherwise we write a_o = 0
- recurrence sets in b: let

$$r_{\lozenge} = \sum_{x \, \lozenge \text{-rec } b} x \,, \quad r_{\square} = \sum_{x \, \square \text{-rec } b} x$$

- Definition:
 - flight ξ is recurrent in b if ξ_n≤ img(r_k; q*) for some n∈N
 - flight \mathcal{E} is eternally recurrent in b if $\mathcal{E}_{\mathbb{N}} < \operatorname{img}(r_{-} : \sigma^{**})$





let b be a basin in the q-dynamics:

- b □-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_{0} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{0} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

- Definition:
 - flight ξ is **recurrent in** b if $\xi_n \leq \operatorname{img}(r_o; q^*)$ for some $n \in \mathbb{N}$
 - flight ξ is eternally recurrent in b if $\xi_{\mathbb{N}} \leq \operatorname{img}(r_{\square}; q^{-*})$



let b be a basin in the q-dynamics:

- b □-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_{\circ} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\circ} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

- Definition:
 - flight ξ is recurrent in b if ξ_n≤ img(r_o; q*) for some n∈N
 flight ξ is eternally recurrent in b if ξ_N ≤ img(r_o; q**)



recurrence and attractors: basic facts

let b be a basin in the q-dynamics:

- b□-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_{\circ} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\circ} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

then: $r_{\square} \le r_{\lozenge}$ and $img(r_{\square}; q^*) = r_{\square}$

Definition:

flight ξ is recurrent in b if ξ_n≤ img(r_o; q*) for some n∈N
 flight ξ is eternally recurrent in b if ξ_N ≤ img(r_□; q**)



let b be a basin in the q-dynamics:

- b□-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_{\circ} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\circ} = 0$
- recurrence sets in b: let

$$\mathit{r}_{\Diamond} \, = \sum_{\mathit{x}\,\Diamond\operatorname{-rec}\,\mathit{b}} \mathit{x}\,, \quad \mathit{r}_{\Box} \, = \sum_{\mathit{x}\,\Box\operatorname{-rec}\,\mathit{b}} \mathit{x}$$

then: $r_{\square} \leq r_{\Diamond}$ and $img(r_{\square}; q^*) = r_{\square}$

Definition:

flight ξ is recurrent in b if ξ_n≤ img(r_o; q*) for some n∈N
 flight ξ is eternally recurrent in b if ξ_N ≤ img(r_□; q**)



let b be a basin in the q-dynamics:

- b□-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_{\circ} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\circ} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

- Definition:
 - flight ξ is **recurrent in** b if $\xi_n \leq \operatorname{img}(r_{\wedge}; q^*)$ for some $n \in \mathbb{N}$
 - flight ξ is **eternally recurrent in** b if $\xi_{\mathbb{N}} \leq \operatorname{img}(r_{\square}; q^{\vee *})$



let b be a basin in the q-dynamics:

- b□-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of b, when it exists, otherwise we write $a_{\square} = 0$
- a_{\circ} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\circ} = 0$
- recurrence sets in b: let

$$r_{\Diamond} = \sum_{x \, \Diamond \text{-rec } b} x \,, \quad r_{\Box} = \sum_{x \, \Box \text{-rec } b} x$$

- Definition:
 - flight ξ is **recurrent in** b if $\xi_n \leq \operatorname{img}(r_{\wedge}; q^*)$ for some $n \in \mathbb{N}$
 - flight ξ is eternally recurrent in b if $\xi_{\mathbb{N}} \leq \operatorname{img}(r_{\square}; q^{**})$





recurrence and attractors: basic facts

let b be a basin in the q-dynamics:

- b □-attracts b
- $a \square$ -attracts $b \Rightarrow a \lozenge$ -attracts b
- a_{\square} is the *unique* unavoidable attractor of *b*, when it exists, otherwise we write $a_{\square} = 0$
- a_{\circ} is the *unique* potential attractor of b, when it exists, otherwise we write $a_{\circ} = 0$
- recurrence sets in b: let

$$\mathit{r}_{\Diamond} \, = \sum_{\mathit{x}\,\Diamond\operatorname{-rec}\,\mathit{b}} \mathit{x}\,, \quad \mathit{r}_{\Box} \, = \sum_{\mathit{x}\,\Box\operatorname{-rec}\,\mathit{b}} \mathit{x}$$

