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Central Schemes for Balance Laws

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Abstract

A short review on central schemes for the numerical solution of hyperbolic systems of conservation laws is given. The main focus of the talk concerns the construction of high order central schemes in one and two space dimensions, and the extension of central schemes to systems with source term. The treatment of stiff source and the construction of well-balanced central scheme will be addressed.

1 Introduction

We consider central schemes for the numerical solution of balance equations of the form

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(x,u(x,t))}{\partial x} = R(x,u(x,t)) \tag{1}$$

$u \in I\!R^m$; $f, R : I\!R \times I\!R^m \to I\!R^m$, $A = \nabla_u f(x,u)$ has real eigenvalues and a basis of eigenvectors. Such systems appear in many contexts in the applications. We just mention a few examples such as shallow water equation [27], discrete kinetic models [6], gas in Extended Thermodynamics [22], hydrodynamical models of semiconductors [1].

Central schemes for the numerical solution of conservation laws have been widely developed in the last decade. The first order prototype of such schemes is Lax-Friedrichs scheme. Such scheme is more dissipative than first order Godunov scheme, but it is certainly easier to implement, since it does not require the knowledge of the characteristic decomposition of the system or the knowledge of the (exact or approximate) solution to the Riemann problem.

Second order central schemes on a staggered grid have been derived independently by Nessyahu and Tadmor (NT) [23] for one space dimension and by Sanders and Weiser (SW) [25] for one and two space dimensions. SW schemes do not require the solution of the Riemann problem, but they require the computation of the eigenvalue and eigenvectors of the system.

NT scheme is very simple and efficient. It provides sharp shock resolution and can be easily used for systems for which the characteristic decomposition is not known. Only an estimate of the largest eigenvalue of the Jacobian matrix is necessary, in order to satisfy a suitable CFL condition.

Central schemes are derived by integrating the conservation law over a staggered cell in space-time (see Figure 1), obtaining

$$\begin{aligned}\Delta x\, \bar{u}_{j+1/2}^{n+1} &= \int_{x_j}^{x_{j+1}} u(x,t_n)\,dx - \int_{t_n}^{t^{n+1}} \left(f(x_{j+1},u(x_{j+1},t)) - f(x_j,u(x_j,t))\right) dt \\ &\quad + \int_{x_j}^{x_{j+1}} \int_{t_n}^{t^{n+1}} R(x,u(x,t))\,dx\,dt. \end{aligned} \tag{2}$$

In order to transform the above relation (which is exact) in a numerical scheme, one has to approximate the space and time integrals.

The first step is to perform a piecewise polynomial reconstruction of the function $u(x,t^n)$ in each cell from cell averages. The reconstruction step is crucial, since space accuracy and non-oscillatory properties of the scheme depend mainly on this step.

Next, one has to compute the numerical flux function by an approximation of the average flux on the boarder of the staggered cell.

NT scheme is obtained by piecewise linear non-oscillatory reconstruction from cell averages, and by the mid-point rule to compute the integral. Forward Euler scheme is used to compute the predictor value of the field at the node of the midpoint rule. A generalization of NT scheme to systems with source is obtained by including the effect of the source in the predictor, and by using a quadrature formula to approximate the source term. The scheme can be summarized in the two line formula

$$\begin{aligned} u_j^{n+1/2} &= u_j^n - \frac{\lambda}{2} f_j' + \frac{\Delta t}{2} R(x_j, u_j^{\text{pred}}), \\ u_{j+1/2}^{n+1} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') - \lambda(f(x_{j+1}, u_{j+1}^{n+1/2}) - f(x_j, u_j^{n+1/2})) \\ &\quad + \frac{\Delta t}{2}(R(x_j, u_j^{n+1/2}) + R(x_{j+1}, u_{j+1}^{n+1/2})), \end{aligned} \tag{3}$$

where $\lambda = \Delta t/\Delta x$, and $u_j'/\Delta x$, $f_j'/\Delta x$ denote first order approximation of space derivatives. Several *slope limiters* can be used to make the scheme non oscillatory (see [23] and references therein). The source in the predictor can be computed explicitly ($u_j^{\text{pred}} = u_j^n$), or implicitly ($u_j^{\text{pred}} = u_j^{n+1/2}$). The implicit time discretization guarantees A-stability for the source term (the scheme reduces in fact to the implicit midpoint scheme if the flux is zero).

