On the impulse formulation of the Euler Equations

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Abstract

The impulse formulation of the incompressible Euler equations is briefly described, and its Hamiltonian structure is shown. A numerical scheme for the solutions of the impulse variable, which is based on the Hamiltionian structure of the equations, is presented. Some stability issues are discussed.

1 Introduction

A recent formulation of the incompressible Euler equation has attracted some interest in the last years. The formulation is based on a gauge freedom in the choice of the field that describes the evolution of an incompressible flow. The purpose of this talk is to show some of the properties of the impulse (or *velicity* equations in the so called *geometric gauge*, and to exploit the Hamiltonian structure of the impulse equation by studying the interaction of an incompressible fluid with a membrane, both from an analytical and a numerical point of view.

Most of the material presented is discussed in detail in [6, 7]. Here we discuss a particular aspect which concerns the choice of the blob in the so called *velicity-blob method*.

Let us consider the incompressible Euler's equation in $\mathbf{R}^3$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla \mathcal{P}}{\rho}\,, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0\,, \tag{2}$$

where ρ, $\mathbf{u}$, and $\mathcal{P}$ are the fluid density, velocity, and pressure. From now on we shall set $\rho = 1$.

Taking the curl of (1) one has

$$\frac{\partial \omega}{\partial t} + \nabla \times (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{0} \tag{3}$$

where $\omega = \nabla \times \mathbf{u}$ denotes the vorticity of the fluid. By using a simple vector identity, (3) can be written as

$$\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{u} \times \omega) = \mathbf{0}. \tag{4}$$

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Let us introduce a new vector field $\mathbf{p}$ with the following property

$$\nabla \times \mathbf{p} = \omega \,. \tag{5}$$

Substituting the last equation into (4) we obtain

$$\nabla \times [\frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{p})] = \mathbf{0}\,, \tag{6}$$

which implies

$$\frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{p}) = \nabla \Psi\,, \tag{7}$$

where Ψ is an arbitrary scalar function. With a proper choice of the gauge, $\Psi = -\mathbf{u} \cdot \mathbf{p}$, the equations provide a Hamiltonian description of the fluid flow [5], [8]. Equation (7) in component form is

$$\frac{\partial p^\alpha}{\partial t} + u^\beta \frac{\partial p^\alpha}{\partial x^\beta} = -p^\beta \frac{\partial u_\beta}{\partial x_\alpha}\,,\ \ \alpha, \beta = 1, 2, 3, \tag{8}$$

and repeated indices are summed. In order to get a closed description of the fluid motion in terms of $\mathbf{p}$, we need a relation between $\mathbf{p}$ and $\mathbf{u}$. From $\nabla \times \mathbf{p} = \nabla \times \mathbf{u}$ it follows that

$$\mathbf{u} = \mathbf{p} - \nabla \phi\,, \tag{9}$$

where

$$\Delta \phi = \nabla \cdot \mathbf{p} \tag{10}$$

because of incompressibility. Equations (8), (9) and (10) constitute a closed system of equations for $\mathbf{p}(\mathbf{x}, t)$. An explicit expression of $\mathbf{u}$ in terms of $\mathbf{p}$ can be obtained by using the Green's function $G(\mathbf{x})$ of the Laplacian operator. In $\mathbf{R}^3$ and with vanishing flow at infinity it is

$$G(\mathbf{x}) = -\frac{1}{4\pi |\mathbf{x}|}\,. \tag{11}$$

From (9) and (10) one has:

$$u^\alpha(\mathbf{x}) = p^\alpha(\mathbf{x}) - \frac{\partial}{\partial x^\alpha} \int G(\mathbf{x} - \mathbf{y}) \frac{\partial p^\beta}{\partial y^\beta}\, d\mathbf{y}\,. \tag{12}$$

The evolution equations of the fluid can be written as

$$\frac{D x^\alpha}{Dt} = u^\alpha\,, \tag{13}$$

$$\frac{D p^\alpha}{Dt} = -p^\beta \frac{\partial u^\beta}{\partial x^\alpha}\,, \tag{14}$$

where $D/Dt = \partial/\partial t + u^\alpha \partial/\partial x^\alpha$ denotes the Lagrangian derivative and Eq. (14) is Eq. (8) in Lagrangian form.