- Definition:
 - flight ξ is **recurrent in** b if $\xi_n \leq \operatorname{img}(r_{\wedge}; q^*)$ for some $n \in \mathbb{N}$
 - flight ξ is eternally recurrent in b if $\xi_{\mathbb{N}} \leq \operatorname{img}(r_{\square}; q^{-*})$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

Proof idea

- arrange the x-orbit in a x-rooted tree:
- modes are labeled by individual states in the x-orbit
 criticism of y are its successor states, imply: all
- prune the nodes labeled by states in $img(r_{\wedge}; q^*)$
- use König's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin *b*, and an additional individual state $x \le b$ with $\lim_{x \to a} (x : a) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The *q*-dynamics is **finitary** if the *q* relation is image-finite or individual states, *i.e.* img (*x*; *q*) represents a finite state whenever *x* represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the x-orbit in a x-rooted tree:
- nodes are labeled by individual states in the x-orbit of children of y are its successor states. Img(y, q)
- prune the nodes labeled by states in $img(r_o; q^*)$
- use König's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and ar additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the x-orbit in a x-rooted tree:
- lacktriangledown prune the nodes labeled by states in $\mathrm{img}(r_{\diamond}\;;q^*)$
- use König's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin *b*, and ar additional individual state $x \le b$ with $\text{img}(x : a) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

Definition The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state

Lemma 1 Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the x-orbit in a x-rooted tree:
- prune the nodes labeled by states in $img(r_0; q^*)$ • use Könia's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin *b*, and ar additional individual state $x \le b$ with $\text{img}(x : a) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ img $(x; q^{\ge n}) \le \operatorname{img}(r_{\lozenge}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the x-orbit in a x-rooted tree:
- nodes are labeled by individual states in the x-orbit
 children of y are its successor states: img(y; q)
- prune the nodes labeled by states in $img(r_s; q^*)$
- use König's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and ar additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ img $(x; q^{\ge n}) \le \operatorname{img}(r_{\lozenge}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the *x*-orbit in a *x*-rooted tree:
 - nodes are labeled by individual states in the x-orbit
 - children of y are its successor states: img(y; q)
- prune the nodes labeled by states in $img(r_s; q^*)$
- use König's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and an additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

Proof idea:

- arrange the *x*-orbit in a *x*-rooted tree:
 - nodes are labeled by individual states in the x-orbit
 - children of y are its successor states: img(y; q)
- prune the nodes labeled by states in $img(r_{\diamond}; q^*)$
- use König's Lemma

Remark the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and ar additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the *x*-orbit in a *x*-rooted tree:
 - nodes are labeled by individual states in the x-orbit
 - children of y are its successor states: img(y; q)
- prune the nodes labeled by states in $img(r_{\land}; q^*)$
- use König's Lemma
- **Remark** the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and ar additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ img $(x; q^{\ge n}) \le \operatorname{img}(r_{\lozenge}; q^*)$, then there is a nonrecurrent x-flight in b

Proof idea:

- arrange the *x*-orbit in a *x*-rooted tree:
 - nodes are labeled by individual states in the x-orbit
 - children of y are its successor states: img(y; q)
- prune the nodes labeled by states in img(r_⋄; q*)
- use König's Lemma

Remark the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and ar additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ img $(x; q^{\ge n}) \le \operatorname{img}(r_{\lozenge}; q^*)$, then there is a nonrecurrent x-flight in b

Proof idea:

- arrange the *x*-orbit in a *x*-rooted tree:
 - nodes are labeled by individual states in the *x*-orbit
 - children of y are its successor states: img(y; q)
- prune the nodes labeled by states in img(r_△; q*)
- use König's Lemma

Remark the *q*-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and ar additional individual state $x \le b$ with $\operatorname{img}(x;q) = \xi_{\mathbb{N}}$



Lemma 1: existence of nonrecurrent flights

may hamper existence of the unavoidable attractor (see later)

- **Definition** The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img (x;q) represents a finite state whenever x represents an individual state
- **Lemma 1** Let b be a basin in a finitary q-dynamics. If there exists $x \le b$ such that for no $n \in \mathbb{N}$ $\operatorname{img}(x; q^{\ge n}) \le \operatorname{img}(r_{\diamond}; q^*)$, then there is a nonrecurrent x-flight in b

