High order central schemes for hyperbolic systems of conservation laws

Franca Bianco, Gabriella Puppo, and Giovanni Russo

Abstract. In this talk we present third and fourth order central schemes for the approximate solution of quasilinear systems of conservation laws. The schemes are an extension of the second order Nessyahu-Tadmor scheme, and are based on a ENO reconstruction from cell averages, and a numerical compuation of the flux on cell boundaries, efficiently obtained by Runge-Kutta schemes with Natural Continuous Extension.

Here we focus on the linear stability analysis of the scheme. The exact CFL condition for linear third and fourth order schemes are derived.

1. Introduction

Central schemes for conservation laws have been a subject of active research in recent years [8, 10, 8, 7, 5, 6, 2]. The main advantage of central schemes over upwind schemes is that they do not require the solution of Riemann problems, or the computation of characteristic velocities of the system.

These features make the central scheme approach very attractive for those systems for which the solution to the Riemann problem is complicated, or when there is no simple analytical expression for the eigenvalues of the Jacobian matrix. This is the case of systems arising, for example, in semiconductor modeling [9, 1]. In that case the NT scheme, suitably modified to incorporate source terms, has been successfully used.

The schemes that we present can be viewed as an extension of the second order Nessyahu-Tadmor scheme [8]. A third order scheme has been presented by Liu and Tadmor [7]. They show that the scheme is *Number of Extrema Decreasing* (NED), and it gives good numerical results both on the scalar equation and on the Euler equations.

The main focus of our work is the development of third and fourth order schemes, which are robust and efficient, so that they can be easily implemented for several systems of conservation laws. The user has to provide only a subroutine for the computation of the flux vector and an estimate of the eigenvalues (necessary to satisfy the stability condition).

This goal is obtained by the combination of two main ingredients: high order ENO reconstruction from cell averages (which provides high order space accuracy and shock capturing capability), and Runge-Kutta schemes with Natural Continuous Extension (NCE) for the integration of the flux (which provides stability and high order accuracy in time, without requiring the computation of the Jacobian or the Hessian of the system).

The paper is divided in two parts. In the first part we describe the general features of the method, and we summarize the main results presented in [3]. The second part of the paper is original, and deals with the study of the linear stability analysis of the linear schemes. The analysis is relevant for the computation of the exact CFL condition, and helps developing modification to the original ENO stencil selection mechanism.

2. Description of the method

We describe the method in the case of scalar equation. The extension to systems is recalled later.

Let us consider the scalar conservation law:

$$u_t + f_x(u) = 0, \tag{1}$$

on an interval I, with suitable boundary conditions. We consider for simplicity a uniform grid on I of points $\{x_j\}$, $j = 0, \cdots, N$, with $x_{j+1} - x_j = h$. Let k be the time step, with $u_j^n = u(x_j, t^n)$, $t^n = nk$. Finally, w will denote the computed solution of (1). At time t^n, we start from a piecewise constant function $\overline{w}_j^n$, representing the cell averages of the computed solution w at time t^n, namely:

$$\overline{w}_j^n = \frac{1}{h}\int_{-h/2}^{h/2} w(x_j + y, t^n)\, dy. \tag{2}$$

From the values $\{\overline{w}_j^n\}_{j=0}^N$ we reconstruct the point values of the function $w(x, t^n)$, via a suitable non linear piecewise polynomial interpolation. The reconstruction we use is due to Harten et al. [4].

Let $\mathcal{R}(\overline{w}^n;\ x)$ be the reconstruction operator, where $\overline{w}^n$ is the vector with components $\overline{w}_j^n$, $j = 0, \cdots, N$. Then:

$$w(x, t^n) := \mathcal{R}(\overline{w}^n;\ x) \tag{3}$$

is the function defined on I which will be used as initial data for the n-th time step. The reconstruction $\mathcal{R}(\overline{w}^n;\ x)$ is piecewise polynomial in the sense that:

$$\mathcal{R}(\overline{w}^n;\ x) \in P_j^m(x) \qquad \text{for } x \in \left[x_j - \frac{h}{2}, x_j + \frac{h}{2}\right],$$

where P_j^m is the space of polynomials of degree m defined on the interval $[x_j - h/2, x_j + h/2]$. Note that in general R will have jump discontinuities at the points $x_j \pm h/2$.

