Central schemes for conservation laws with application to shallow water equations

Giovanni Russo[1]

Department of Mathematics and Computer Science
University of Catania
Viale A. Doria 6
95125 Catania
Italy
russo@dmi.unict.it
http://www.dmi.unict/~russo

Summary. An overview is given of finite volume central schemes for the numerical solution of systems of conservation and balance laws. Well balanced central schemes on staggered grid for the Saint-Venant model or river flow are considered. A scheme which is well balanced for channels with variable cross section is introduced. Finally, a scheme which preserves non static equilibria is introduced, and some numerical results are presented.

1 Introduction

The numerical solution of hyperbolic systems of conservation laws has been a challenging and fascinating field of research for the last decades.

The solution of conservation laws may develop jump discontinuities in finite time, and the uniqueness of the (weak) solution is guaranteed only by recurring to some additional selection rule, such as entropy condition (see, for example, [Daf00] for a recent account of the theory of hyperbolic systems

of conservation laws). The dissipation mechanism of a quasilinear hyperbolic system is concentrated at the shocks, and its effect can be described in terms of the balance laws and entropy condition.

The schemes more commonly used in this context are the so called *shock capturing schemes*. At variance with *front tracking methods*, such schemes solve the field equations on a fixed grid, and the shocks are identified by the regions with large gradients. Among shock capturing schemes, the most commonly used are finite volume schemes, in which the basic unknowns represent the cell average of the unknown field. In finite difference schemes, the basic unknown represents the pointwise value of the field at the grid node.

The necessity of high accuracy and sharp resolution of the discontinuities encouraged the development of high order schemes for conservation laws.

Most modern high order shock capturing schemes are written in conservation form (in this way the conservation properties of the system are automatically satisfied), and are based on two main ingredients: the numerical flux function, and the non-oscillatory reconstruction. High order accuracy in the smooth regions, sharp resolution of discontinuities, and absence of spurious oscillations near shocks strongly depend on the characteristics of these two essential features (see for example the books by [LeV92], [GR96] or the lecture notes in [CIME00]).

Among finite volume methods, we distinguish between semidiscrete and fully discrete schemes. The first ones are obtained by integrating the conservation law in a spatial cell, by using a numerical flux function at the edge of the cell, and by providing a suitable reconstruction of the field at the two sides of each edge of the cell in terms of the cell averages. In this way one obtains a set of ordinary differential equations that can be then solved by some ODE solver such as Runge-Kutta. High order semidiscrete schemes are described, for example, in the chapter 4 of [CIME00] by Shu.

Alternatively, fully discrete schemes are obtained by integrating the conservation law on a cell in space-time. The flux function appearing in the

scheme will be consistent, to the prescribed order of accuracy, with the time average of the flux at the edge of the cell in one time step. A second order fully discrete method can be obtained, for example, by combining second order Lax-Wendroff method with a first order method by suitable *flux limiter*, which prevents formation of spurious oscillations (see [LeV92] for some examples).

Another distinction can be made between upwind and central schemes. Roughly speaking, we say that a scheme is *upwind* if it makes extensive use of the characteristic information of the system, so that the scheme can take into account the direction of propagation of the signal, while a scheme is *central* if characteristic information is not used. The prototype of upwind schemes is first order upwind, or its version for quasilinear systems, which is first order Godunov method, based on the solution of the Riemann problem at cell edges. The prototype of central scheme is first order Lax-Friedrichs scheme, which requires neither Riemann solvers nor characteristic decomposition.

Generally speaking, upwind-based methods guarantee sharper resolution than central schemes for the same order of accuracy and grid spacing, but are usually more expensive, and more complicated to implement. For this reason, central schemes have attracted a lot of attention in the last fifteen years. Following the original work of Nessyahu and Tadmor [NT90], where a second order, shock capturing, finite volume central scheme on a staggered grid in space-time was introduced and analyzed, several extensions and generalizations have been performed about central schemes, both fully discrete and semidiscrete (see [Rus02] and references therein for a review on central schemes).

The distinction between the *upwind* and the *central* world is not sharp, and in fact some characteristic information can be used to improve the performance of central schemes. For example, by using different estimates of the negative and positive characteristic speeds, Kurganov et al. [KNP01] improved the original semidiscrete central scheme [KT00]. The latter, in turn, is related to the finite volume schemes used by Shu, when a Local Lax-Friedrics

flux (also called Rusanov flux) is used ([CIME00], chapter 4). Qiu and Shu [QS02] showed that by using a reconstruction in characteristic variables for the computation of staggered cell average in central schemes, one eliminates the spurious oscillations produced by high order central schemes for the integration of Euler equation of gas dynamics near discontinuities.

Although semidiscrete schemes are very attractive because of their flexibility and simplicity of use, fully discrete schemes provide sometimes better performance with the same grid spacing. Because of this reason, it is of interest to consider the use of fully discrete central schemes for systems with source term.

High order central schemes on staggered grid for conservation laws have been derived. See, for example, [LPR] and [PPR] for recent results in this field, and [QS02] for a comparison between semidiscrete and fully discrete high order central schemes.

When a source is introduced in the system (i.e. when dealing with a quasilinear system of *balance laws*) then several new interesting problems arise in extending shock capturing schemes for conservation laws to this new case.

Straightforward extensions can be obtained by integrating the source term in space (for semidiscrete schemes) and in space and time (for fully discrete schemes) and using a suitable quadrature formula to compute the contribution of the source.

Another general technique that is commonly used for systems with source is based on the fractional step (also called time-splitting) method.

Both approaches, however, perform poorly in two cases, which require a more detailed "ad hoc" treatment.

One case concerns the problem of hyperbolic systems with stiff source. Here the source has to be treated by an implicit scheme, to avoid an excessive restriction on the time step due to the small characteristic time of the source term. The flux, on the other hand, is in general not stiff, and an explicit scheme is certainly more convenient because the nonlinearity of the space

discretization (mainly due to the non linear reconstruction) makes an implicit treatment excessively too expensive.

Implicit-Explicit (IMEX) time discretization are a natural choice for this kind of problems. See, for example, [LRR00] for a second order IMEX scheme based on central discretization in space, and [KC03] for a review on recently developed IMEX schemes.

Another important problem consists in the integration of systems in which the source term is nearly balanced by flux gradients. In this case the solution is a small perturbation of a stationary one. For this problem it would be desirable to construct numerical schemes that maintain the stationary solutions at a discrete level.

Such schemes are often called *well balanced*, after the paper by Greenberg and Le Roux [GLR96], and their development and analysis has interested many researchers in the last decade [Go96], [LeV98], [GHS03], [Jin01], [KL02], [BPV03], although some ideas based on characteristic decomposition where introduced earlier (see [Roe91] for linear problems and [BV94] for the extension to quasilinear problems). See also [To01] for a detailed explanation of various numerical methods for shallow water equations.

