Central Schemes for Systems of Balance Laws

Salvatore Fabio Liotta, Vittorio Romano, Giovanni Russo

Abstract. Several models in mathematical physics are described by quasilinear hyperbolic systems with source term, which in several cases may become stiff. Here a suitable central numerical scheme for such problems is developed and application to shallow water equations, Broadwell model and Extended Thermodynamics are mentioned.

The numerical methods are a generalization of the Nessyahu-Tadmor scheme to the non-homogeneous case. We propose two ways for treating the production term. The first is obtained by including the cell averages of the productions, while the second family of schemes is obtained by a splitting strategy.

1. Introduction

In several problems of mathematical physics, hyperbolic systems of balance laws arise. In particular we mention hyperbolic systems with relaxation, such as discrete velocity models in kinetic theory [8], gas with vibrational degrees of freedom [21], hydrodynamical models for semiconductors [2, 1], radiation hydrodynamics [14].

Lately the development of high-order shock-capturing methods for conservation laws has become an interesting area or research. However, most schemes deal almost exclusively with the homogeneous case. The extension to systems with a source term has been studied in [18, 11, 5] where a method of line approach, together with splitting techniques has been used. Here we consider the extension of second order central schemes to the non homogeneous case. The aim is to provide a general-purpose robust scheme for systems of balance laws [16, 19].

The main advantage of central schemes is their flexibility, in fact they do not require the knowledge of the characteristic structure of the system, and the (exact or approximate) solution to the Riemann problem, at variance with upwind-based schemes.

There are systems with relaxation for which the analytical expression of the eigenvalues and eigenvectors is not known. Typical examples are given by monoatomic gas in Extended Thermodynamics [15], and some hydrodynamical models for electron transport in semiconductors.

Explicit central schemes for balance laws with source have been considered in [7], and applied to the shallow water equations.

In this paper we consider two families of central schemes, which extend the Nessyahu-Tadmor (NT) scheme [16] to systems with source. The first family is obtained by including the source term in the integration over the cell in space-time. Explicit and implicit formulations are considered. The implicit schemes are able to treat stiff source terms. However a degradation of the accuracy is observed in this case.

The second family is based on a splitting strategy. A new splitting scheme is presented. For non stiff source, the new scheme gives second order accuracy with first order evaluation of the relaxation step.

Application of the schemes to the shallow water equations, the Broadwell model, and the equations for a monoatomic gas in Extended Thermodynamics are presented in [13].

The plain of the paper is the following. In section 2, a sketch of the mathematical models is given. In sections 3 and 4 the numerical schemes obtained by including the cell average of the production term and the numerical scheme based on a splitting strategy respectively are presented.

2. The mathematical models

Here we present some significant physical models represented by hyperbolic systems of balance laws. First we show a case with a non stiff source, the shallow water equations with a smooth profile of the bottom. Then we present some examples of relaxation systems.

We recall that the system

$$\partial_t U + \partial_x F(U) = -\frac{1}{\varepsilon} R(U), \quad U \in \mathcal{R}^N,$$

is said a *relaxation system* [22, 6] if there exists a constant matrix $\mathcal{Q}$ with rank $n < N$ such that $\mathcal{Q}R(U) = 0 \ \forall \ U$ which gives n conserved quantities v. One assumes that each such v uniquely determines a local equilibrium value $U = \mathcal{E}(v)$ satisfying $R(\mathcal{E}(v)) = 0$ and such that

$$\mathcal{Q}\mathcal{E}(v) = 0, \forall v.$$

Associated with the original system there are n conservation laws which represent the local equilibrium subsystem

$$\partial_t \mathcal{Q}U + \partial_x(\mathcal{Q}F(U)) = 0, \quad \text{with } U = \mathcal{E}(v) \ .$$

When ε is small compared to the time scale determined by the characteristic speed of the system, we have a stiff relaxation term.

2.1. Non stiff problems: the shallow water equations

The evolution equations governing the flow of shallow water waves into a channel with a fixed bottom elevation described by $B(x)$, in the one dimensional case are given by the system [22, 20]

$$h_t + (hu)_x = 0, \tag{1}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x, \tag{2}$$

where g is the gravitational constant, h represents the fluid depth and u is the velocity. $B(x) + h(x,t)$ gives the top surface. It is straightforward to check that the equations (1)-(2) form a hyperbolic system with characteristic velocities

$$\lambda = u \pm \sqrt{gh}.$$

Stationary solutions are given by

$$g(h(x) + B(x)) + \frac{1}{2}u^2 \quad = \quad \text{constant.}$$

The next example we consider is the Broadwell model. It describes a 2-D (3-D) gas composed of particles with four (six) discrete velocities with binary collision law and spatial variation in only one direction. For the 2-D gas the evolution equations read

$$\partial_t \rho + \partial_x m = 0, \tag{3}$$

$$\partial_t m + \partial_x z = 0, \tag{4}$$

$$\partial_t z + \partial_x m = \frac{1}{\varepsilon}(\rho^2 + m^2 - 2\rho z), \tag{5}$$

where ε is the mean free path. The dynamical variables ρ and m are the density and the momentum respectively, while z represents the flux of momentum. A description of the Broadwell model can be found, for example, in [5].

