Mathematical Models and Methods in Applied Sciences

A REMARK TO THE PAPER "THE PARAMETER DEPENDENCE OF THE COEFFICIENT IN A MODEL FOR CONSTANT PRESSURE STEAM INJECTION IN SOIL"

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A simple and accurate approximation is obtained to the numerical solution of the integral equation presented in[1] by using nonlinear optimization

1. Introduction

In a recent paper[1] the authors study an integral equation that arises in a model for constant-pressure steam injection in soil.

The equation may be written as

$$\lambda = 2y\exp(\lambda^2 y^2)\int_{\lambda y}^{\infty}\exp(-s^2)\,ds \tag{1.1}$$

The authors present several interesting theoretical and computational results about this equation. In particular they show that Eq. (1.1) defines a unique function $\lambda(y)$ which is smooth, strictly increasing and strictly concave in $[0,+\infty)$.

Following estimates on the Mills ratio, the authors present some bounds on the magnitude of $\lambda(y)$.

More precisely, they found a lower bound, $l(y)$, and an upper bound, $u(y)$, defined as:

$$l(y) = \begin{cases} \sqrt{\dfrac{2\pi}{1+4y^2+\sqrt{1+8y^2+8(4-\pi)y^4}}}\,y & \text{if } y \le y_1 \\[2ex] \sqrt{\dfrac{2\alpha_1}{1+2(\alpha_1-1)y^2+\sqrt{1+4(\alpha_1-1)y^2+4y^4}}}\,y & \text{if } y > y_1 \end{cases}$$

and

$$u(y) = \begin{cases} \sqrt{\dfrac{2\pi}{1+4y^2+\sqrt{1+8y^2+4\pi^2\alpha_2^{-2}y^4}}}\,y & \text{if } y \le y_2 \\[2ex] \sqrt{\dfrac{8}{1+6y^2+\sqrt{1+12y^2+4y^4}}}\,y & \text{if } y > y_2 \end{cases}$$

with

$$\alpha_1 = \frac{\pi(13-4\pi)}{4+2\pi-\pi^2}, \quad \alpha_2 = \pi\sqrt{\frac{4-\pi}{16+2\pi^2-11\pi}},$$

$$y_1 = \sqrt{\frac{(4-\pi)(\pi-3)}{(7-2\pi)(4+2\pi-\pi^2)}}, \quad y_2 = \sqrt{\frac{4-\pi}{2(\pi-3)(\pi-2)}}.$$

By using these estimates, the authors construct an approximation $\tilde{\lambda}(y)$ of $\lambda(y)$ defined by the harmonic mean of $l(y)$ and $u(y)$:

$$\tilde{\lambda}(y) = \frac{2u(y)l(y)}{u(y)+l(y)}$$

and they show that this approximation gives a relative error which is less than 4×10^{-3}.

A numerical evaluation of the function shows that the approximation introduced by $\tilde{\lambda}(y)$ is indeed much better than what predicted by theoretical estimates. Here we show that by nonlinear optimization, a simpler expression can be derived, which gives even more accurate results.

2. The minimization problem

We observe that the curve $(y, \lambda(y))$ has a parametric representation that can be explicitly written in terms of the complementary error function.

Let $\mu = \lambda y$. In the referred paper[1] it is proved that $\lambda(y)$ defines a unique smooth strictly increasing function of y in $[0, +\infty)$. As a consequence, $\mu = y\lambda(y)$ is a strictly increasing function in $[0, +\infty)$ with image in $[0, +\infty)$. Therefore the relation $\mu = y\lambda(y)$ defines a unique inverse function $y = \hat{y}(\mu)$ in $[0, +\infty)$ with image in $[0, +\infty)$. An explicit expression of $\hat{y}(\mu)$ can be given as follows.

The complementary error function is defined as:

$$\operatorname{erfc}(\mu) = \frac{2}{\sqrt{\pi}} \int_\mu^\infty \exp(-s^2)\,ds$$

Multiplying (1.1) by y one obtains:

$$\mu = 2y^2 \exp(\mu^2) \int_\mu^\infty \exp(-s^2)\,ds, \tag{2.2}$$

and therefore the parametric expression for the curve $(y, \lambda(y))$ is

$$\begin{aligned} y &= \hat{y}(\mu) \equiv \sqrt{\frac{\mu}{\sqrt{\pi}\exp(\mu^2)\,\mathrm{erfc}(\mu)}} \\ \lambda &= \hat{\lambda}(\mu) \equiv \mu/\hat{y}(\mu) \end{aligned}$$

The functions $l(y)$ and $u(y)$ are piecewise of the form

$$\lambda = g(y; \mathbf{a})$$

where $\mathbf{a} = (a_1, \ldots, a_5)$ and

$$g(y; \mathbf{a}) = \frac{y}{\sqrt{a_1 + a_2 y^2 + \sqrt{a_3 + a_4 y^2 + a_5 y^4}}} \tag{2.3}$$

Here we use a numerical procedure in order to determine a set nonnegative parameters (not all zero) $(a_1, \ldots, a_5)$ that give a "good" approximation to the function $\lambda(y)$. Let $V_n \equiv (\mathbf{R}^+ \cup \{0\})^n \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector of $\mathbf{R}^n$.

The analytical problem is stated as follows:

Problem 0. Compute $\mathbf{a} \in V_5$ such that

$$\mathcal{E}(\mathbf{a}) \equiv \sup_{y>0} \left| \frac{g(y; \mathbf{a})}{\lambda(y)} - 1 \right|$$

is minimum.