Most well-balanced schemes have been derived either for semidiscrete or for fully discrete schemes on a non staggered grid. However, staggered central schemes are attractive since they have an automatic mechanism for controlling spurious oscillations. In many cases, they allow better resolution than the non staggered counterpart. It is therefore attractive to explore the possibility of constructing well-balanced schemes on a staggered grid.

An example of well-balanced central scheme on a staggered grid was presented at the HYP2000 conference, and is briefly described in [Rus00]. A well balanced second order central scheme for the Saint-Venant equations that preserves static equilibria was derived.

Here we extend the result by presenting a well-balanced central scheme on a staggered grid that works also for channels of variable cross section, and a scheme that preserves non static equilibria (in the case of subcritical flow).

The rest of the section is devoted to a brief review of hyperbolic systems of conservation laws and conservative schemes for their numerical approximation.

The next section is a review of shock capturing central schemes for balance laws. Section 3 presents central schemes that preserve static solutions, with application to the Saint-Venant model of shallow water. Section 4 is quite techincal, and describes in detail the derivation of a central scheme for the Saint-Venant equations that preserves stationary, non static, equilibria. Some applications of the schemes are illustrated in Sections 3 and 4.

1.1 Hyperbolic systems

Let us consider a hyperbolic system of balance laws. It takes the form

$$\frac{\partial u}{\partial t} + \frac{Df(x,u)}{Dx} = R(x,u), \tag{1}$$

where $u \in R^m$; $f, R : \mathbb{R}^m \to \mathbb{R}^m$, $A = \partial f/\partial u$ has real eigenvalues and a basis of eigenvectors. Here $Df/Dx = \partial f/\partial x + Au_x$.

Such system may develop discontinuities in finite time (shocks) and therefore one has to abandon the hope of finding regular solutions (strong solutions), and one looks for weak solutions.

An admissible discontinuity that propagates in the media has to satisfy the so called jump conditions, which can be derived directly from the balance law, written in the original integral form. Such conditions, also called *Rankine-Hugoniot conditions* can be written in the form

$$-V_\Sigma [\![u]\!] + [\![f]\!] = 0, \tag{2}$$

where V_Σ denotes the speed of the moving discontinuity Σ, and for any function $h(x,t)$, $[\![h]\!] \equiv (h^+ - h^-)$ denotes the jump of the quantity across the discontinuity Σ.

A function u that satisfies Eq. (1) in the regions of regularity and conditions Eq. (2) at discontinuities is a weak solution of the balance equation. However, uniqueness is not guaranteed for such solution. In order to restore uniqueness of the solution, one has to resort to additional selection rules. Entropy condition, for example, is often used to select the unique solution of a conservation law. For a review of modern theory of hyperbolic systems of conservation laws see, for example, the book by Dafermos [Daf00].

Conservation form, jump conditions, and entropy conditions are used as guidelines in the development of modern shock capturing schemes, in order to guarantee that the numerical solution of the scheme converges to the unique entropic solution when the grid is refined.

1.2 Numerical schemes

A numerical scheme for balance laws has to admit the possibility of capturing discontinuous solutions, providing a correct shock speed. For a system of conservation laws (i.e. in the case of zero source term) it is desirable that a numerical scheme maintains the conservation properties of the exact solution of the system. For both purposes it is essential that the scheme is written in conservation form.

The so called *shock captuting finite volume schemes* are usually derived by integrating the system of balance laws on a suitable region of space-time.

Let us divide space into equally spaced cells $I_j \equiv [x_{j-1/2}, x_{j+1/2}]$ of size Δx, centered at x_j, $j \in \mathbb{Z}$.

Semidiscrete schemes are obtained by integrating the conservation law (1) in each cell in space, and approximating the flux function at the boarder of the cell by a suitable *numerical flux function* that depends on the values of the field across the edge of the cell. These values, in turn, are obtained by a suitable reconstruction of the function from the cell averages. In this way, one obtains a system of ordinary differential equations for the cell averages, of the form

$$\frac{d\bar{u}_j}{dt} = -\frac{F_{j+1/2} - F_{j-1/2}}{\Delta x} + \bar{R}_j,$$

where $\bar{u}_j$ is an approximation of the cell average

$$\bar{u}_j \approx \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)\, dx,$$

and

$$F_{j+1/2} = F(u^-_{j+1/2}, u^+_{j+1/2})$$

is the *numerical flux function*, $u^{\pm}_{j+1/2}$ are the reconstructed values of the field across the edge $x_{j+1/2}$.

For an account of modern semidiscrete high order schemes for conservation laws, see for example, the chapter by Shu in [CIME00].

Fully discrete schemes are obtained by integrating the conservation law on a suitable cell in space-time. Integrating the equation on a cell $I_j \times [t^n, t^{n+1}]$ one typically obtains a scheme of the form

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}(F_{j+1/2} - F_{j-1/2}) + \Delta t\, \bar{R}^{n+1/2}_j$$

where $F_{j+1/2}$ is the so called numerical flux function, which is consistent, to the prescribed order of accuracy, with the time average of the flux f at the edge $x_{j+1/2}$ of the cell, and $\bar{R}^{n+1/2}_j$ is an approximation of the space-time cell average of the source.

The piecewise constant solution $\tilde{U}^n(x) = \sum_j \chi_j(x) U^n_j$, $\chi_j(x)$ being the characteristic function of interval I_j, satisfies discrete jump conditions, and if it converges to a function $U(x,t)$ as $\Delta x \to 0$, then $U(x,t)$ is a weak solution of (1) (Lax-Wendroff theorem, see [RM67]). A nice description of a fully discrete schemes of this form is presented, for example, in [LeV92].

2 Staggered central schemes for balance laws

A different family of schemes is obtained by integrating the conservation laws on a staggered grid in space-time (see Fig.1)

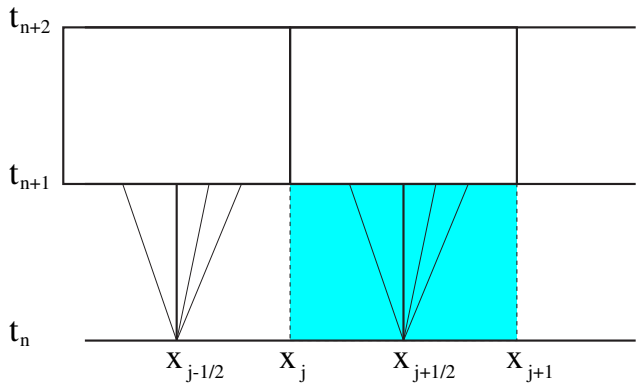

Fig. 1. Staggered grid in space-time and central scheme

After integration one obtains

$$\Delta x\, \bar{u}^{n+1}_{j+1/2} = \int_{x_j}^{x_{j+1}} u(x,t^n)\,dx - \int_{t^n}^{t^{n+1}} (f(u_{j+1}(t)) - f(u_j(t)))\,dt + \int_{x_j}^{x_{j+1}} dx \int_{t^n}^{t^{n+1}} dt\, R(x,u(x,t)),$$

with $u_j \equiv u(x_j,t)$. The above formula is exact for piecewise smooth solutions. To convert the formula in a numerical scheme one has to

(i) reconstruct $u(x,t^n)$ from $\bar{u}^n_j$ and use it to compute $\bar{u}^n_{j+1/2}$,

(ii) approximate integrals in time by a quadrature formula,

(iii) compute approximation of $u_j(t)$ on the quadrature nodes,

(iv) approximate the integral of the source on the cell in space-time by a quadrature formula.