The characteristic velocities are constant

$$\lambda = -1, 0, 1.$$

As $\varepsilon \to 0$, z is given by a local Maxwellian distribution

$$z = z_E(\rho, m) = \frac{1}{2\rho}(\rho^2 + m^2) \tag{6}$$

and we get the fluid dynamic limit

$$\partial_t \rho + \partial_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \partial_x \left(\frac{1}{2}(\rho + \rho u^2)\right) = 0,$$

with $u = m/\rho$.

3. Non splitting schemes

Let us consider the initial value problem

$$u_t + f(u)_{,x} = g(u), \quad (7)$$
$$u(x,0) = \Phi(x), \quad -\infty < x < \infty \quad (8)$$

with $u \in \mathbf{R}^m$ and $f : \mathbf{R}^m \to \mathbf{R}^m$.

We discretize space-time with staggered cells. Following [16], at each time level we reconstruct a piecewise linear approximation of the form

$$L_j(x,t) = u_j(t) + \frac{x - x_j}{\Delta x} u'_j \qquad x_{j-1/2} \le x \le x_{j+1/2},$$

where $u'_j/\Delta x$ is a first order approximation of the space derivative at x_j. By integrating Eq.(7) over the cell $[x_j, x_{j+1}] \times [t^n, t^{n+1}]$, we obtain

$$\begin{aligned} u^{n+1}_{j+1/2} &= \frac{1}{2}(u^n_j + u^n_{j+1}) + \frac{1}{8}(u'_j - u'_{j+1}) + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} g(u(x,t))\,dxdt \\ &+ \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} f(u(x_j,t)) - f(u(x_{j+1},t))\,dt. \end{aligned} \quad (9)$$

Different schemes are obtained by suitable discretization of the integrals of fluxes and source. The integral of the flux is discretized by midpoint rule

$$\int_{t^n}^{t^{n+1}} f(u(x_j,t))\,dt \approx \Delta t\, f(u^{n+1/2}_j).$$

The predictor $u^{n+1/2}_j$ is obtained by Taylor expansion in an explicit way

$$u^{n+1/2}_j = u^n_j + \frac{\Delta t}{2}\left(g(u^n_j) - \frac{f'_j}{\Delta x}\right),$$

or in an implicit way

$$u^{n+1/2}_j = u^n_j + \frac{\Delta t}{2}\left(g(u^{n+1/2}_j) - \frac{f'_j}{\Delta x}\right),$$

which is appropriate for stiff problems in order to avoid restriction on the time step.

The values of $u'_j/\Delta x$ and $f'_j/\Delta x$ are a first order approximation of the space derivatives of the field and of the flux. They can be computed in several ways [9, 16]. The simplest choice is

$$u'_j = \mathrm{MM}(u_{j+1} - u_j, u_j - u_{j-1}), \quad (10)$$

where $\mathrm{MM}(x,y)$ is the min mod function, defined by

$$\mathrm{MM}(x,y) = \begin{cases} sgn(x)\cdot\min(|x|,|y|) & \text{if } sgn(x) = sgn(y), \\ 0 & \text{otherwise .} \end{cases}$$

This approximation has the drawback of accuracy degradation near local extrema. Better results are obtained by the UNO-like derivative [9]

$$u'_j = \mathrm{MM}(u_j - u_{j-1} + \frac{1}{2}D_{j-1/2}, u_{j+1} - u_j - \frac{1}{2}D_{j+1/2}), \tag{11}$$

where

$$D_{j+\frac{1}{2}} = MM(u_{j+2} - 2u_{j+1} + u_j,\, u_{j+1} - 2u_j + u_{j-1}). \tag{12}$$

In the original NT scheme, the time step Δt must satisfy a stability condition

$$\Delta t \cdot \max_x \rho(A(u(x,t_n))) < \frac{1}{2}\Delta x \tag{13}$$

where $A = \partial f/\partial u$ is the Jacobian matrix, and ρ denotes the spectral radius. This condition ensures that the generalized Riemann problems with piecewise smooth data at time t_n do not interact during the time step Δt. The effect of the presence of the source on the stability restriction is presently under investigation.