The solution is updated on a staggered grid. By integrating the conservation law (1) on the cell $[x_j, x_{j+1}] \times [t^n, t^n + k]$, we obtain

$$\begin{aligned} \overline{w}^{n+1}_{j+1/2} &= \frac{1}{h}\left\{ \int_0^{h/2} w(x_j + y, t^n)\, dy + \int_{-h/2}^{0} w(x_{j+1} + y, t^n)\, dy \right\} \\ &\quad - \frac{1}{h}\int_0^k \left[f\left(w(x_{j+1}, t^n + \tau)\right) - f\left(w(x_j, t^n + \tau)\right)\right] d\tau, \end{aligned} \tag{4}$$

If the time step is subjected to the CFL condition $k \le h/(2\max|f'(u)|)$, we can assume that $w(x, t^n + \tau)$ is smooth at x_{j+1} and x_j, since the discontinuities starting at t^n from the staggered grid points $x_{j+1/2}$ have not had the time to reach the cell boundaries.

Then the time integrals can be approximated by a quadrature formula, say:

$$\frac{1}{k}\int_0^k f\left(w(x_j, t^n + \tau)\right)\, d\tau \simeq \sum_{l=0}^{L} f\left(w(x_j, t^n + k\tau_l)\right)\omega_l, \tag{5}$$

where τ_l and $\omega_l \in [0,1]$ are the knots and weights of the quadrature formula. Simpson's rule is enough for third and fourth order schemes.

Since w is smooth at x_j, we can evaluate w at the intermediate times $t^n + k\tau_l$ through Taylor expansion or with a Runge Kutta method.

In the following time step we repeat a similar process and go back to the original grid.

2.1. Reconstruction.

Reconstruction is a key step in high resolution schemes. The algorithm we consider here was introduced in [4] and it has been widely implemented, see [12] and references therein.

The basic idea of ENO reconstruction is that by a suitable choice of the stencil, the interpolation polynomial will have only small oscillations. The original technique for the selection of the stencil is very sensitive to the data, and this may cause a deterioration of the accuracy. In order to overcome this problem, two modified stencil have been used, one developed by Shu, and an original one (MC) proposed by the authors [3].

2.2. Evaluation of the fluxes.

To compute the time integrals of the fluxes in (5), we need to evaluate the function $f(w(x_j, t^n + k\tau_l))$ at the different instants $k\tau_l \in [0, k]$, $l = 0, \cdots, L$. This can be obtained by Taylor expansion [7], or by Runge-Kutta schemes. Here we describe the latter approach.

The evaluation of the field at the j-th grid point can be written as

$$\left\{ \begin{array}{rcl} y'(\tau) &=& F(\tau, y(\tau)) = -f_x\left(y(x_j, t^n + \tau)\right) \\ y(\tau = 0) &=& w(x_j, t^n). \end{array} \right. \tag{6}$$

Thus the computation of the i-th Runge-Kutta flux requires the evaluation of the x-derivative of f at the intermediate time $t = t^n + c_i k$, where c_i are the coefficients of the RK scheme.

Therefore we must compute all grid values of f at the intermediate time, and perform a piecewise polynomial interpolation of these data to maintain high accuracy and control over oscillations in the evaluation of f_x.

For a method of order $m = 4$, we need a third order accurate three stage Runge-Kutta method. Thus we need to compute three polynomial interpolations for each node τ_l in the quadrature formula appearing in (5).

Fortunately a great saving in computational time can be obtained with the use of Natural Continuous Extensions (NCE) of a Runge-Kutta scheme.