The celebrated second order Nessyahu-Tadmor (NT) scheme [NT90], is obtained (for $R = 0$) by approximating:

(i) $u(x,t^n)$ by a piecewise linear function,

(ii) integrating the flux by midpoint rule,

(iii) using first order Taylor expansion for the computation of $u(x_j, t^n + \Delta t/2)$.

Its generalization to a system of balance laws (with no explicit dependence of flux and source on x) can be written as a simple two line predictor-corrector scheme

$$\begin{aligned} u_j^{n+1/2} &= u_j^n - \tfrac{\lambda}{2} f'_j + \tfrac{\Delta t}{2} R(u_j^{n+\beta}) && \text{predictor} \\ u_{j+1/2}^{n+1} &= \tfrac{1}{2}(u_j^n + u_{j+1}^n) + \tfrac{1}{8}(u'_j - u'_{j+1}) - \lambda\big(f(u_{j+1}^{n+1/2}) \\ &\quad - f(u_j^{n+1/2})\big) + \tfrac{\Delta t}{2}(R(u_j^{n+\beta}) + R(u_{j+1}^{n+\beta}) && \text{corrector} \end{aligned} \tag{3}$$

where $\lambda = \Delta t/\Delta x$ denotes the mesh ratio, and $u'_j/\Delta x$, $f'_j/\Delta x$ are first order approximation of space derivatives.

NT scheme can be made (discretely) entropic and Total Variation Diminishing (TVD). These properties depend on the reconstruction of the derivatives. In order to avoid spurious oscillations, suitable *slope limiters* for u' and f' are required. The simplest choice is given by the so called *MinMod* limiter, which is defined as

$$\mathrm{MinMod}(a,b) = \begin{cases} \mathrm{sign}(a)\min(|a|,|b|) & \text{if } ab > 0 \\ 0 & \text{if } ab \le 0 \end{cases}$$

Therefore u'_j can be computed as

$$u'_j = \mathrm{MinMod}(u_{j+1} - u_j, u_j - u_{j-1}),$$

and f'_j can be computed either using the minmod function or by $f'_j = A(u_j)u'_j$.

Better slope limiters (e.g. Harten's UNO limiter) can be used. For an account of different slope limiters see, for example, [LeV92] or [NT90].

Note that the contribution of the source term can be completely explicit ($\beta = 0$) or implicit ($\beta = 1/2$). Both cases will result in a second order scheme in space and time. The time restriction due to the flux term, in absence of

source, is the Courant-Friedrichs-Lewy (CFL) condition, which for the NT scheme reads

$$\lambda C_{\max} \leq \frac{1}{2}. \tag{4}$$

where $C_{\max}$ is the maximum spectral radius of the Jacobian matrix A on the computational domain.

High order central schemes for conservation laws ($R \equiv 0$) are obtained by using high order non-oscillatory reconstruction, such as WENO, and higher order time evolution, such as Runge-Kutta schemes with Natural Continuous Extension (see [LPR]) or Central Runge-Kutta [PPR].

The time step restriction due to the source term depends on the use of explicit ($\beta = 0$) or implicit ($\beta = 1/2$) predictor, and on the stiffness of the source, i.e. on its relaxation time. If the restriction introduced by explicit treatment of the source is more severe than the CFL condition (4), then it is preferable to use an implicit discretization of the predictor step. Note that the above scheme with $\beta = 1/2$ is a simple example of Implicit-Explicit (IMEX) time discretization, obtained by coupling explicit RK2 scheme (modified Euler scheme) with an A-stable second order scheme (midpoint method). In the case of very stiff relaxation terms, the above scheme is not suitable, and an L-stable scheme is needed for a proper treatment of the source. An example of IMEX central scheme for hyperbolic systems with stiff sources is presented in [LRR00]. Other examples of IMEX schemes applied to relaxation systems are shown in [PR04] and references therein.

We remark here that finite volume schemes are not suitable for high order approximation of hyperbolic systems with stiff source, because the averaging of the source couples all the cells, making implicit schemes expensive. Finite difference schemes are more natural in this case, because the pointwise value of the function rather than its cell average is used as basic unkwnon, and therefore the cells are decoupled (at the level of the source term). See [CIME00], Chap. 4, for illustration of high order finite difference schemes in conservation form.

3 A well-balanced scheme that preserves static equilibria

In this section we develop a well-balanced central scheme for a system of balance laws, with particular application to the Saint-Venant equations of shallow water.

Consider a problem in which the solution of the following system

$$\frac{\partial u}{\partial t} + \frac{Df(x,u)}{Dx} = R(x,u)$$

is a small deviation from the stationary solution $\tilde{u}(x)$, for which

$$\frac{\partial f(\tilde{u})}{\partial x} = R(\tilde{u}). \tag{5}$$

In this case, fractional step schemes or a scheme of the form Eq. (3) perform poorly, because they will not preserve equilibria (5), even at a discrete level.

We consider here the specific case of the Saint Venant model of shallow water equations. Let us start with the one dimensional equations in a channel of constant cross section. The equations can be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{6}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{1}{2}gh^2\right) = -ghB_x, \tag{7}$$

where $h(x,t)$ denotes the water depth, $q(x,t)$ is the water flux, $B(x)$ denotes

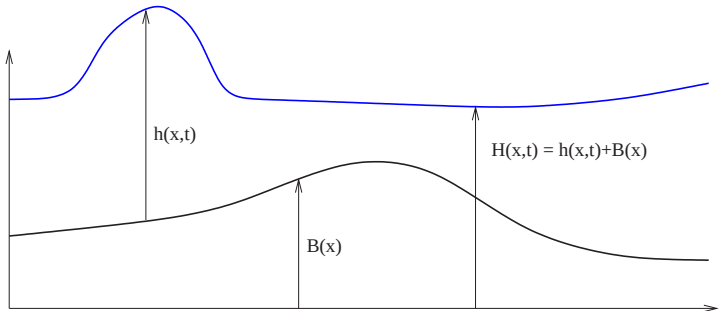

Fig. 2. Water height and bottom profile of the Saint-Venant model of shallow water

the bottom profile, and g is the constant gravity acceleration.

Several well-balanced schemes have been developed in the literature, which satisfy different requirements. We mention here the works by Greenbern and Le Roux [GLR96], Gosse [Go96], LeVeque [LeV98], Perthame et al. [BPV03], Jin [Jin01], Kurganov and Levy [KL02], Bouchut et al. [ABBKP04], Gallouet et al. [GHS03], just to mention a few names.