Several quadrature formulas have been tested for discretizing the integral of the source term. An example of explicit scheme is given by using the mid-point rule.

$$I_g = \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} g(u(x,t))\, dxdt \approx g(u^{n+1/2}_{j+1/2})\Delta x \Delta t, \tag{14}$$

with

$$u^{n+1/2}_{j+1/2} = \frac{u^{n+1/2}_j + u^{n+1/2}_{j+1}}{2}, \tag{15}$$

where the values $u^{n+1/2}_j$ are computed in the predictor step. An example of semi-implicit scheme is given by

$$I_g \approx \hat{g}^{n+1/2}_{j+1/2} = \left[\frac{g(u^n_j) + g(u^n_j)}{2} + g(u^{n+1}_{j+1/2})\right]\frac{\Delta x \Delta t}{2}, \tag{16}$$

which leads to the second order semi-implicit scheme

$$\begin{aligned} u^{n+1}_{j+1/2} - \frac{\Delta t}{2} g(u^{n+1}_{j+1/2}) &= \frac{1}{2}(u^n_j + u^n_{j+1}) + \frac{1}{8}(u'_j - u'_{j+1}) \\ -\lambda\left[f(u^{n+1/2}_{j+1}) - f(u^{n+1/2}_j)\right] &+ \frac{\Delta t}{4}\left[g(u^n_j) + g(u^n_{j+1})\right]. \end{aligned} \tag{17}$$

Here $\lambda = \Delta t/\Delta x$ denotes the mesh ratio.

A fully implicit scheme is obtained by the (first order) quadrature formula

$$I_g \approx g(u^{n+1}_{j+1/2})\Delta x \Delta t.$$

Limitation of the the non splitting scheme for stiff source

Here we show, by and explicit example, that the semi-implicit non splitting scheme may fail when dealing with a stiff relaxation term.

Let us consider a single equation with a source term of relaxation type

$$g(u) = -\frac{R(u)}{\varepsilon}$$

We want to analyze the non splitting schemes when $\varepsilon \to 0$.

We shall consider the simple case

$$g(u) = -\frac{u-u_0}{\varepsilon}, \quad \text{with } u_0 = \text{constant.}$$

The scheme with the implicit predictor step reads

$$\begin{aligned} u_{j+1/2}^{n+1} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u'_j - u'_{j+1}) - \lambda\left[f(u_{j+1}^{n+1/2}) - f(u_j^{n+1/2})\right] \\ &\quad -\frac{\Delta t}{4}\left[\frac{u_j^n - u_0}{\varepsilon} + \frac{u_{j+1}^n - u_0}{\varepsilon}\right] - \frac{\Delta t}{2}\frac{u_{j+1/2}^{n+1}}{\varepsilon} \\ u_j^{n+1/2} &= u_j^n - \left[\frac{f'(u_j^n)}{\Delta x} + \frac{u_j^{n+1/2} - u_0}{\varepsilon}\right]\frac{\Delta t}{2}. \end{aligned}$$

When $\varepsilon \to 0$, we get

$$u_j^{n+1/2} = u_0, \quad u_{j+1/2}^{n+1} = 2u_0 - \frac{u_j^n + u_{j+1}^n}{2}.$$

For the initial data

$$u_j^0 = u_0 + \delta, \quad \text{with } \delta = \text{constant,}$$

one has

$$\begin{cases} u_j^{2n+1} = u_0 - \delta, \\ u_j^{2n+2} = u_0 + \delta, \end{cases} \qquad n = 0, 1, 2, \cdots$$

Therefore the scheme does not converge.

Such a counterexample suggests that the quadrature formula (14) is not adequate in the case of a stiff source.

Accuracy of the schemes

Standard truncation analysis can be performed for non-stiff source. The scheme based on semi-implicit evaluation of the source integral is second order accurate in space and time. When the source is stiff, however, the fully implicit scheme is preferable. The latter is first order accurate for non-stiff source, but gives a second order scheme for the limit system obtained as $\varepsilon \to 0$. A possible way in order to obtain a uniform second order scheme could be to combine the implicit and semi-implicit schemes. Such approaches are presently under investigation [13, 17].

4. Splitting schemes.

The basic idea is to integrate first the relaxation system (relaxation step),

$$\frac{du}{dt} = g(u), \tag{18}$$

for a time step Δt, and then the homogeneous system (convection step), using the output of the previous step as initial condition,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \tag{19}$$

We use NT scheme with a staggered grid for the convection step. A complete convection step of step-size Δt on the original grid is obtained as a sequence of two intermediate steps of step-size $\Delta t/2$ on staggered grid.