The properties of NCE's which are essential for their application to our scheme are described and proved in [13].

At each time step, we apply the Runge-Kutta scheme only once, and we obtain all intermediate values $w(x_j, t^n + k\tau_l)$ through the evaluation of the appropriate NCE.

2.3. Systems.

We consider the system of conservation laws:

$$\mathbf{u}_t + \mathbf{f}_x(\mathbf{u}) = 0, \tag{7}$$

where $\mathbf{u}$ and $\mathbf{f}$ are vectors with M components.

We apply our scheme componentwise. At each time step, we start from the array of cell averages, $\{\overline{w}^n_{j,i}\}, j = 1, \cdots, N;\ i = 1, \cdots, M$. We apply the reconstruction step at each component. The stencil chosen in general will be different for each component. The evaluation of the space contribution appearing in (4) is now straightforward.

For the time evolution, we must compute all components of the Runge-Kutta fluxes $g_i^{(j)}, i = 1, \cdots, M$, before computing the successive fluxes $g_i^{(j+1)}$. No differentiation of the flux function $\mathbf{f}$ is required. We only need an estimate of the maximum characteristic velocity to satisfy the CFL condition.

2.4. Stability analysis.

Linear stability analysis of the schemes is performed, in order to identify the linearly stable central schemes and compute the critical Courant number of the scheme. The former information will be used as a guideline for the choice of the stencil in the non linear schemes.

Let us consider a generic central scheme of third and fourth order, applied to the linear equation:

$$u_t + u_x = 0,$$

Such scheme will take the form

$$\begin{aligned}\overline{w}^{n+1}_{j+\frac{1}{2}} &= \sum_{l=0}^{m-1}(\frac{1}{2})^{l+1}\frac{1}{(l+1)!}[\tilde{D}^l_j(\overline{w}^n)+(-1)^i\tilde{D}^l_{j+1}(\overline{w}^n)] \\ &- \frac{\lambda}{6}\{[w^n_{j+1}+4w^{n+\frac{1}{2}}_{j+1}+w^{n+1}_{j+1}]-[w^n_j+4w^{n+\frac{1}{2}}_j+w^{n+1}_j]\},\end{aligned} \qquad (8)$$

where the discrete space derivatives of the field, $\tilde{D}^l_j$, are obtained from cell averages $\{\overline{w}^n_j\}$ by deconvolution [3, 4], and $\lambda = k/h$ denotes the mesh ratio.

A detailed truncation analysis of the schemes shows that with our approach it is necessary to use a m degree interpolation polynomial in the reconstruction step to obtain a method of order m [3].

We shall consider separately third and fourth order schemes.

2.4.1. Third order schemes. The different stencils will be labeled by the value of $il(j)-j$, where $il(j)$ is the leftmost point of the stencil. The stencil of third order schemes is formed by four points, and must include points j and $j+1$, therefore the possible stencils are (-2), (-1), and (0).

From deconvolution we obtain:

$$\tilde{D}^0_j(\overline{u}) = D^0_j(\overline{u}) - \frac{1}{24}D^2_j(\overline{u}), \quad \tilde{D}^1_j(\overline{u}) = D^1_j(\overline{u}), \quad \tilde{D}^2_j(\overline{u}) = D^2_j(\overline{u})$$

where $D^0_j(u) = u_j$, and $D^k_j(u)$, $k \geq 1$, denotes the numerical approximation of k-th derivative of u (times h^k). They are obtained by taking the derivatives of the interpolating poliniomal of the particular stencil chosen. We list them here:

Stencil (-2)

$$\begin{aligned}D^1_j(u) &= \frac{1}{6}(u_{j-2}-6u_{j-1}+3u_j+2u_{j+1}) \\ D^2_j(u) &= u_{j-1}-2u_j+u_{j+1}.\end{aligned}$$