Usual requirement for a well-balanbced scheme is the preservation of static equilibria, which means a stationary solution of Eq. (5), for which the fluid does not flow, i.e. $q = 0$.

In addition, other common requirements are:

i) preservation of all equilibria,

ii) preservation of non-negativity of h,

iii) capability of treating dry zones (i.e. zones for which $h = 0$),

iv) numerical entropy condition.

A recent scheme derived by Bouchut and collaborators [ABBKP04] is able to fulfill all these requirements.

Let us consider, as a test, the initial condition

$$q(x,0) = q_0, \quad h(x,0) = \begin{cases} 1.01 \text{ if } |x - 0.2| < 0.05 \\ 1 \quad \text{otherwise} \end{cases}, \tag{8}$$

with $q_0 = 0$, and let the bottom profile be given by

$$B(x) = \begin{cases} 1 + \cos(10x - 5) \text{ if } |x - 0.5| < 0.1 \\ 0 \qquad\qquad\qquad\quad \text{otherwise} \end{cases}. \tag{9}$$

If we use scheme (3) (with $\beta = 0$) then we obtain the solution shown in Fig.(3), where the total height $H = h + B$ is reported at initial time (dashed line) and final time $t = 0.7$. In all calculations we set the gravity constant g to 1. The number of grid points used in the calculation is $N = 200$. The dashed line represents the initial condition.

Flat boundary conditions have been used here and for all calculations presented in the paper.

The spurious effect present in the center of the computational domain is due to the fact that the scheme is not able to preserve stationary solutions of Eqs.(6,7).

How can we construct well balanced schemes that preserve equilibria? Several approaches have been considered in the literature.

The paper by Kurganov and Levy [KL02], for example, provides an example of a semidiscrete central scheme that preserves static equilibria. The scheme they present, however, can not be straightforwardly generalized in the case of staggered grids, because NT scheme is not able to preserve solutions of the equation

$$\frac{\partial u}{\partial t} = 0,$$

unless $u \equiv$ constant.

Taking this into account, a well balanced central scheme that preserves static solutions has been derived and presented at the hyperbolic conference in Magdeburg [Rus00].

The guidelines in the development of such scheme are:

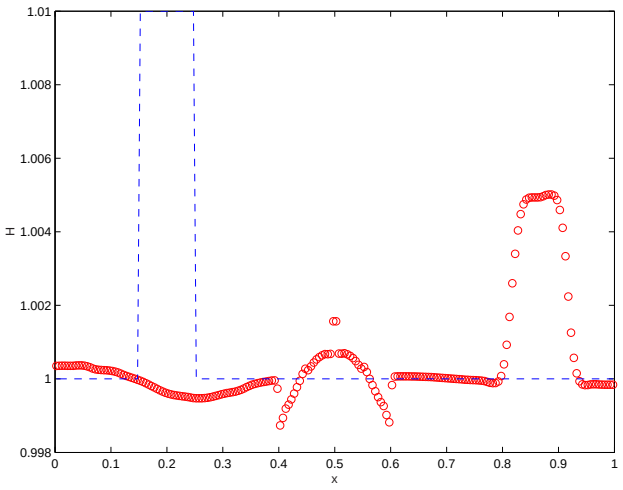

Fig. 3. Numerical solution of system (6, 7), with the use of scheme (3), with $\beta = 0$). 200 grid points.

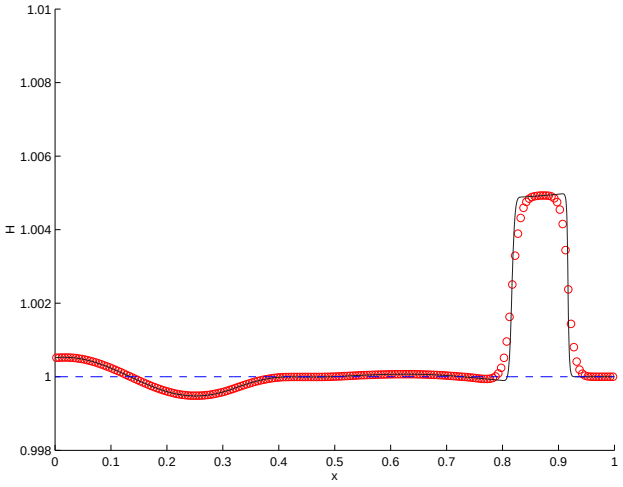

Fig. 4. Numerical solution of system (6,7), with the well-balanced central scheme (10-11). The thin line represents the reference solution and is obtained by the well-balanced scheme with $N = 1600$ gridpoints.

1) reformulate the problem using $H = h + B$ as conserved variable
2) compute predictor by non conservative form using $f' = \delta f + A(u)u'$, where $\delta f \approx \Delta x\, \partial f/\partial x$,
3) in the corrector use suitable approximation of functions and space derivatives.

The last requirements are obtained for example, by setting (at even time steps)

$$B_j = \frac{1}{2}\left(B(x_j + \Delta x/2) + B(x_j - \Delta x/2),\right)$$
$$\frac{B'_j}{\Delta x} = \frac{B(x_j + \Delta x/2) - B(x_j - \Delta x/2)}{\Delta x}.$$

The scheme applied to SV equations takes the form

predictor:

$$H_j^{n+1/2} = H_j^n - \frac{\lambda}{2} q'_j,$$

$$q_j^{n+1/2} = q_j^n - \frac{\lambda}{2}, \left(2v_j^n q_j' - (v_j^n)^2(H_j' - B_j') + gH_j'(H_j - B_j)\right) \tag{10}$$

corrector:

$$\begin{aligned} H_{j+1/2}^{n+1/2} &= H_{j+1/2}^n - \lambda(q_{j+1}^{n+1/2} - q_j^{n+1/2}), \\ q_{j+1/2}^{n+1/2} &= q_{j+1/2}^n - \lambda(\psi_{j+1}^{n+1/2} - \psi_j^{n+1/2}) \\ &\quad - g\frac{\lambda}{2}(H_j^{n+1/2}B_j' + H_{j+1}^{n+1/2}B_{j+1}'), \end{aligned} \tag{11}$$

where the staggered cell averages $H_{j+1/2}^n$ and $q_{j+1/2}^n$ are computed as in scheme (3), $v \equiv q/(H-B)$, $\psi_j^{n+1/2} \equiv \psi(H_j^{n+1/2}, q_j^{n+1/2})$ and

$$\psi(H,q) \equiv \frac{q^2}{H-B} + \frac{1}{2}H(H-B)$$

It is easy to check that $H = \text{const}$, $q = 0$ is a solution for this scheme.

The scheme is applied to problem (6, 7, 8), and the numerical results are shown in Fig.3. Note that no spurious profile appears where $B \neq 0$.