The simple splitting scheme is only first order accurate. A different splitting strategy is proposed which uses implicit Euler scheme for the source. The scheme is second order accurate for non stiff source and reduces to first order in the stiff case.

In developing the numerical schemes we keep in mind the following guidelines:

- truncation error analysis is used to obtain second order accuracy in the rarefied regime (ε=O(1));
- the collision step is well posed $\forall\varepsilon$ and it relaxes to a local Maxwellian as $\varepsilon \to 0$;
- the scheme should be unconditionally stable in the collision step;
- the limiting scheme obtained as $\varepsilon \to 0$ is a consistent scheme for the equilibrium subsystem.

We accept our schemes only if the above conditions are satisfied.

Truncation analysis is performed on the linear system:

$$\partial_t U + AU + BU = 0 \tag{20}$$

where $U \in \mathbf{R}^m$ and A, B are constant matrices.

Following the approach used in [5], we write our splitting scheme as a combination of convection and relaxation steps:

$$\begin{aligned}
U_1 &= U^n - \alpha\Delta t B U_1 && (21)\\
U_2 &= U_1 - \tilde{\alpha}\Delta t A U_1 + \frac{1}{2}\tilde{\alpha}^2\Delta t^2 A^2 U_1 && (22)\\
U_3 &= U_2 - \beta\Delta t B U_3 - \gamma\Delta t B U_1 && (23)\\
U_4 &= U_3 - \tilde{\beta}\Delta t A U_3 + \frac{1}{2}\tilde{\beta}^2\Delta t^2 A^2 U_3 && (24)\\
U_5 &= \xi U_1 + \eta U_4 && (25)\\
U^{n+1} &= U_5 - \mu\Delta t B U^{n+1} && (26)
\end{aligned}$$

By imposing that the local truncation error is $O(\Delta t^3)$, the set of constraints constitutes a system of 7 algebraic equations in 8 unknowns, which can be explicitly solved in terms of the parameter β. The following family of schemes is obtained:

$$
\begin{aligned}
U_1 &= U^n - \alpha \Delta t B U_1 && (27)\\
U_2 &= U_1 - \beta \Delta t B U_2 - \gamma \Delta t B U_1 && (28)\\
U_3 &= \mathcal{T}(U_2) && (29)\\
U^{n+1} &= U_3 - \frac{1}{2}\Delta t B U^{n+1} && (30)
\end{aligned}
$$

where $\alpha = \beta/(2\beta - 1)$, $\gamma = 1/2 - \alpha - \beta$, and the symbol $\mathcal{T}(U)$ denotes a robust second order accurate scheme (in space and time). In our case it will represent the Nessyahu-Tadmor scheme.

We remark that all the conditions used as guidelines are satisfied. Second order accuracy (for non-stiff relaxation) is obtained by one second order convection and two first order relaxation steps.

Accuracy test of the splitting scheme

Here we want to investigate the accuracy of the splitting scheme. For our numerical experiments, we shall consider the Brodwell model.

We start our simulations by considering a smooth solution of periodic type. We integrate the system (3)-(5) with the following initial data

$$
\begin{aligned}
\rho(x,0) &= 1 + a_\rho \sin\frac{2\pi x}{L},\\
u(x,0) &= \frac{1}{2} + a_u \sin\frac{2\pi x}{L},\\
m(x,0) &= \rho(x,0)u(x,0),\\
z(x,0) &= z_E(\rho(x,0), m(x,0))\theta_M
\end{aligned}
$$

with θ_M = constant, z_E being the local Maxwellian (6). For $\theta_M = 1$ the initial value of z is Maxwellian. Several values of ε have been considered. The goal of the test is to check uniform convergence of the scheme with respect to ε.

The system has been integrated for $t \in [0, t_f]$. The values of the parameters used in the computation are:

$$L = 20, \quad t_f = 30, \quad a_\rho = 0.3, \quad a_u = 0.1.$$

A mesh 200, 400 and 800 point grid has been considered. We call Δx_1, Δx_2 and Δx_3 the relative values of the mesh size.

The convergence rate is computed from the error according to the formula

$$CR_i = \frac{\log(error_i/error_{i+1})}{\log(\Delta x_i/\Delta_{i+1})}$$

where $error_i$ is the error obtained comparing the solutions with Δx_i and Δ_{i+1}.

In Table 1 we report the density convergence rate in the L_1, L_2 and L_∞ norms for the mesh ratio $\lambda \equiv \Delta t/\Delta x = 0.5$. Similar results have been obtained for m and z.