Stencil (-1)

$$\begin{aligned}D^1_j(u) &= \frac{1}{6}(-2u_{j-1}-3u_j+6u_{j+1}-2u_{j+2}) \\ D^2_j(u) &= u_{j-1}-2u_j+u_{j+1}.\end{aligned}$$

Stencil (0)

$$\begin{aligned}D^1_j(u) &= \frac{1}{6}(-11u_j+18u_{j+1}-9u_{j+2}+2u_{j+3}) \\ D^2_j(u) &= \frac{1}{3}(6u_j-15u_{j+1}+12u_{j+2}-3u_{j+3}).\end{aligned}$$

From deconvolution we obtain:

$$\begin{aligned}\tilde{D}^0_j(\overline{u}) &= D^0_j(\overline{u}) - \frac{1}{24}D^2_j(\overline{u}) \\ \tilde{D}^1_j(\overline{u}) &= D^1_j(\overline{u}) \\ \tilde{D}^2_j(\overline{u}) &= D^2_j(\overline{u}).\end{aligned}$$

Here we consider three different schemes, namely *Timeux*, RK2-2, and NCERK2. The schemes differ in the computation of the predictor values $w_j^{n+1/2}$ and w_j^{n+1}. The first scheme uses Taylor expansion (as in [7]), the second uses two steps of Runge-Kutta 2, one for $w_j^{n+1/2}$ and one for w_j^{n+1}, and the third scheme uses RK2 for the computation of w_j^{n+1}, and NCE to evaluate $w_j^{n+1/2}$.

Scheme NCERK2 is the one that gives the best results in terms of robustness and efficiency (in the case of systems), therefore we describe in detail only the analysis for this one.

In this case, w^{n+1} is computed by RK2 scheme, and $w^{n+1/2}$ is computed using the corresponding NCE of degree 2. Only two evaluations of the function are needed. The predictor values are given by

$$\begin{aligned} w_j^{n+\beta} &= w_j^n + k[b_1(\beta)F(w^n;j) + b_2(\beta)F(K_1;j)], \quad \beta = 1/2, 1 \\ K_1, j &= w_j^n + \beta k F(w^n; j), \end{aligned} \tag{9}$$

where $b_i(\beta)$ are the coefficients of the Natural Continuous Extension of Runge-Kutta schemes [13].

Here and in the following, K_1 denotes the vector with components $K_{1,j}$. Similarly, w^n and $\overline{w}^n$ are the vectors with components w_j^n and $\overline{w}_j^n$. Furthermore we recall that $w_j^n = D_j^0 = D_j^0(w^n)$, and

$$\begin{aligned} kF(w^n;j) &= -\lambda \tilde{D}_j^1(\overline{w}^n), \\ kF(K_1;j) &= -\lambda D_j^1(K_1). \end{aligned}$$

Substituting in (9) we obtain

$$w_j^{n+\beta} = \tilde{D}_j^0(\overline{w}^n) - \lambda(b_1(\beta)\tilde{D}_j^1(\overline{w}^n) + b_2(\beta)D_j^1(K_1)), \quad \beta = 1/2, 1.$$

2.4.2. FOURTH ORDER SCHEMES. In this case the stencil contains five points. The possible stencils are: (-3), (-2), (-1), and (0).

Stencil (-3)

$$\begin{aligned} D_j^1(u) &= \frac{1}{12}u_{j-3} - u_{j-1} + \frac{2}{3}u_j + \frac{1}{4}u_{j+1} \\ D_j^2(u) &= -\frac{3}{4}u_{j-3} - \frac{2}{3}u_{j-2} + \frac{3}{2}u_{j-1} - 2u_j + \frac{11}{12}u_{j+1} \\ D_j^3(u) &= -\frac{3}{4}u_{j-3} + 2u_{j-2} - \frac{3}{2}u_{j-1} + \frac{1}{4}u_{j+1} \end{aligned}$$