It is possible to show [7] that these equations are the Hamilton equations obtained by taking the functional derivatives of the kinetic energy of the fluid, expressed as a functional of the fluid trajectories:

$$\begin{aligned} \mathbf{x} &= \tilde{\mathbf{x}}(\xi, \mathbf{p}_0, t) = \mathbf{X}(\xi, t)\,, \\ \mathbf{p} &= \tilde{\mathbf{p}}(\xi, \mathbf{p}_0, t) = \mathbf{P}(\xi, t)\,, \end{aligned} \tag{15}$$

where ξ is the Lagrangian coordinate of the fluid particle

$$\xi = \mathbf{x}(\xi, \mathbf{p}_0, 0) = \mathbf{X}(\xi, 0)\,. \tag{16}$$

Inviscid incompressible fluid flow has several invariants. One of them is the kinetic energy E_k. Other two important invariants are the total impulse, and the angular momentum [3], [4]. They can be defined in terms of velicity as [3].

$$\mathbf{I} = \int \mathbf{P} d\xi\,, \tag{17}$$

$$\Omega = \int \mathbf{x} \times \mathbf{P} d\xi\,. \tag{18}$$

The numerical method described below has the property of preserving such invariants.

2 The velicity-blob method

A numerical scheme can be obtained from the Hamiltonian formulation, either by discretizing the equations of motion, or by discretizing the Hamiltonian. The second approach has the advantage of providing a scheme that has an underlying Hamiltonian structure, and therefore it preserves exactly some of the invariants of the motion. In this case, in fact, the equations of motion will be the Hamilton equations corresponding to a Hamiltonian of a set of N fluid particles, and they will presenrve the discrete version of the invariant, such as energy and momentum. If one wants to preserve such invariants after time discretization, then one needs to use a simplectic integrator for the dynamical system of the N particles. As we shall see, for practical purposes an accurate standard explicit ODE solved (such as, for example, Runge-Kutta of order four) is more than enough, because the time error is much smaller than the error introduced by the spatial discretization.

A discretization of the flow equation is obtained as follows. The velicity field is approximated by a collection of velicity blobs:

$$\mathbf{p}(\mathbf{x}, t) \approx \mathbf{p}_{N,\delta}(\mathbf{x}, t) = \sum_{i=1}^{N} g_\delta(\mathbf{x} - \mathbf{x}_i(t))\mathbf{p}_i(t) V_i\,, \tag{19}$$

where $g_\delta(\mathbf{x}) = g(\mathbf{x}/\delta)/\delta^3$ is a suitable blob function, and $\mathbf{x}_i$, $\mathbf{p}_i$, and V_i are the location, velicity and volume associated to particle i. The evolution equations for $\mathbf{x}_i$ and $\mathbf{p}_i$ are given by

$$\frac{dx_i^\alpha}{dt} = \tilde{u}_\delta^\alpha(\mathbf{x}_i)\,, \tag{20}$$

$$\frac{dp_i^\alpha}{dt} = -p_i^\beta \frac{\partial \tilde{u}_\delta^\beta}{\partial x_i^\alpha}\,, \tag{21}$$

with

$$\tilde{u}_\delta^\alpha = \sum_{j=1}^{N} \{V_j p_j^\alpha g_\delta(\mathbf{x} - \mathbf{x}_i) - V_j\, p_j^\beta \frac{\partial}{\partial x^\beta} \frac{\partial G_\delta(\mathbf{x} - \mathbf{x}_j)}{\partial x^\alpha}\}. \tag{22}$$