One can see that the scheme is not uniformly accurate in ε and as $\varepsilon \to 0$ the scheme become of the first order.

Tab.1 : convergence rate for ρ, CFL 1/2

ρ	$\varepsilon = 10^2$	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
L_1	2.22183	2.13676	1.22621	1.00854	1.00829	1.00829
L_2	2.09765	2.18820	1.23911	1.011839	1.01814	1.01814
L_∞	1.67444	2.28726	1.23950	0.998505	0.998191	0.998191

Application of the above schemes to other physical cases are presented in [13].

References

[1] A. M. Anile and O. Muscato, *Improved hydrodynamical model for carrier transport in semiconductors*, Phys. Rev B, **51**. (1995), 16728–16740.

[2] A. M. Anile and S. Pennisi, *Thermodynamic derivation of the hydrodynamical model for charge transport in semiconductors*, Phys. Rev B, **46**, (1992), 13186–13193.

[3] A. M. Anile, V. Romano, and G. Russo, *Extended Hydrodynamical Model of Carrier Transport in Semiconductors*, submitted to SIAM J. of Appl. Math.

[4] J. D. Au, *Nonlinear Closure in Molecular Extended Thermodynamics with Application to the Shock Wave Structure Solution*, in Supplemento ai Rendiconti del Circolo Matematico di Palermo 45, Proceedings International Conference on Wave and Stability, A. Greco and S. Rionero ed.s, Palermo, 1995.

[5] R. E. Caflisch, S. Jin and G. Russo, *Uniformly Accurate Schemes for Hyperbolic Systems with Relaxation*, SIAM J. Num. Anal., **34**, (1997), 246–281.

[6] G. Q. Chen, C. D. Levermore and T. P. Liu, *Hyperbolic Conservation Laws with Stiff Relaxation Terms and Entropy*, Comm. Pure Appl. Math., **Vol. XLVII**, (1994), 787–830.

[7] G. Erbes, *A high-resolution Lax-Friedrichs scheme for hyperbolic conservation laws with source terms. Application to the shallow water equations*, preprint.

[8] R. Gatignol, Lectures Notes in Physics, vol. 36, Springer-Verlag, **1975**.

[9] A. Harten and S. Osher, *Uniformly High-Order Accurate Nonosciollatory Schemes. I*, SIAM J. Numer. Anal. **24**, (1987), 279–309.

[10] I-Shi Liu, *Method of Lagrange Multipliers for Exploitation of the entropy principle*, Arch. Rat. Mech. Anal., **46**, (1972), 131–148.

[11] S. Jin, *Runge-Kutta Methods for Hyperbolic Conservation Laws with Stiff Relaxation Terms*, J. Comput. Phys., **122**, (1995), 51–67.

[12] D. Jou, J. Casas-Vazquez and G. Lebon, *Extended irreversible thermodynamics*, Springer-Verlag, Berlin, **1993**.

[13] S. F. Liotta, V. Romano, and G. Russo, *Central Methods for Systems of Balance Laws, with Application to Continuous Mechanics*, in preparation.

[14] D. Mihalas & B. Mihalas, **Foundations of Radiation Hydrodynamics**, Oxford UP, **1984**.

[15] I. Müller e T. Ruggeri, *Extended thermodynamics*, Springer-Verlag, Berlin, **1993**.

[16] H. Nessyahu and E. Tadmor, *Non-Oscillatory Central Differencing for Hyperbolic Conservation Laws*, J. Comput. Phys., **87**, (1990), 408–448.

[17] L. Pareschi, M. Ronconi, G. Russo, *Uniformly accurate central Schemes for Hyperbolic Systems with Relaxation*, in preparation.

[18] R. B. Pember, *Numerical Methods for Hyperbolic Conservation Laws with Stiff Relaxation I. Spurious Solutions*, SIAM J. Appl. Math., **53**, (1993), 1293–1330.

[19] R. Sanders and W. Weiser, *High Resolution Staggered Mesh Approach for Nonlinear Hyperbolic Systems of Conservation Laws*, J. Comput. Phys., **101**, (1992), 314–329.

[20] J. J. Stoker, *Water Waves*, Interscciennce Publishers, New York, **1974**.

[21] W. Vincenti and C. Kruger, Introduction to Physical Gas Dynamics, Krieger Publishing Company, Malabar, Florida **1986**.

[22] G. B. Whitham, *Linear and non linear waves*, Wiley, New York, **1974**.

Politecnico di Bari, Sede di Taranto
Viale del Turismo 8, 74100 Taranto, Italy
E-mail address: romano@dipmat.unict.it