Stencil (-2)

$$\begin{aligned} D_j^1(u) &= \frac{1}{12}u_{j-2} - \frac{2}{3}u_{j-1} + \frac{2}{3}u_{j+1} - \frac{1}{12}u_{j+2} \\ D_j^2(u) &= -\frac{1}{12}u_{j-2} + \frac{4}{3}u_{j-1} - \frac{5}{2}u_j + \frac{4}{3}u_{j+1} - \frac{1}{12}u_{j+2} \\ D_j^3(u) &= -\frac{1}{2}u_{j-2} + u_{j-1} - u_{j+1} + \frac{1}{2}u_{j+2} \end{aligned}$$

Stencil (-1)

$$\begin{aligned}
D_j^1(u) &= -\frac{1}{4}u_{j-1} - \frac{5}{6}u_j + \frac{3}{2}u_{j+1} - \frac{1}{2}u_{j+2} + \frac{1}{12}u_{j+3} \\
D_j^2(u) &= \frac{11}{12}u_{j-1} - \frac{5}{3}u_j + \frac{1}{2}u_{j+2} - \frac{1}{12}u_{j+3} \\
D_j^3(u) &= -\frac{3}{2}u_{j-1} + 5u_j - 6u_{j+1} + 3u_{j+2} - \frac{1}{2}u_{j+3}
\end{aligned}$$

Stencil (0)

$$\begin{aligned}
D_j^1(u) &= -\frac{25}{12}u_j + 4u_{j+1} - 3u_{j+2} + \frac{4}{3}u_{j+3} - \frac{1}{4}u_{j+4} \\
D_j^2(u) &= \frac{35}{12}u_j - \frac{26}{3}u_{j+1} + \frac{19}{2}u_{j+2} - \frac{14}{3}u_{j+3} + \frac{11}{12}u_{j+4} \\
D_j^3(u) &= -\frac{5}{4}u_j + 4u_{j+1} - \frac{9}{2}u_{j+2} + 2u_{j+3} - \frac{1}{4}u_{j+4}
\end{aligned}$$

From deconvolution we obtain:

$$\tilde{D}_j^0(u) = D^0(\overline{u}) - \frac{1}{24}D_j^2(\overline{u}), \quad \tilde{D}_j^1(u) = D^1(\overline{u}) - \frac{1}{24}D_j^3(\overline{u})$$

$$\tilde{D}_j^2(u) = D^2(\overline{u}), \quad \tilde{D}_j^3(u) = D^3(\overline{u})$$

As in the case of third order schemes, we consider three schemes, namely *Timeux*, RK3-2, and NCERK4. The first is based on a Taylor expansion, the second on two steps of Runge-Kutta 3, and the last uses one step of RK4 with NCE of degree 3.

Only the analysis for scheme NCERK4 is shown in detail. By applying Runge-Kutta 4 with NCE to (6) one has

$$\begin{aligned}
w_j^{n+\beta} &= w_j^n + k[b_1(\beta)F(w^n;j) + b_2(\beta)F(K_1;j) + b_3(\beta)F(K_2;j) \\
&\quad + b_4(\beta)F(K_3;j)], \qquad \beta = 1/2, 1 \\
K_{1,j} &= w_j^n + \frac{k}{2}F(w^n;j) \\
K_{2,j} &= w_j^n + \frac{k}{2}F(K_1;j) \\
K_{3,j} &= w_j^n + kF(K_2;j)
\end{aligned}$$

where the classical RK4 scheme has been used with $\mathbf{b}^T = (1,2,2,1)/6$; the terms $b_i(\beta), i = 1, \ldots, 4$ are the NCE polynomials, and

$$\begin{aligned}
kF(w^n;j) &= -\lambda\tilde{D}_j^1(\overline{w}^n) \\
kF(K_s;j) &= -\lambda D_j^1(K_s), \qquad s = 1, 2, 3.
\end{aligned}$$