Here $G_\delta = G * g_\delta$ is the solution of the Poisson equation

$$\Delta G_\delta = g_\delta \,. \tag{23}$$

These equations are obtained by substituting $\mathbf{p}(\mathbf{x}, t)$ by $\mathbf{p}_{N,\delta}(\mathbf{x}, t)$ and $G(\mathbf{x}, t)$ by $G_\delta(\mathbf{x}, t)$ in (13), (14) and (12). These are the Hamilton equations corresponding to the Hamiltonian of a system with $3N$ degrees of freedom:

$$\mathcal{H}_F(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N) = \frac{1}{2}\sum_{j=1}^{N} V_j \mathbf{p}_j \cdot \tilde{\mathbf{u}}_\delta(\mathbf{x}_j)\,. \tag{24}$$

Because of the incompressibility of the fluid, the volume V_i is constant in time. Typically (but not necessarily) at initial time $t = 0$, the points are located on a regular grid of size h. In this case $V_i = h^3$, $\mathbf{x}_i(0) = \xi_i$ and $\mathbf{p}_i(0) = \mathbf{P}(\xi_i, 0)$.

In analogy to the continuous case the kinetic energy, the total impulse and the angular momentum can be defined for the discrete system as

$$E_{k_N} = \frac{1}{2}\sum_{i=1}^{N} V_i\, \mathbf{p}_i \cdot \tilde{\mathbf{u}}_\delta(\mathbf{x}_i)\,, \tag{25}$$

$$I_N = \frac{1}{2}\sum_{i=1}^{N} V_i \mathbf{p}_i \tag{26}$$

$$\Omega_N = \sum_{i=1}^{N} V_i \mathbf{x}_i \times \mathbf{p}_i\,. \tag{27}$$

It can be shown [3] that these quantities are invariants of motion for system (20), (21). The equations of motion have been derived for a fluid in $\mathbf{R}^3$. Similar equations hold in $\mathbf{R}^2$. In this case, of course, it is $g_\delta(\mathbf{x}) = g(\mathbf{x}/\delta)/\delta^2$ and $G(\mathbf{x}) = \log(|\mathbf{x}|)/(2\pi)$.

We consider radial blobs: $g(\mathbf{x}) = \tilde{g}(r)$, $r = |\mathbf{x}|$.

This method has some analogies to the *vortex-blob* methods. A discussion of the latter, together with some properties of the blob functions, is discussed, for example, in [1].

2.1 Positivity of the blob

In this section we discuss some features concerning the stability of the velicity blob method, and in particular its connection to the positivity of the Fourier transform of the blob.

Let $g_\delta(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$, denote the blob function and $\hat{g}_\delta(\zeta)$ its Fourier transform. We remind that the following property hold

$$\hat{g}_\delta(\zeta) \geq 0 \iff \mathcal{E} = \int \mathbf{u}(\mathbf{x}) g_\delta(\mathbf{x} - \mathbf{y}) \mathbf{u}(\mathbf{y}) d\mathbf{x} d\mathbf{y} \;\geq 0,\; \forall\, \mathbf{u} \in L_2(\mathbf{R}^n). \tag{28}$$

In fact it is

$$\mathcal{E} = \int \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}_\delta(\mathbf{x})\, d\mathbf{x} = \int \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}_\delta(\mathbf{x} + \mathbf{z})\, d\mathbf{x}\,|_{\mathbf{z}=\mathbf{0}} = \mathcal{F}(\mathbf{0}), \tag{29}$$

where $\mathcal{F}(\mathbf{z}) = \mathbf{u} \star \mathbf{u}_\delta\,(\mathbf{z}) = \int \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}_\delta(\mathbf{x} + \mathbf{z})\, d\mathbf{x}$ is the correlation function of $\mathbf{u}(\mathbf{x})$ and $\mathbf{u}_\delta(\mathbf{x})$. The Fourier transform of $\mathcal{F}(\mathbf{z})$ is given by

$$\hat{\mathcal{F}}(\zeta) = |\hat{\mathbf{u}}(\zeta)|^2 \hat{g}_\delta(\zeta) \tag{30}$$

and therefore $\mathcal{E} = \mathcal{F}(\mathbf{0}) = \int |\hat{\mathbf{u}}(\zeta)|^2\, \hat{g}_\delta(\zeta) d\zeta$. From the arbitrariness of $\mathbf{u}$ the equivalence follows.