- arrange the x-orbit in a x-rooted tree:
 - nodes are labeled by individual states in the *x*-orbit
 - children of y are its successor states: img(y; q)
- prune the nodes labeled by states in img(r_△; q*)
- use König's Lemma
- **Remark** the q-finitarity hypothesis is fairly essential: consider an antiflight ξ , with ξ_0 the only fixed point in basin b, and an additional individual state $x \le b$ with $\mathrm{img}(x ; q) = \xi_{\mathbb{N}}$



Lemma 2: nonexistence of the unavoidable attractor

here is a **sufficient condition**

Definition A flight ξ is antiflight-free if

no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
 - there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_I occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of $\xi_{\mathbb{N}}$ is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0\) occurs infinitely often in the x-orbit; then the set of path lengths in the tree would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\)nig's Lemma, which entails that \(\xi_0\) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if

no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q^{-} -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_j occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of $\xi_{\mathbb{N}}$ is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q^{*}) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0 \) occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\)nig's Lemma, which
 entails that \(\xi_0 \) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is **antiflight-free** if

no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q-dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_j occur infinitely offen the x-orbit, whereas
 (ii) no infinite subset of ξ_N is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0 \) occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\)nig's Lemma, which
 entails that \(\xi_0 \) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if

no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in \(\xi_N\) is removable from the unavoidable attracting set \(\xi_N\) viz. for no \(x \leq 0\) may any \(\xi_N\) occur infinitely often in the \(x-\text{orbit}\), whereas
 (ii) no infinite subset of \(\xi_N\) is removable from \(\xi_N\) without losing the unavoidable attracting set property.
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0 \) occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\) nig's Lemma, which
 entails that \(\xi_0 \) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if

no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_j occur infinitely of the x-orbit, whereas
 (ii) no infinite subset of ξ_N is removable from b without being the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0\) occurs infinitely often in the x-orbit; then the set of path lengths in the tree would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\) nig's Lemma, which entails that \(\xi_0\) is the target of an antiflight, against the hypotesis.



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

- **Definition** A flight ξ is antiflight-free if
 - no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight
- **Lemma 2** For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if
 - (i) the converse q-dynamics is finitary, and
 - (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_j occur infinitely often in the x-orbit, whereas
 (ii) in infinite subset of ξ_N is removable from b without losing the unavoidable attracting set property.
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q^{*}) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume ξ₀ occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by König's Lemma, which
 entails that ξ₀ is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if

no individual state in $\xi_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in *b*, under the *q*-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
- (i) every individual state in E_N is removable from the unavoidable attracting set b, viz. for no x ≤ b mayany E_j occur infinitely often in the x-orbit, whereas
 (ii) in infinite subset of E_N is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume ξ₀ occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by König's Lemma, which
 entails that ξ₀ is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if no individual state in $\mathcal{E}_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in E_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any E_i occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of ξ_N is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q") in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0 \) occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\) nig's Lemma, which
 entails that \(\xi_0 \) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

- **Definition** A flight ξ is antiflight-free if no individual state in $\mathcal{E}_{\mathbb{N}}$ is the target of an antiflight
- **Lemma 2** For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if
 - (i) the converse q -dynamics is finitary, and
 - (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_i occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of ξ_N is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q") in a finitel' branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0\) occurs infinitely often in the x-orbit; then the set of path lengths in the tree would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\) nig's Lemma, which entails that \(\xi_0\) is the target of an antiflight, against the hypotesis.



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if no individual state in $\mathcal{E}_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_i occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of $\xi_{\mathbb{N}}$ is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q") in a finitel branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0 \) occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\) nig's Lemma, which
 entails that \(\xi_0 \) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if no individual state in $\mathcal{E}_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in $\xi_{\mathbb{N}}$ is removable from the unavoidable attracting set b, viz. for no $x \leq b$ may any ξ_i occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of $\xi_{\mathbb{N}}$ is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0 \) occurs infinitely often in the x-orbit; then the set of path lengths in the tree
 would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\)nig's Lemma, which
 entails that \(\xi_0 \) is the target of an antiflight, against the hypotesis



Lemma 2: nonexistence of the unavoidable attractor

here is a sufficient condition

Definition A flight ξ is antiflight-free if no individual state in $\mathcal{E}_{\mathbb{N}}$ is the target of an antiflight