Variable cross section

Saint Venant equations for a channel of variable cross section have the form

$$\begin{aligned} &\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \\ &\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial H}{\partial x} = 0, \end{aligned}$$

where $A(x,t)$ is the cross section of the part of channel occupied by the water, and Q is the flux. The above approach can be extended to the case of channel with rectangular cross section, $A(H,x) = hW(x)$. A well-balanced scheme is obtained as follows. Let $A = W(x)h(x,t) = W(x)(H(x,t) - B(x))$. Reformulate the problem using H and $q = Q/W$ as unknown conservative variable. This choice will not alter the jump conditions. Then the system becomes

$$\begin{aligned} &\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = -q\frac{W_x(x)}{W(x)}, \\ &\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{H - B(x)} + \frac{g}{2}H(H - 2B(x))\right) \\ &\qquad = -gHB_x(x) - \frac{q^2}{H-B(x)}\frac{W_x(x)}{W(x)}. \end{aligned} \tag{12}$$

The scheme used before preserves static solutions of this system, because the two additional terms vanish as $q = 0$.

As an application of the scheme we consider two test cases. The initial conditions are given by Eq. (8), and the bottom profile by Eq. (9).

The results corresponding to these two cases are reported in Fig. 5 The first corresponds to the choice

$$W(x) = 1 - B(x),$$

while in the second one it is

$$W(x) = \frac{1}{1 - B(x)}.$$

Notice that in the second case the area of the cross section of the channel corresponding to the static solution $H = 1$ is basically constant, and this reduces the amplitude of the reflected wave, while in the first case the cross sectional area at the center of the channel becomes narrower than the case of constant cross section, resulting in a larger reflected wave.

4 A well-balanced scheme for subcritical flows

The approach described before can not be directly applied to the case in which the stationary solution is not static. Even if the scheme preserves the equilibrium

$$\frac{Df}{Dx} = R$$

at a discrete level, Nessyahu-Tadmor scheme does not preserve a solution of the trivial equation $\partial u/\partial t = 0$ unless u is also constant in space.

One possibility to obtain a scheme that preserves stationary solutions is to perform a change of variables in such a way that in the new variables the equilibrium is represented by constants.

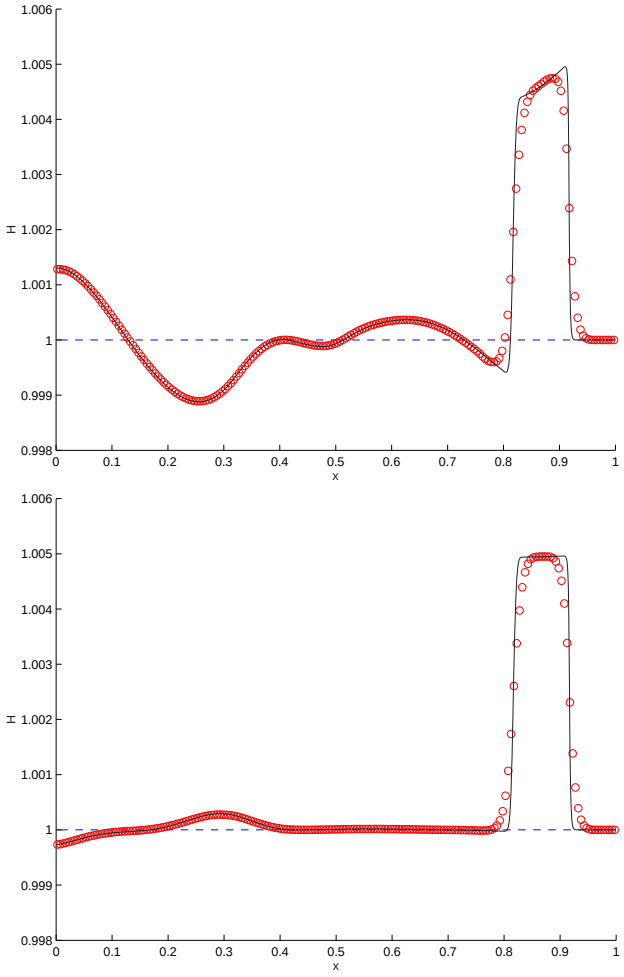

Fig. 5. Numerical solution of Saint-Venant equations with variable cross section, at time $t = 0.7$. The thin dotted line represents the reference solution obtained by 1600 points. Upper: $W(x) = 1 - B(x)$; lower: $W(x) = (1 - B(x))^{-1}$

Scalar equation

Let us consider the scalar case first. Let us denote by $\tilde{u}(x)$ a stationary solution, i.e.

$$\frac{Df(x,\tilde{u})}{Dx} = R(x,\tilde{u}). \tag{13}$$

Then we look for a solution of the form

$$u(x,t) = \tilde{u}(x)v(x,t).$$

The equation for u becomes

$$\tilde{u}(x)\frac{\partial v}{\partial t} + \frac{Df(x,\tilde{u}v)}{Dx} = R(x,\tilde{u}v).$$

Integrating on a staggered cell in space time one has

$$\begin{aligned}\Delta x\, \bar{u}^{n+1}_{j+1/2} = &\int_{x_j}^{x_{j+1}} \tilde{u}(x)v^{n+1}(x)\,dx = \int_{x_j}^{x_{j+1}} \tilde{u}(x)v(x,t^n)\,dx \\ &- \int_{t^n}^{t^{n+1}} (f(x_{j+1},\tilde{u}_{j+1}v_{j+1}(t)) - f(x_j,\tilde{u}_j v_j(t)))\,dt \\ &+ \int_{x_j}^{x_{j+1}} dx \int_{t^n}^{t^{n+1}} dt\, R(x,\tilde{u}(x)v(x,t)).\end{aligned} \tag{14}$$

A second order discretization of the conservation equations is obtained as follows. Let us define the quarter cell values of the equilibrium solution (which coincide with their average to second order)

$$\tilde{u}_{j\pm1/4} \equiv \tilde{u}(x_j \pm \Delta x/4) = \tilde{u}(x_{j\pm1/4}).$$

The cell average at time t^{n+1} (left hand side of Eq. (14)) is discretized as

$$\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \tilde{u}(x)v^{n+1}(x)\,dx \approx \frac{1}{2}(\tilde{u}_{j+1/4} + \tilde{u}_{j+3/4})\bar{v}^{n+1}_{j+1/2}.$$

The first term on the right hand side (staggered cell average) is discretized as follows. Assume $v^n(x)$ is approximated by a piecewise linear function

$$v^n(x) \approx \sum_j \chi_j(x)L_j(x),$$

where χ_j is the characteristic function of the $j-th$ interval, and

$$L_j(x) = v_j^n + v_j'(x - x_j)/\Delta x.$$

Here $v_j'/\Delta x$ denotes a first order approximation of the space derivative of $v^n(x)$, and v_j^n denotes an approximation of the pointwise value of $v^n(x)$ (which agrees with its cell average to second order).