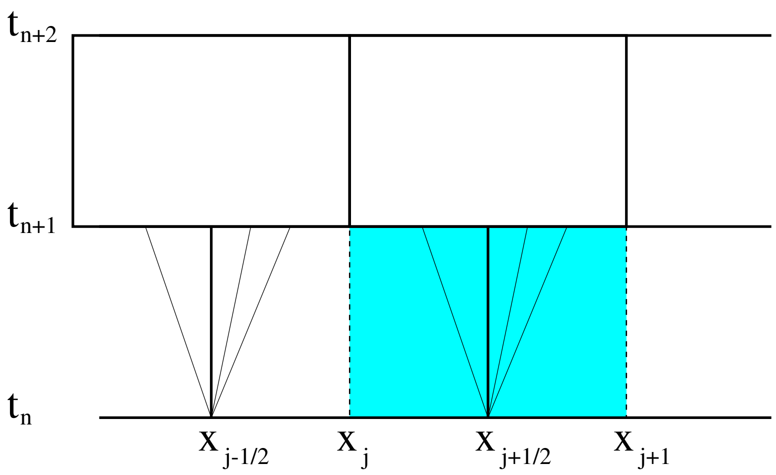

Figure 1: Integration over the staggered cell in space-time and derivation of central schemes

In absence of source ($R \equiv 0$) the above scheme reduces to the well-known NT scheme.

NT method has been extended to two space dimensions by several authors. Arminjon et al. [2] and Jiang and Tadmor [10] propose two dimensional central schemes on a staggered rectangular grid.

Subsequently, Arminjon et al. construct a two dimensional central scheme on unstructured grids, and apply it to flow past an airfoil [3].

Central schemes have been used for incompressible two dimensional Euler equations in the mixed velocity-vorticity formulation [17] and in primitive variables [11].

One of the drawbacks of the original formulation is the staggering of the mesh. A non staggered central version of the original NT scheme has been proposed in [9], in which the authors reconstruct at each time step the numerical solution on the original grid.

A genuinely non-staggered version of central scheme has been recently proposed by Kurganov and Tadmor [13]. In the paper the authors propose two new schemes. A fully discrete scheme, which makes use of the information on the local maximum characteristic speed to improve accuracy (by reducing the local numerical viscosity), and a fully discrete scheme, which allow a great flexibility on the time discretization. Such flexibility can be exploited, for example, in several problems with source.

We conclude the section by remarking that there are modern “upwind-based” shock capturing schemes that do not require the solution of the Riemann problem or the knowledge of the characteristic structure of the system (see, for example, [20]).

2 High order central schemes

Accurate numerical computation require a very large number of grid points, or the use of accurate schemes. A third order central scheme has been presented in [21]. Here we briefly describe a systematic procedure to obtain high order staggered central schemes (in the absence of source). A similar procedure can be used to obtain high order semidiscrete central schemes [12].

High order shock capturing central schemes on a staggered grid can be constructed from Eq. (2) by using

i) high order reconstruction (for accurate evaluation of space integrals, pointwise values from cell averages, and space derivatives),

ii) high order evaluation of the integral of the flux.

The first goal can be achieved by reconstruction techniques similar to those used in the context of upwind schemes, i.e. Essentially Non Oscillatory (ENO) [8] or Weighted ENO [19, 26] reconstruction.

ENO idea, originally introduced by Harten and Osher, is very simple and effective. It may be stated as follows: if you want to compute the first derivative at a point, you can use either left or right difference. Just use the smallest one. If you want to compute the second derivative, you need a three point stencil. Therefore you have to add one point to the two-point stencil that was used for the computation of the first derivative. You may add it to the right or to the left. Of these two possible stencils, choose the one that gives you the smallest second derivative. And so on, you add one point at time to construct high order derivatives. A detailed explanation of this procedure is given in [8]. Such idea has been successfully used in a large number of contexts.

ENO reconstruction has been used in [4] for the construction of high order central schemes. The reconstruction is used for three purposes: to compute the space integral appearing in Eq. (2), to reconstruct the pointwise value at the center of the cell from cell averages, and to compute space derivative of the flux. Pointwise values are needed, for example, in the evaluation of the predictor values that are used for the computation of time integral of the flux at the boarder of the cell. They differ from cell averages by a term which is $O(\Delta x^2)$, where Δx denotes the grid size. For second order schemes this difference is not relevant, but for third or higher order schemes, it is necessary to take this into account.

A more efficient reconstruction technique is provided by Weighted ENO [19]. The reconstruction is a weighted sum of low order polynomials, constructed using adjacent stencil. Fifth order upwind schemes can be constructed by taking a linear combination of three parabola,

$$R_j(x) = w_{-1}P_{-1}(x) + w_0 P_0(x) + w_1 P_1(x)$$

Parabola $P_i, i = -1, 0, 1$ is obtained by imposing that the cell average of the parabola equals the cell average of the solution in cells $j + i + \ell$, $\ell = -1, 0, 1$.