Lemma 2 For any basin *b* in the *q*-dynamics, $a_{\square} = 0$ if

- (i) the converse q -dynamics is finitary, and
- (ii) there is a nonrecurrent antiflight-free flight in b, under the q-dynamics

- if ξ is a nonrecurrent antiflight-free flight, in b, then:
 - (i) every individual state in ξ_N is removable from the unavoidable attracting set b, viz. for no x ≤ b may any ξ_i occur infinitely often in the x-orbit, whereas
 - (ii) no infinite subset of $\xi_{\mathbb{N}}$ is removable from b without losing the unavoidable attracting set property
- it's enough to prove (i) for ξ₀; much like in the proof of Lemma 1, arrange the ξ₀-orbit (under q) in a finitely branching tree, where individual states may only occur once in any given path, by nonrecurrence of ξ; then
- by contradiction, assume \(\xi_0\) occurs infinitely often in the x-orbit; then the set of path lengths in the tree would be unbounded, so the tree should be infinite, thus having an infinite path by K\(\tilde{o}\)nig's Lemma, which entails that \(\xi_0\) is the target of an antiflight, against the hypotesis



Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

Proof idea

• for each x, find $x' \le \operatorname{img}(x; q^h) \setminus \operatorname{img}(x; q^m)$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:



 iterate the previous procedure and show injectivity of the resulting flight map





Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

Proof idea

• for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{**})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:



 iterate the previous procedure and show injectivity of the resulting flight map





Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

Proof idea:

• for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{**})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:



 iterate the previous procedure and show injectivity of the resulting flight map





Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

- for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{**})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:
 - (i) $\xi_0 = \lambda$, $\xi_{n+1} = \lambda$, $\xi_{n+1} = i \text{ rig}(\xi_i, \xi_j)$, for $0 \le i \le n$; (ii) $\xi_i \le i \text{mg}(x; q^{**})$, for $0 < i \le n$; (iii) $\xi_i = \xi_i \Leftrightarrow i = i$ for $0 \le i \le n + 1$
- iterate the previous procedure and show injectivity of the resulting flight map





Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

- for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{\circ *})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:
 - (i) $\xi_0 = X$, $\xi_{n+1} = X'$, $\xi_{i+1} \le \text{img}(\xi_i; q)$, for $0 \le i \le n$;
 - (ii) $\xi_i \leq \operatorname{img}(x; q^{*})$, for $0 < i \leq n$;
 - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \le i, j \le n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map





start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

- for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{\circ *})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:
 - (i) $\xi_0 = x$, $\xi_{n+1} = x'$, $\xi_{i+1} \le \text{img}(\xi_i; q)$, for $0 \le i \le n$;
 - (ii) $\xi_i \leq \operatorname{img}(x; q^{*})$, for $0 < i \leq n$;
 - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \le i, j \le n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map





start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

- for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{\circ *})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:
 - (i) $\xi_0 = x$, $\xi_{n+1} = x'$, $\xi_{i+1} \le \text{img}(\xi_i; q)$, for $0 \le i \le n$;
 - (ii) $\xi_i \leq \operatorname{img}(x; q^{\vee *})$, for $0 < i \leq n$;
 - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \le i, j \le n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map





start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

- for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{\circ *})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:
 - (i) $\xi_0 = x$, $\xi_{n+1} = x'$, $\xi_{i+1} \le \text{img}(\xi_i; q)$, for $0 \le i \le n$;
 - (ii) $\xi_i \leq \operatorname{img}(x; q^{*})$, for $0 < i \leq n$;
 - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \le i, j \le n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map





start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \le b$ is the origin of a flight

- for each x, find $x' \le \operatorname{img}(x; q^+) \setminus \operatorname{img}(x; q^{\circ *})$ and a finite sequence of n+2 individual states $(\xi_i \mid 0 \le i \le n+1)$, for some $n \ge 0$, that satisfies the following requirements:
 - (i) $\xi_0 = x$, $\xi_{n+1} = x'$, $\xi_{i+1} \le \text{img}(\xi_i; q)$, for $0 \le i \le n$;
 - (ii) $\xi_i \leq \operatorname{img}(x; q^{*})$, for $0 < i \leq n$;
 - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \le i, j \le n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map





Theorem: recurrence and attractors

characterizes both *existence* and *extent* of attractors

Theorem In any basin *b* with the *q*-dynamics:

- (i) $a_{\square} = \operatorname{img}(r_{\diamond}; q^*)$ if the q-dynamics is finitary and every flight is recurrent otherwise $a_{\square} = 0$ if the converse q^* -dynamics is finitary and if there is a nonrecurrent antiflight-free flight, under the q-dynamics
- (ii) $a_5 = r_{\Box}$ if every flight is eternally recurrent otherwise $a_5 = 0$

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses



Theorem: recurrence and attractors

characterizes both existence and extent of attractors

Theorem In any basin *b* with the *q*-dynamics:

- (i) $a_{\square} = \operatorname{img}(r_{\Diamond}; q^*)$
- if the q-dynamics is finitary and every flight is recurrent otherwise $a_\square=0$ if the converse q-dynamics is finitary and if there is a nonrecurrent antiflight-free flight, under the q-dynamics
- (ii) $a_0 = r_{\Box}$ if every flight is eternally recurrent, otherwise $a_{\perp} = 0$

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses



Theorem: recurrence and attractors

characterizes both existence and extent of attractors

Theorem In any basin b with the q-dynamics:

- (i) $a_{\square} = \operatorname{img}(r_{\diamond}; q^*)$ if the q-dynamics is finitary and every flight is recurrent otherwise $a_{\square} = 0$ if the converse q -dynamics is finitary and if there is a nonrecurrent antiflight-free flight, under the q-dynamics
- (ii) $a_{\Diamond} = r_{\Box}$ if every flight is eternally recurrent, otherwise $a_{\Diamond} = 0$

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses



Theorem: recurrence and attractors

characterizes both existence and extent of attractors

Theorem In any basin b with the q-dynamics:

- (i) a_□ = img(r_⋄; q*)
 if the q-dynamics is finitary and every flight is recurrent,
 otherwise a_□ = 0 if the converse q -dynamics is finitary
 and if there is a nonrecurrent antiflight-free flight, under
 the q-dynamics
- (ii) $a_{\diamond} = r_{\square}$ if every flight is eternally recurrent, otherwise $a_{\diamond} = 0$

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence bypotheses





Theorem: recurrence and attractors

characterizes both existence and extent of attractors

Theorem In any basin b with the q-dynamics:

- (i) a_□ = img(r_⋄; q*)
 if the q-dynamics is finitary and every flight is recurrent,
 otherwise a_□ = 0 if the converse qˇ-dynamics is finitary
 and if there is a nonrecurrent antiflight-free flight, under
 the q-dynamics
- (ii) $a_{\Diamond} = r_{\Box}$ if every flight is eternally recurrent, otherwise $a_{\Diamond} = 0$

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by



basic definitions and facts existence of nonrecurrent flights nonexistence of the unavoidable attractor flights in absence of eternal recurrence characterization of recurrence and attractors

Theorem: recurrence and attractors

characterizes both existence and extent of attractors

Theorem In any basin b with the q-dynamics:

- (i) a_□ = img(r_◊; q*)
 if the q-dynamics is finitary and every flight is recurrent,
 otherwise a_□ = 0 if the converse q dynamics is finitary
 and if there is a nonrecurrent antiflight-free flight, under
 the q-dynamics
- (ii) $a_{\diamond} = r_{\square}$ if every flight is eternally recurrent, otherwise $a_{\diamond} = 0$

Remarks:

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses



basic definitions and facts existence of nonrecurrent flights nonexistence of the unavoidable attractor flights in absence of eternal recurrence characterization of recurrence and attractors

Theorem: recurrence and attractors

characterizes both existence and extent of attractors

Theorem In any basin b with the q-dynamics:

- (i) a_□ = img(r_◊; q*)
 if the q-dynamics is finitary and every flight is recurrent,
 otherwise a_□ = 0 if the converse q dynamics is finitary
 and if there is a nonrecurrent antiflight-free flight, under
 the q-dynamics
- (ii) a_◊ = r_□ if every flight is eternally recurrent, otherwise a_◊ = 0

Remarks:

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses



- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 - that is: replace exact inclusion of orbits in the attracting set with approximate inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings
 - at least for the deterministic case





- definability of weaker notions of recurrence
 - that is: replace exact occurrence of a state in its own trajectory with approximate occurrence
- definability of weaker notions of attraction
 - that is: replace exact inclusion of orbits in the attracting set with approximate inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings
 - at least for the deterministic case





- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 - that is: replace exact inclusion of orbits in the attracting set with approximate inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings
 - at least for the deterministic case





Moving from structureless to topological state spaces:

- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 that is: replace exact inclusion of orbits in the attracting se
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings





Moving from structureless to topological state spaces:

- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 - that is: replace **exact** inclusion of orbits in the attracting set with **approximate** inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings





Moving from structureless to topological state spaces:

- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 - that is: replace **exact** inclusion of orbits in the attracting set with **approximate** inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings





Moving from structureless to topological state spaces:

- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 - that is: replace **exact** inclusion of orbits in the attracting set with **approximate** inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings





- definability of weaker notions of recurrence
 - that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence
- definability of weaker notions of attraction
 - that is: replace **exact** inclusion of orbits in the attracting set with **approximate** inclusion
- generalization of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings
 - at least for the deterministic case





References: state transition dynamics



V. Manca, G. Franco, G. Scollo

State transition dynamics: basic concepts and molecular computing perspectives.

Chapter 2 in: M. Gheorghe (Ed.)

Molecular Computational Models: Unconventional Approaches

Idea Group, Hershey, PA, USA (2005) 32-55.



G. Scollo, G. Franco, V. Manca

A relational view of recurrence and attractors in state transition dynamics.

Proc. RelMiCS/AKA 2006

Manchester, 29 August – 2 September, 2006.

Springer, LNCS (2006) to appear.





References: state transition dynamics



V. Manca, G. Franco, G. Scollo State transition dynamics: basic concepts and molecular

computing perspectives.

Chapter 2 in: M. Gheorghe (Ed.)

Molecular Computational Models: Unconventional Approaches

Idea Group, Hershey, PA, USA (2005) 32-55.



G. Scollo, G. Franco, V. Manca

A relational view of recurrence and attractors in state transition dynamics.

Proc. RelMiCS/AKA 2006

Manchester, 29 August – 2 September, 2006.

Springer, LNCS (2006) to appear.





References: background for further study

P. Kůrka

Topological and symbolic dynamics. Cours Spécialisés 11, Société Mathématique de France (2003).

G. Schmidt & Th. Ströhlein

Relations and Graphs

Discrete Mathematics for Computer Scientists,

EATCS Monographs in Theoretical Computer Science, Springer (1993).



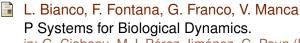
References: background for further study

- P Kůrka Topological and symbolic dynamics. Cours Spécialisés 11.
 - Société Mathématique de France (2003).
 - G. Schmidt & Th. Ströhlein Relations and Graphs Discrete Mathematics for Computer Scientists. EATCS Monographs in Theoretical Computer Science, Springer (1993).





References: metabolic P systems



in: G. Ciobanu, M.J. Pérez-Jiménez, G. Paun (Eds.) Applications of Membrane Computing Springer, Natural Computing Series (2006) 81–126.

- L. Bianco, F. Fontana, V. Manca
 P Systems with reaction maps.

 Int'l J. of Foundations of Computer Science,
 vol. 17:1, 2006, 27–48.
 - V. Manca
 Topics and Problems in Metabolic P Systems.
 Fourth Brainstorming on Membrane Computing
 Sevilla, 2006.



References: metabolic P systems



Springer, Natural Computing Series (2006) 81–126.

L. Bianco, F. Fontana, V. Manca
P Systems with reaction maps.

Int'l J. of Foundations of Computer Science,
vol. 17:1, 2006, 27–48.

V. Manca
Topics and Problems in Metabolic P Systems.
Fourth Brainstorming on Membrane Computing
Sevilla, 2006.



References: metabolic P systems

L. Bianco, F. Fontana, G. Franco, V. Manca P Systems for Biological Dynamics.

in: G. Ciobanu, M.J. Pérez-Jiménez, G. Paun (Eds.) Applications of Membrane Computing Springer, Natural Computing Series (2006) 81–126.

L. Bianco, F. Fontana, V. Manca
P Systems with reaction maps.

Int'l J. of Foundations of Computer Science,
vol. 17:1, 2006, 27–48.

V. Manca
Topics and Problems in Metabolic P Systems.
Fourth Brainstorming on Membrane Computing
Sevilla, 2006.