In each cell one has

$$\bar{u}_j^n \approx \frac{1}{2}(\tilde{u}_{j-1/4} + \tilde{u}_{j+1/4})v_j^n.$$

Staggered cell average at time n are computed as

$$\begin{aligned}&\int_{x_j}^{x_{j+1}} \tilde{u}(x)v^n(x)\,dx = I_j^R + I_{j+1}^L \\ &= \int_{x_j}^{x_{j+1/2}} \tilde{u}(x)L_j(x)\,dx + \int_{x_{j+1/2}}^{x_{j+1}} \tilde{u}(x)L_{j+1}(x)\,dx.\end{aligned}$$

To second order accuracy, the integrals are evaluated as

$$\begin{aligned}I_j^R &= \Delta x\,\tilde{u}_{j+1/4}\left(\frac{1}{2}v_j^n + \frac{1}{8}v_j'\right), \\ I_j^L &= \Delta x\,\tilde{u}_{j-1/4}\left(\frac{1}{2}v_j^n - \frac{1}{8}v_j'\right),\end{aligned}$$

and therefore the staggered cell average is

$$\begin{aligned}\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \tilde{u}(x)v^n(x)\,dx \approx &\frac{1}{2}(\tilde{u}_{j+1/4}v_j^n + \tilde{u}_{j+3/4}v_{j+1}^n) \\ &+ \frac{1}{8}(\tilde{u}_{j+1/4}v_j' - \tilde{u}_{j+3/4}v_{j+1}').\end{aligned}$$

Remark

A better approximation of the staggered cell value can be obtained by using

$$\int_0^{h/2} \tilde{u}(x_j + \xi)\xi\,d\xi = \frac{1}{8}\tilde{u}_{x_j+h/3}h^2 + O(h^4),$$

however, this requires the storage of an additional value of the stationary solution, and it does not improve the overall order of accuracy.

The contribution of flux and source is computed by a predictor-corrector type scheme

$$\frac{1}{\Delta x}\int_0^{\Delta t} f(x_j, u_j(t^n+\tau)\,d\tau \approx f(x_j, \tilde{u}(x_j)v_j^{n+1/2})$$

Predictor step

Can be computed by a non conservative scheme

$$u_j^{n+1/2} = u_j^n - \frac{\lambda}{2}\left(\delta f_j + \frac{\partial f}{\partial u}u_j'\right) + \frac{\Delta t}{2}R(x_j, u_j^n)$$

where $\delta f_j/\Delta x$ denotes a first order approximation of the space derivative of f and

$$u_j' = \tilde{u}_j' v_j^n + u_j v_j'$$

Here

$$u_j' = \Delta x \left.\frac{du}{dx}\right|_{(x_j,u_j)}, \quad v_j' = \Delta x \left.\frac{\partial v}{\partial x}\right|_{(x_j,u_j)} + O(\Delta x^2), \quad \tilde{u}_j' = \Delta x \frac{d\tilde{u}}{dx}$$

Once the stationary equation 13 is solved for $\tilde{u}$, the quantity $\tilde{u}'$ is obtained from the differentiation of Eq. (13) as

$$A\tilde{u}' = \Delta x\left(R(x,\tilde{u}) - \frac{\partial f}{\partial x}\right).$$

Corrector

The contribution of the source term is obtained by a suitable quadrature formula

$$\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} R(x,u)\,dx\,dt \approx \Delta t\,\tilde{R}(\tilde{u}_j v_j^{n+1/2}, \tilde{u}_{j+1}v_{j+1}^{n+1/2}).$$

The formula has to be consistent with the well balanced property of the scheme, i.e.

$$\lambda(f(x_j,\tilde{u}_j) - f(x_{j+1},\tilde{u}_{j+1})) + \Delta t\,\tilde{R}(\tilde{u}_j,\tilde{u}_{j+1}) = 0;$$

in fact, this relation can be used to define the function $\tilde{R}$ in the numerical scheme.

The extension to system of equations is obtained by repeating the above steps component by component.

Let us apply this technique to the shallow water equations. We consider 1D shallow water equations in the form

$$\begin{aligned}&\frac{\partial H}{\partial t}+\frac{\partial q}{\partial x}=0\\&\frac{\partial q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q^2}{H-B(z)}+\frac{1}{2}gH(H-2B(x))\right)=-gHB_x.\end{aligned}$$

The stationary solution is $q=q_0$ and $H=\tilde{H}(x)$, obtained by solving the equation

$$\frac{\partial}{\partial x}\left(\frac{q_0^2}{\tilde{H}-B(z)}+\frac{1}{2}g\tilde{H}(\tilde{H}-2B(x))\right)=-g\tilde{H}B_x.$$

Integrating the equation one has

$$\frac{q_0^2}{2(\tilde{H}-B)^2}+g\tilde{H}=gH_0+\frac{q_0^2}{2(H_0-B_0)^2}.$$

Therefore $\tilde{H}$ is obtained as solution of a cubic equation.

If, in the whole channel, the stationary flow is subcritical, i.e. if

$$|u|<\sqrt{gh},$$

then the cubic equation has only one real solution. This is the case we consider.

The quantities $\tilde{H}_j$, $\tilde{H}_{j+1/2}$, $\tilde{H}_{j+1/4}$, $\tilde{H}_{j-1/4}$ are precomputed and stored at the beginning of the calculation.

Particular care has to be used in the approximation of the derivatives. Here we distinguish between predictor and corrector steps.

Predictor:

Use non conservative form of the equation. In particular

$$q_j^{n+1}=q_j^n-\frac{\lambda}{2}K,$$

with

$$K=2\frac{q}{h}q_j'-\left(\frac{q^2}{h^2}-gh\right)(H'-B')+gHB',$$

$$H' = v'\tilde{H} + v\tilde{H}'.$$

Here $h = H - B, \quad B' = B_x \Delta x, \tilde{H}' = \tilde{H}_x \Delta x$, and $\tilde{H}_x$ is computed exactly differentiating the equation for $\tilde{H}$.

Corrector:

The source has to be computed with a well-balanced formula. For example, for the flux one has:

$$\begin{aligned} q_{j+1/2}^{n+1} = q_{j+1/2}^{n} - \lambda(f_{j+1}^{n+1/2} - f_{j}^{n+1/2}) \\ + (\tilde{H}_j v_j^{n+1/2} + \tilde{H}_{j+1} v_{j+1}^{n+1/2}) S_{j+1/2}, \end{aligned}$$

with

$$S_{j+1/2} = \frac{f(\tilde{H}(x_j), q_0) - f(\tilde{H}(x_{j+1}), q_0)}{\tilde{H}_j + \tilde{H}_{j+1}}.$$

Remark

In the new scheme the derivative of the bottom are computed exactly, and not by finite difference, as in the old well balanced scheme (10,11).