Third order schemes		
Scheme	Stencil	Stability region
RK2-2	j-2,j-1,j,j+1	λ^*=0.35099
	j-1,j,j+1,j+2	λ^*=0.43403
	j,j+1,j+2	unstable
NCERK2	j-2,j-1,j,j+1	λ^*=0.348086
	j-1,j,j+1,j+2	λ^*=0.435831
	j,j+1,j+2,j+3	unstable
Timeux	j-2,j-1,j,j+1	λ^*=1/2
	j-1,j,j+1,j+2	λ^*=1/2
	j,j+1,j+2,j+3	unstable

TABLE 1. Stable stencils for third order schemes

After substitution, the expression of the predicted values is given by:

$$\begin{aligned} w_j^{n+\beta} &= \tilde{D}_j^0(\overline{w}_j^n) - \beta\lambda[b_1(\beta)\tilde{D}_j^1(\overline{w}^n) + b_2(\beta)D_j^1(K_1) \\ &\quad + b_3(\beta)D_j^1(K_2) + b_4(\beta)D_j^1(K_3)], \qquad \beta = 1/2, 1 \\ K_{1,j} &= \tilde{D}_j^0(\overline{w}^n) - \frac{\lambda}{2}\tilde{D}_j^1(\overline{w}^n) \\ K_{2,j} &= \tilde{D}_j^0(\overline{w}^n) - \frac{\lambda}{2}D_j^1(K_1) \\ K_{3,j} &= \tilde{D}_j^0(\overline{w}^n) - \lambda D_j^1(K_2) \end{aligned}$$

Now we are ready to express (8) in terms of $\overline{w}^n$.

The amplification factor is obtained by looking for solutions of the form

$$\overline{w}_j^n = \rho^n e^{ij\xi},$$

where $i^2 = -1$. By substituting such expression in the numerical schemes one obtains

$$\overline{w}_{j+\frac{1}{2}}^{n+1} = \rho_\lambda(\xi)e^{i\xi/2}\overline{w}_j^n, \qquad \xi \in [0, 2\pi].$$

Stability is studied by analyzing the function

$$P_\lambda(\xi) = |\rho_\lambda(\xi)|^2.$$

Let λ^* be the maximum value of λ for which

$$\max_{0\le\xi\le2\pi} |P_\lambda(\xi)| \le 1, \tag{10}$$

We say that the scheme is stable if there exists $\lambda^* > 0$.

From the analysis of $P_\lambda(\xi)$ corresponding to the previous schemes one obtains the stability results summarized in Table 1 and 2, for third and fourth order schemes respectively.

The values of λ^* have been computed by solving an algebraic equation obtained from condition (10).

Fourth order schemes		
Scheme	Stencil	Stability region
RK3-2	j-3,j-2,j-1,j,j+1	unstable
	j-2,j-1,j,j+1,j+2	λ^*=9/22
	j-1,j,j+1,j+2,j+3	moder. unstable
	j,j+1,j+2,j+3,j+4	unstable
NCERK4	j-3,j-2,j-1,j,j+1	unstable
	j-2,j-1,j,j+1,j+2	λ^*=9/22
	j-1,j,j+1,j+2,j+3	moder. unstable
	j,j+1,j+2,j+3,j+4	unstable
Timeux	j-3,j-2,j-1,j,j+1	unstable
	j-2,j-1,j,j+1,j+2	λ^*=1/2
	j-1,j,j+1,j+2,j+3	moder. unstable
	j,j+1,j+2,j+3,j+4	unstable

TABLE 2. Stable stencils for fourth order schemes

The results of the analysis have been confirmed by the numerical results obtained by the above schemes.

For third order schemes based on Runge-Kutta, we find that stencil (-2) and (-1) are stable, and the stability region is slightly smaller than the one corresponding to scheme *Tadmor* and the schemes based on Taylor expansion. Stencil (0) is unstable.