The property of the positivity of the blob seems to be crucial to prove the convergence of the velicity-blob method [2]. A singular behavior is observed in the numerical solution of some partiular initial value problem, when a blob with non positive Fourier transform is is used. This is the case observed in [7], when the following blob has been used for two dimensional calculations:

$$g(r) = \begin{cases} \dfrac{7}{2\pi}(1 - 10r^3 + 15r^4 - 6r^5), & \text{if } r \le 1; \\ 0, & \text{if } r > 1, \end{cases}$$

where $r^2 = x^2 + y^2$. This blob is $C^3(R^2)$, infact the parameters it is choosen in such a way that it is positive, and its derivatives (up to third order) vanish in $r = 0$ and $r = 1$. Regular blobs with positive Fourier transform in 2D and 3D are easy to find. An example is a Gaussian blob,

$$g(r) = \frac{1}{(2\pi)^{d/2}} \exp(-r^2/2)$$

where $d = 2, 3$ denotes the space dimension. For computational reasons, it is often desirable to use a blob with compact support, which is computationally more effifient than a gaussian blob. Here we show, with a specific example, that it is possible to use blobs with compact support, which have a positive Fourier transform. Firts we remark that if a blob is radially simmetric, also its Forier transform is radially symmetric. More precisely, one has:

2D case Given a radially symmetric function $f(x) = F(|x|)$ in 2D, its Fourier transform is given by:

$$\begin{aligned} \hat{f}(\xi) &= \int_{R^2} f(x) \exp(-ix \cdot \xi)\, dx && (31) \\ &= \int_0^\infty dr \int_0^{2\pi} d\theta\, F(r) \exp(-ir|\xi| \cos\theta) && (32) \\ &= 2\pi \int_0^\infty dr\, rF(r) J_0(r|\xi|) = \hat{F}(|\xi|) && (33) \end{aligned}$$

where $J_0(x)$ denotes the Bessel function of order 0.

3D case In the three dimensional case one has:

$$\begin{aligned} \hat{f}(\xi) &= \int_{R^3} f(x) \exp(-ix \cdot \xi)\, dx && (34) \\ &= 2\pi \int_0^\infty dr \int_0^{pi} d\theta\, rF(r) \exp(-ir|\xi| \cos\theta) \sin\theta && (35) \\ &= 2\pi \int_0^\infty dr\, rF(r) \frac{\sin(r|\xi|)}{|\xi|} && (36) \end{aligned}$$

Let us consider now the specific blob function

$$g(r) = \begin{cases} c_d(5(1-r)^4 - 4(1-r)^5)), & \text{if } r \le 1; \\ 0, & \text{if } r > 1, \end{cases}$$

where $d = 2, 3$, $c_2 = 7/\pi$, and $c_3 = 21/(2\pi)$. These blobs are only $C^1(R^d)$, but their Fourier transform is positive, as it can be checked by direct calculation. In three dimension, an explicit calculation shows that

$$G(\zeta) \equiv \frac{|\xi|^8 \hat{g}(\xi)}{2520} = \zeta^2(4 - \cos\zeta) - 24(1 - \cos\zeta) + 9\zeta\sin\zeta$$

where $\zeta = |\xi|$. It is easy to check that $G(\zeta) > 0 \forall \zeta > 0$, and that $G(\zeta) = O(\zeta^8$ as $\zeta \to 0$.

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