4.1 Numerical results

We consider here a case where the solution is a small perturbation of a stationary non static equilibrium. The initial condition is given by Eq. (8), but with $q_0 = 0.17$. The numerical solution for the total height at two different times is reported in Fig.6, where only 50 grid points have been used. The main difference between the old well balanced scheme and the new one that preserves non static equilibria is noticeable near the middle of the channel, where the bottom is higher, and the water profile lowers. To enhance the effect, in Figure 4.1 we report the quantity

$$I(x,t) = \frac{q^2}{2(H-B)^2} + gH,$$

which is invariant at equilibrium. Notice the spurious effect near the center of the channel in the numerical result obtained by scheme (10,11). We remark,

however, that these effects are rather small, and quickly disappear as the grid is refined. Figure 4.1, for example, represents the same quantity with 200 grid points, and the effect is barely noticeable.

In all cases, the reference solution is obtained by the old well balanced scheme with 1600 points.

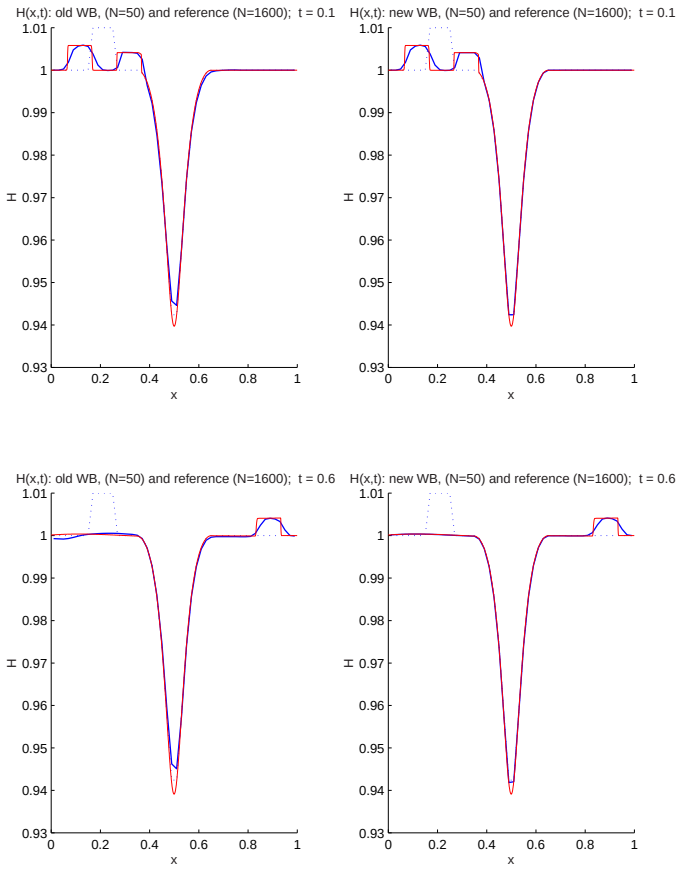

Fig. 6. Small perturbation of stationary, non static, equilibria. $H(x,t)$ at different times: $t = 0.1$ (top) and $t = 0.6$ (bottom). Well balanced central scheme that preserves static equilibria (left), and scheme that preserves non static equilibria (right). Number of grid points $N = 50$.

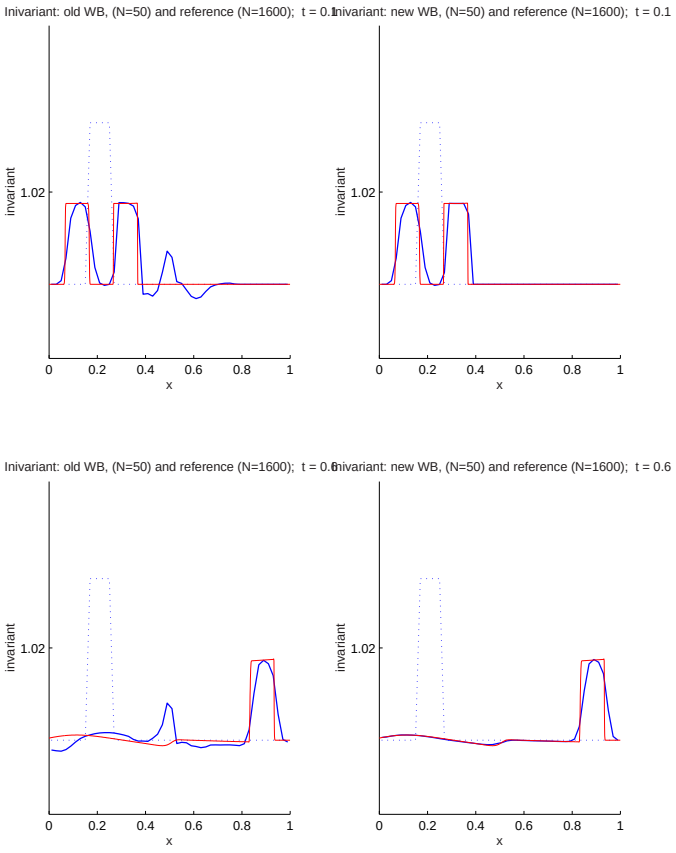

Fig. 7. Plot of the equilibrium invariant at different times. Time $t = 0.1$ (top) and $t = 0.6$ (bottom). Old well balanced scheme (left) and new well balanced scheme (right). Number of grid points $N = 50$.

5 Conclusions

Staggered central schemes can be used for

- systems with stiff source (implicit-explicit schemes)
- quasi-stationary flows (well balanced schemes)