Because of the parametric representation, the problem can be stated as follows:

Problem 1. Compute $a_1, \ldots, a_5$ such that

$$\tilde{\mathcal{E}}(\mathbf{a}) \equiv \sup_{\mu>0} |\mathrm{err}(\mu; \mathbf{a})|$$

is minimum, where

$$\mathrm{err}(\mu; \mathbf{a}) \equiv \left| \frac{g(\hat{y}(\mu); \mathbf{a})}{\hat{\lambda}(\mu)} - 1 \right|.$$

The solution of the above minimization problems, however, does not satisfy the asymptotic behavior of the function $\lambda(y)$. By imposing the conditions

$$\lim_{y \to 0^+} \frac{g(y; \mathbf{a})}{\lambda(y)} = 1, \quad \lim_{y \to \infty} \frac{g(y; \mathbf{a})}{\lambda(y)} = 1$$

one obtains the following relations on the parameters:

$$a_3 = \left(\frac{1}{\pi} - a_1 \right)^2, \quad a_5 = (1 - a_2)^2,$$

and therefore, the optimal approximation of the form (2.3) that satisfies the asymptotic conditions is obtained by the following minimization problem:

Problem 2. Compute $(a_1, a_2, a_4) \in V_3$ such that

$$\hat{\mathcal{E}}(a_1, a_2, a_4) \equiv \sup_{\mu>0} \left|\mathrm{err}(\mu; a_1, a_2, (1/\pi - a_1)^2, a_4, (1-a_2)^2)\right|$$

is minimum.

3. Numerical optimization

We consider a discrete version of Problem 1. Take discrete values of the parameter μ :

$$\mu_i = \exp(\xi_0 + i\Delta\xi), i = 1, \ldots, N$$

where the discretization parameters $\xi_0, \Delta\xi, N$ are suitably chosen. Such discretization has been used in order to be approximately uniform in a logarithmic scale.

Next we compute the set of parameters $(a_1, \ldots, a_5)$ that minimize

$$\mathcal{E}_N^{(1)}(a_1, \ldots, a_5) \equiv \sup_{i=1}^{N} |\mathrm{err}(\mu_i; (a_1, \ldots, a_5)|$$

by using the simplex method. The computation has been performed by Matlab, using the built-in function `fmins`. A sequence of discrete problems has been solved, with an increasing number of discretization points, so that the values of the parameters obtained with a certain number of points is used as a guess for the more refined problem. The final result is obtained with $\xi_0 = -7, \Delta\xi = 0.01, N = 1200$.

The following optimal values have been found for the parameters:

$$\bar{\mathbf{a}} = (0.209169, 0.830417, 0.0119059, 0.0972485, 0.0287879)$$

Let us denote by $\lambda_1(y)$ the approximation obtained with this set of parameters: $\lambda_1(y) = g(y, \bar{\mathbf{a}})$. The maximum relative error for these parameters is 3.985×10^{-5}.

The discrete version of Problem 2 is solved with the same technique, i.e. compute a_1, a_2, a_4 that minimize

$$\mathcal{E}_N^{(2)}(a_1, a_2, a_4) \equiv \sup_{i=1}^{N} \left|\mathrm{err}(\mu_i; (a_1, a_2, (1/\pi - a_1)^2, a_4, (1-a_2)^2)\right|$$

The solution of the problem gives the following values of the parameters:

$$(a_1, a_2, a_4) = (0.209705, 0.832136, 0.0963156).$$

Let use denote by $\lambda_2(y)$ the approximation corresponding to these parameters. In this case the maximum relative error is 5.27×10^{-5}. The error is therefore about 30% larger than the previous one, but the solution is asymptotically correct as $y = 0$ or $y \to \infty$.

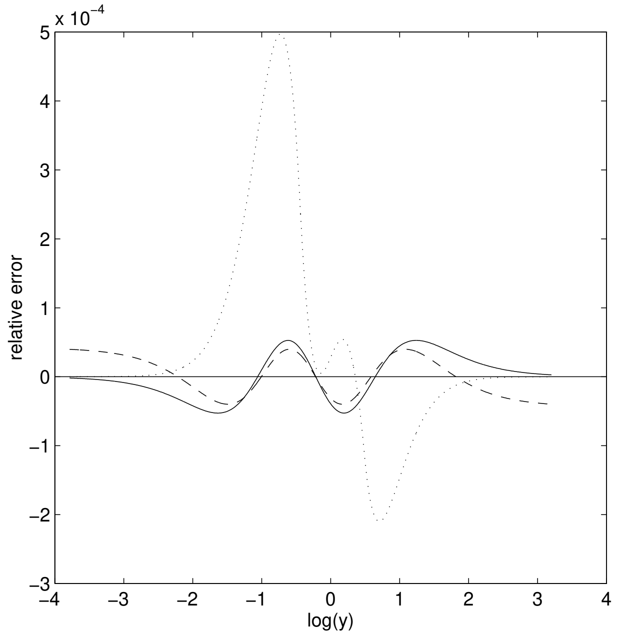

Fig. 1. Relative errors of the approximations $\tilde{\lambda}$, λ_1 and λ_2 vs log(y).

The results of the computations are summarized in Figure 1, where the relative error of the different approximations are plotted on a semilogarithmic scale. The continuous line represents the relative error of the optimal solution $\lambda_2(y)$ that satisfies the asymptotic conditions. The dashed line represents the relative error of the optimal solution $\lambda_1(y)$, and the dotted line is the relative error of the approximation $\tilde{\lambda}(y)$. The maximum relative error of the latter is 4.97×10^{-4}. Note that this is much smaller than the estimate obtained in[1], and that the simple solution $\lambda_2(y)$, which has the same asymptotic properties, gives an error which is one order of magnitude smaller.

References

1. B. H. Gilding and Shuanhu Li, *The parameter dependence of the coefficient in a model for constant-pressure steam injection in soil*, *Mathematical Models and Methods in Applied Sciences*, in press.