The weights of the sum are chosen in such a way that high order approximation is guaranteed for smooth function. The fact that fifth order is obtained only with a parabolic reconstruction is not surprising, since the three weights (whose sum is one) introduce two extra degrees of freedom, that can be tuned in order to achieve high accuracy in the computation of averages or derivatives. The effective stencil of function R_j is in fact five points. When the function is not smooth, the weights will try to select the least oscillatory parabola. Third and fourth order Central WENO schemes have been presented in [15]. Numerical test confirm high resolution property of such schemes.

A compact third order CWENO reconstruction is obtained by using two linear and parabolic reconstruction [16]. Such a scheme has a very compact stencil. This property is particularly desirable for computations in several dimension.

The integral of the flux can be performed by using a quadrature formula in time, and by evaluating the function u at the nodes of the quadrature formula. This is obtained by solving the equation in time

$$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x}$$

at each node x_j. This can be obtained either by Taylor expansion, or by the use of Runge-Kutta methods. Taylor expansion can be effective in the case of the scalar conservation equation, but it may be very expensive when applied to systems, since it requires the solution of the Jacobian and possibly the Hessian matrix of the flux function f. Runge-Kutta schemes with Natural Continuous Extension [28] provide a uniform approximation of the numerical solution on the whole time step, and can be used to compute the predictor value of the field at the nodes of the quadrature formula with just one RK step.

3 Stiff source and well-balanced schemes

A general procedure for the treatment of systems with source of the form (1) is the use of a fractional step method, such as the well known Strang splitting. The performance of such approach, however, degrades when the source becomes stiff, and splitting schemes become first order in the stiff limit.

A suitable splitting strategy can be devised for upwind-based schemes, that guarantees second order accuracy, both in the stiff and in the non stiff limit [5]. The same splitting strategy can not be applied to schemes based on NT convection step.

On the other hand, NT central scheme can be naturally extended to treat problems with a source term, as illustrated in the introduction.

For stiff source, the situation is more delicate, since A-stability is not enough. A scheme that gives second order accuracy in both stiff and non stiff limit is proposed and analyzed in [18]. The scheme can be written as two-stage predictor corrector scheme:

$$\begin{aligned}
u_j^{n+1/2} &= u_j^n - \frac{\lambda}{2} f_j' + \frac{\Delta t}{2} g(u_j^{n+1/2}) \\
u_j^{n+1/3} &= u_j^n - \frac{\lambda}{3} f_j' + \frac{\Delta t}{3} g(u_j^{n+1/3}) \\
u_{j+1/2}^{n+1} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') - \lambda(f(u_{j+1}^{n+1/2}) - f(u_j^{n+1/2})) \\
&\quad + \frac{\Delta t}{8}(3g(u_j^{n+1/3}) + 3g(u_{j+1}^{n+1/3}) + 2g(u_{j+1/2}^{n+1}))
\end{aligned}$$

Note that both predictors and corrector are implicit. It can be shown that the scheme is second order both in the stiff and non stiff limit.

Well balanced central schemes There are problems of systems with source in which the solution is a small deviation from the equilibrium solution. In such case, it is essential that the numerical scheme preserves the equilibrium solution at a discrete level, otherwise the truncation error may be of the same order of the signal one is interested in, unless an extremely fine mesh is used to integrate the equations.

At equilibrium, the vector $\tilde{u}(x)$ satisfies

$$\frac{\partial f(x, \tilde{u}(x))}{\partial x} = R(x, \tilde{u}(x)). \tag{4}$$

A typical problem arises, for example, when solving the 1D shallow water equations. The system is given by

$$\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0, \\
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{1}{2} g h^2\right) &= -ghB_x,
\end{aligned}$$

where $h(x,t)$ denotes the water depth, $q = hv$ denotes the water flux, v is the velocity, $B(x)$ is the bottom profile, and g is the gravitation acceleration constant. This problem has been studied by several authors [14, 7], and suitable upwind schemes have been designed that are able to maintain the equilibrium condition (4) at a discrete level. The result of a naive implementation of a central scheme with relaxation (second order splitting) , when applied to the shallow water equations, is given in Figure (2). The spurious profile in the middle is due to the bottom profile

$$B(x) = \begin{cases} \frac{1}{4}(1 + \cos \pi(10x - 5) & \text{if } |10x - 5| < 1 \\ 0 & \text{if } |10x - 5| \geq 0 \end{cases}$$