For fourth order schemes based on Runge-Kutta, only the central stencil (-2) is stable. Stencil (-1) is moderately unstable for $\lambda < 0.4251$, and this instability is observed only after long integration time. Stencil (-3) and (0) are unstable.

Note that for the third order schemes based on ENO reconstruction there are two central stencils (i.e. (-2) and (-1)), and those are both stable. If equation $u_t - u_x = 0$ is considered, then the results for stencil (-2) and (-1) for third order schemes are reversed. In case of systems, because one does not want to do upwinding, the most restrictive CFL condition has to be used. For the fourth order schemes based on ENO there is one central stencil, and it is stable. In both cases, therefore, the stability does not depend on the sign of the characteristic velocity.

2.5. Numerical results.

Several tests have been performed, and the results are presented in [3]. In particular the scalar equation has been used to test the accuracy of the schemes, and several test problems in gas dynamics have been considered, in order to study the shock capturing and high resolution properties of the schemes. The best performance, in terms of accuracy and efficiency, has been obtained by schemes NCERK2 and NCERK4. The first is a third order scheme based on MC-ENO reconstruction with piecewise cubic polynomials, and flux evaluation obtained by Runge-Kutta 2, with NCE of degree 2. The second is a fourth order scheme based on MC-ENO reconstruction with piecewise polynomials of degree 4, a nd Runge-Kutta 4 with NCE of degree 3. Both schemes show the prescribed accuracy and sharp shock resolution.

References

[1] Anile A.M., Romano V., and Russo G., *Hydrodynamical models for semiconductors based on Extended Thermodynamics*, SIAM J. APpl. Math., submitted.

[2] Arminjon P., Viallon M.-C. and Madrane A., *A Finite Volume Extension of the Lax-Friedrichs and Nessyahu-Tadmor Schemes for Conservation Laws on Unstructured Grids*, International Journal of Computational Fluid Dynamics, **9**, (1997), 1–22.

[3] Bianco F., Puppo G., and Russo G., *High order central scheme for hyperbolic system of conservation laws*, in press on SIAM J. Sci. Comput.

[4] Harten A., Engquist B., Osher S., and Chakravarthy S. *Uniformly high order accurate essentially non-oscillatory schemes III*, J. Comp. Phys., **71**, (1987), 231–303.

[5] Jiang, G.S., and Tadmor E., *Non-oscillatory Central Schemes for Multidimensional Hyperbolic Conservation Laws*, CAM Report 96-36, UCLA, October 1996.

[6] Levy D., and Tadmor E., *Non-oscillatory central differencing for the Incompressible 2-D Euler Equations*, CAM Report 96-37, UCLA, October 1996.

[7] Liu X.D., and Tadmor E. *Third order non-oscillatory central scheme for hyperbolic conservation laws*, UCLA preprint.

[8] Nessyahu H., and Tadmor E. *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comp. Phys., **87**, (1990), 408–448.

[9] Romano V., and Russo G. *Numerical solution for hydrodynamical models of semiconductors*, Transactions on Computer-Aided Design of Integrated Circuits and Systems, submitted.

[10] Sanders R., and Weiser A. *High Resolution Staggered Mesh Approach for Nonlinear Hyperbolic Systems of Conservation Laws*, J. Comput. Phys., **101**, (1992), 314–329.

[11] Shu C.W. *Numerical experiments on the accuracy of ENO and modified ENO schemes*, J. Sci. Comp., **5**, Vol. 2, (1990), 127–149.

[12] Shu C.W., and Osher S. *Efficient Implementation of Essentially Non-Oscillatory Shock-Capturing Schemes, II*, J. Comput. Phys. **83**, (1989), 32–78.

[13] Zennaro M., *Natural Continuous Extensions of Runge-Kutta Methods*, Math. Comp. **46**, (1986), 119–133.

Dipartimento di Matematica,
Università dell'Aquila,
Via Vetoio, loc. Coppito
67100 L'Aquila, Italy
E-mail address: `russo@univaq.it`