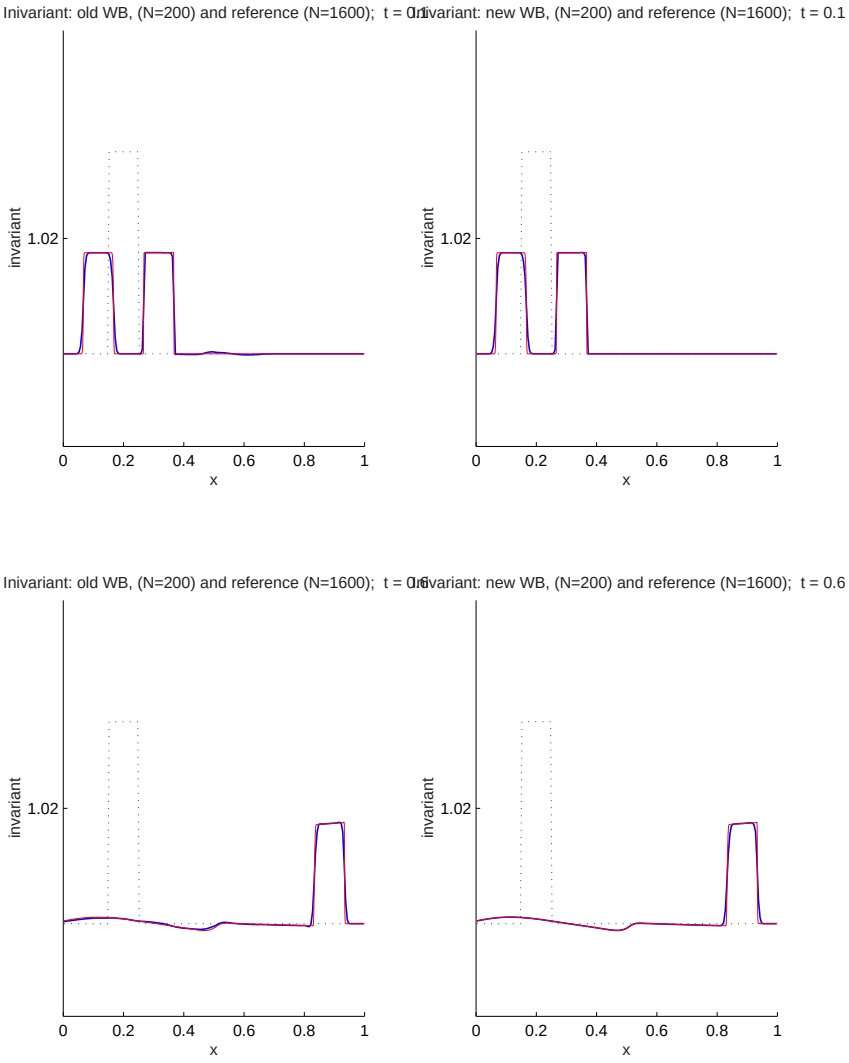

Fig. 8. Same as in the previous figure, but with $N = 200$ grid points

- accurate solutions (high order central schemes)

They may be more effective than non staggered central schemes in some cases (higher resolution with the same number of grid points)

High order schemes can be constructed for problems with stiff source (high order finite difference discretization + IMEX time discretization).

The construction of well-balanced central schemes on staggered grid that preserves static equilibria is possible, but requires the solution of the stationary problem.

Future work in this topic may include the construction of schemes that do not require this information. Second order schemes that preserve only static equilibria perform rather well even when the unperturbed solution is non static. The performance would improve with the increase of the accuracy of the scheme. An interesting problem would be the construction of a third order well-balanced scheme (on staggered or non staggered grid).

References

[Daf00] Dafermos, C.M.: Hyperbolic conservation laws in continuum physics, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 325, Springer-Verlag, Berlin (2000)

[LeV92] LeVeque, R.J.: Numerical Methods for Conservation Laws, Lectures in Mathematics, Birkhäuser Verlag, Basel (1992)

[GR96] Godlewski, E., Raviart, P.-A.: Numerical approximation of hyperbolic systems of conservation laws, Applied Mathematical Sciences, 118, Springer-Verlag, New York (1996)

[CIME00] Cockburn, B., Johnson, C., Shu, C.-W., Tadmor, E.: Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, Lecture Notes in Mathematics, 1697, Springer, Berlin (2000)

[NT90] Nessyahu, H., Tadmor, E.: Non-oscillatory Central Differencing for Hyperbolic Conservation Laws, J. Comput. Phys., **87**, 408–463 (1990)

[Rus02] Russo, G.: Central Schemes and Systems of Balance Laws, in "Hyperbolic Partial Differential Equations, Theory, Numerics and Applications", edited by Andreas Meister and Jens Struckmeier, Vieweg, Göttingen (2002)

[KNP01] Kurganov, A., Noelle, S., Petrova, G.: Semidiscrete central-upwind schemes for hyperbolic conservation laws and Hamilton-Jacobi equations, SIAM J. Sci. Comput., **23**, 707–740 (2001)

[KT00] Kurganov, A., Tadmor, E.: New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations, J. Comput. Phys., **160**, 241–282 (2000)

[QS02] Qiu, J., Shu, C.-W.: On the construction, comparison, and local characteristic decomposition for high-order central WENO schemes, J. Comput. Phys. **183**, 187–209 (2002)

[LPR] Levy, D., Puppo, G., Russo, G.: Central WENO schemes for hyperbolic systems of conservation laws, Math. Model. Numer. Anal. **33**, 547–571 (1999)

[PPR] Pareschi, L., Puppo, G., Russo, G.: Central Runge-Kutta Schemes for Conservation Laws, submitted for publication on SIAM J. Sci. Comput.

[LRR00] Liotta S.F., Romano, V., Russo, G.: Central schemes for balance laws of relaxation type, SIAM J. Numer. Anal., **38**, 1337–1356 (2000)

[KC03] Kennedy, C.A., Carpenter, M.H.: Additive Runge-Kutta schemes for convection-diffusion-reaction equations, Appl. Numer. Math. **44**, 139–181 (2003)

[GLR96] Greenberg, J. M.; Leroux, A.Y.: *A well-balanced scheme for the numerical processing of source terms in hyperbolic equations*, SIAM J. Numer. Anal., **33**, 1–16 (1996)

[Go96] Gosse, L.: A well-balanced scheme using non-conservative products designed for hyperbolic systems of conservation laws with source terms, Math. Models Methods Appl. Sci., **11**, 339–365 (2001)

[LeV98] LeVeque, R.J.: Balancing source terms and flux gradients in high-resolution Godunov methods: the quasi-steady wave-propagation algorithm, J. Comput. Phys., **146**, 346–365 (1998)

[GHS03] Gallouët, T., Hérard, J.-M., Seguin, N.: Some approximate Godunov schemes to compute shallow-water equations with topography, Comput. & Fluids, **32**, 479–513 (2003)

[Jin01] Jin, S.: A steady-state capturing method for hyperbolic systems with geometrical source terms, Math. Model. Numer. Anal., **35**, 631–645 (2001)

[KL02] Kurganov, A., Levy, D.: Central-upwind schemes for the Saint-Venant system, Math. Model. Numer. Anal., **36**, 397–425 (2002)

[BPV03] Botchorishvili, R., Perthame, B., Vasseur, A.: Equilibrium schemes for scalar conservation laws with stiff sources, Math. Comp., **72**, 131–157 (2003)

[Roe91] Roe, P.L.: Upwind differencing schemes for hyperbolic conservation laws with source terms, Nonlinear hyperbolic problems (St. Etienne, 1986), 41–51, Lecture Notes in Math., 1270, Springer, Berlin (1987)

[BV94] Bernudez, A., Vazquez, M.E.: Upwind methods for hyperbolic conservations laws with source terms, Computer Fluids, **23**, 1049-1071 (1994)

[To01] Toro, E.F.: Shock-Capturing Methods for Free-Surface Shallow Flwos, Wiley (2001)

[Rus00] Russo, G,: Central schemes for balance laws. Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000), 821–829, Internat. Ser. Numer. Math., **140**, **141**, Birkhuser, Basel (2001)

[RM67] Richtmyer, R.D., Morton, K.W.: Difference methods for initial-value problems, Second edition, John Wiley & Sons, Inc., New York-London-Sydney (1967)

[PR04] Pareschi, L., Russo, G.: Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation, submitted for publication on Journal of Computational Physics.

[ABBKP04] Audusse, E., Bouchut, F., Bristeau, M.-O., Klein, R., Perthame, B.: A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows, to appear in SIAM J. Sci. Comp., 2004.