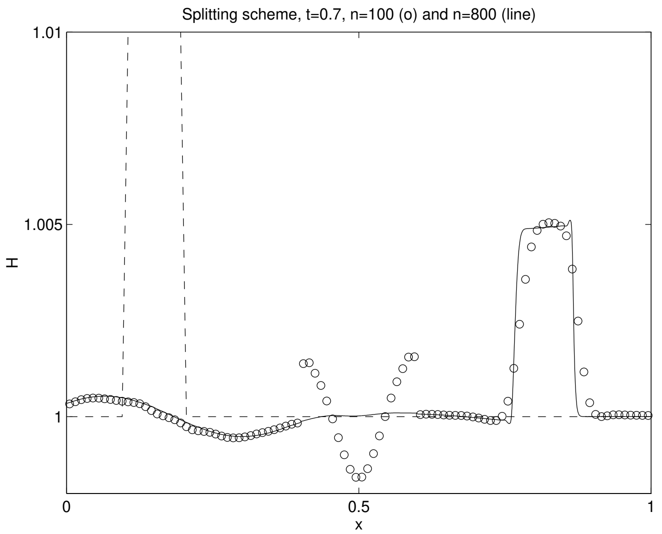

Figure 2: Total height $H(x,t) = h(x,t) + B(x)$ at time $t = 0.7$. The dashed line represents the initial state. The circles denote the numerical solution obtained by a second order splitting scheme with $N = 100$ grid points. The continuous line is obtained with the same scheme, but with $N = 800$ grid points.

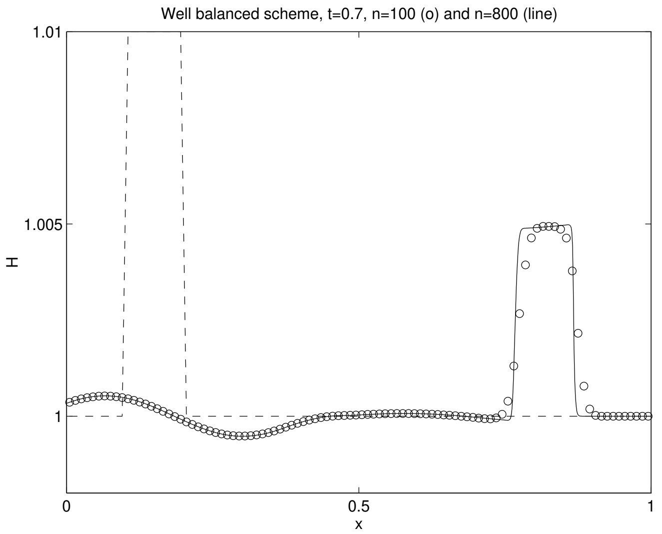

Figure 3: The same problem, solved by the well balanced central scheme. Circles: $N = 100$, continuous line: $N = 800$.

Here we show how to construct a fully discrete well balanced central scheme which is based on NT type scheme with staggered grid.

A well balanced scheme has to preserve equilibria. If $\tilde{u}$ is an equilibrium solution, then the scheme has to preserve, at a discrete level, Eq.(4) and the following equation level

$$\frac{\partial \tilde{u}}{\partial t} = 0.$$

This last requirement is not preserved by NT scheme, unless $\tilde{u}$ is constant. Therefore we make a change of variables that preserves the weak solutions of the system, and such that the equilibria are represented by constants. In the case of shallow water, we use $H = h + B$ as conserved variable. In this way, static solutions will be given by $H =$ constant and $v = 0$.

Requirement (4), at a discrete level, can be stated as

$$f(x_{j+1}, u_{j+1}^{n+1/2}) - f(x_j, u_j^{n+1/2}) = \frac{\Delta t}{2}(R(x_j, u_j^{n+1/2}) + R(x_{j+1}, u_{j+1}^{n+1/2})) \qquad (5)$$

It is easy to check that a static solution preserves condition (5) if the predictor is computed from the discretization of the non conservative form of the equation,

$$\frac{\partial u}{\partial t} = R - Au_x - \partial_x f,$$

where ∂_x denotes the partial derivative with respect to x at constant u (this corresponds to the choice $f' = Au'$ of the NT scheme,) and if a suitable approximation is used for the pointwise values and space derivatives of the bottom profile, i.e.

$$\begin{aligned} B_j &= \frac{1}{2}\left(B(x_j + \Delta x/2) + B(x_j - \Delta x/2),\right) \\ \left.\frac{\partial B}{\partial x}\right|_j &= \frac{B(x_j + \Delta x/2) - B(x_j - \Delta x/2)}{\Delta x}. \end{aligned}$$

The approach presented here can be used to construct well balanced schemes for the Saint Venant approximation of the shallow water equations in a channel of variable cross section, under the assumption that the lateral walls of the channel are vertical, i.e. that the area of the cross section is $A = (H(x,t) - B(x))W(x)$, $W(x)$ being the width of the channel, [24]. More general cases, such as non static equilibria or different cross section, are presently under consideration.

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