A Nonlocal Formulation of Quantum Maximum Entropy Principle Including Fractional Exclusion Statistics

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Contents

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- Theoretical development
- Conclusions
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- We express the Quantum Entropy in terms of the reduced density matrix.
- We include "ab initio" Quantum Statistics Fermions–Bosons–Exclusons.

We develop a Quantum Maximum Entropy Principle (QMEP) for a proper non-local formulation of theory, by recovering the classic MEP when $\hbar \to 0$.

We determine a closed quantum hydrodynamic system (QHD) for the macroscopic variables used as constraints in the QMEP approach.
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Theoretical development

- We consider, in the **Fock space** $N$ identical particles
- The statistical operator **density matrix** $\rho$ with $\text{Tr}(\rho) = 1$.
- **Wave field operators** $\Psi(r) = \sum a_j \varphi_j(r)$, $\Psi^\dagger(r) = \sum a_j^\dagger \varphi_j^*(r)$
- The general Hamiltonian with many-body interactions $[\uparrow]$
  \[
  H = \int d^D r \, \Psi^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(r) \right] \Psi(r) + \\
  \sum_{k=2}^L \frac{1}{k!} \int d^D r_1 \cdots \int d^D r_k \, \Psi^\dagger(r_1) \cdots \Psi^\dagger(r_k) V_k(r_1, \ldots, r_k) \Psi(r_k) \cdots \Psi(r_1)
  \]
- $U(r)$ $\Rightarrow$ one-body external potential
- $V_k(r_1, \ldots, r_k)$ $\Rightarrow$ symmetric $k$-body interaction potential
- The **Reduced Density Matrix** of single particle
  \[
  \langle r \mid \hat{\rho} \mid r' \rangle = \langle \Psi^\dagger(r') \Psi(r) \rangle = \text{Tr} \left( \rho \Psi^\dagger(r') \Psi(r) \right) \quad \text{with} \quad \text{Tr}(\hat{\rho}) = N
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- The general Hamiltonian with **many-body interactions** $H = \int d^Dr \Psi^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(r) \right] \Psi(r) + \sum_{k=2}^L \frac{1}{k!} \int d^Dr_1 \cdots \int d^Dr_k \Psi^\dagger(r_1) \cdots \Psi^\dagger(r_k) V_k(r_1, \ldots, r_k) \Psi(r_k) \cdots \Psi(r_1)$
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  $\langle r | \hat{\varrho} | r' \rangle = \langle \psi^\dagger(r') \psi(r) \rangle = \text{Tr} \left( \rho \, \psi^\dagger(r') \psi(r) \right)$ with $\text{Tr}(\hat{\varrho}) = N$
**Reduced Wigner function** (Klimontovich [2] 1958)

\[
\mathcal{F}_W = \frac{1}{(2\pi \hbar)^D} \int d^D \tau e^{-\frac{i}{\hbar} \tau \cdot \mathbf{p}} \langle \Psi^\dagger (\mathbf{r} - \tau/2) \Psi (\mathbf{r} + \tau/2) \rangle
\]

- Weyl-Wigner transform and Inverse Weyl-Wigner transform

To \( \hat{A}(\hat{r}, \hat{p}) \) correspond \( \Leftrightarrow \) the Phase-Space Function \( \tilde{A}(\mathbf{r}, \mathbf{p}) \)

\[
\mathcal{W}(\hat{A}) = \tilde{A}(\mathbf{r}, \mathbf{p}) \quad \text{and} \quad \mathcal{W}^{-1}(\tilde{A}) = \langle \mathbf{r} | \hat{A} | \mathbf{r}' \rangle
\]

\( \tilde{\varrho}(\mathbf{r}, \mathbf{p}) = \mathcal{W}(\hat{\varrho}) = (2\pi \hbar)^D \mathcal{F}_W(\mathbf{r}, \mathbf{p}) \quad \langle \mathbf{r} | \hat{\varrho} | \mathbf{r}' \rangle = \mathcal{W}^{-1}(\tilde{\varrho}) \)

- Equation of Evolution in the Hartree approximation

\[
i\hbar \frac{\partial}{\partial t} \langle \mathbf{r} | \hat{\varrho} | \mathbf{r}' \rangle = \int d^D \mathbf{r}'' \left\{ \langle \mathbf{r} | \hat{H} | \mathbf{r}'' \rangle \langle \mathbf{r}'' | \hat{\varrho} | \mathbf{r}' \rangle - \langle \mathbf{r} | \hat{\varrho} | \mathbf{r}'' \rangle \langle \mathbf{r}'' | \hat{H} | \mathbf{r}' \rangle \right\}
\]

where \( \hat{H} = \langle \mathcal{H} \rangle \) is the single particle Hamiltonian, being [\dagger]

\[
\mathcal{H} = \left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) \right] + \\
\sum_{k=1}^{L-1} \frac{1}{k!} \int d^D r_1 \cdots \int d^D r_k \psi^\dagger(r_1) \cdots \psi^\dagger(r_k) V_k(r, r_1, \ldots, r_k) \psi(r_k) \cdots \psi(r_1)
\]
**Reduced Wigner function** (Klimontovich [2] 1958)

\[ \mathcal{F}_W = \frac{1}{(2\pi \hbar)^D} \int d^D \tau e^{\frac{i}{\hbar} \tau \cdot \mathbf{p}} \langle \Psi^\dagger (\mathbf{r} - \tau/2) \Psi (\mathbf{r} + \tau/2) \rangle \]

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**Equation of Evolution in the Hartree approximation**

\[ i\hbar \frac{\partial}{\partial t} \langle \mathbf{r} | \hat{\varrho} | \mathbf{r}' \rangle = \int d^D r'' \left\{ \langle \mathbf{r} | \hat{\mathcal{H}} | r'' \rangle \langle r'' | \hat{\varrho} | \mathbf{r}' \rangle - \langle \mathbf{r} | \hat{\varrho} | r'' \rangle \langle r'' | \hat{\mathcal{H}} | \mathbf{r}' \rangle \right\} \]

where \( \hat{\mathcal{H}} = \langle \mathcal{H} \rangle \) is the single particle Hamiltonian, being [1]

\[ \mathcal{H} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) \right] + \sum_{k=1}^{L-1} \frac{1}{k!} \int d^D r_1 \cdots \int d^D r_k \psi^\dagger(r_1) \cdots \psi^\dagger(r_k) V_k(r, r_1, \ldots, r_k) \psi(r_k) \cdots \psi(r_1) \]
Reduced Wigner function (Klimontovich [2] 1958)

\[ \mathcal{F}_\mathcal{W} = \frac{1}{(2\pi\hbar)^D} \int d^D \tau e^{-i \frac{\tau}{\hbar} \cdot \mathbf{p}} \langle \Psi^\dagger(r - \tau/2) \Psi(r + \tau/2) \rangle \]

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To \( \hat{A}(\mathbf{r}, \mathbf{p}) \) correspond \( \Leftrightarrow \) the Phase-Space Function \( \tilde{\mathcal{A}}(\mathbf{r}, \mathbf{p}) \)

\[ \mathcal{W}(\hat{A}) = \tilde{\mathcal{A}}(\mathbf{r}, \mathbf{p}) \quad \text{and} \quad \mathcal{W}^{-1}(\tilde{\mathcal{A}}) = \langle \mathbf{r} | \hat{A} | \mathbf{r}' \rangle \]

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where \( \hat{\mathcal{H}} = \langle \mathcal{H} \rangle \) is the single particle Hamiltonian, being [†]

\[ \mathcal{H} = \left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r})\right] + \sum_{k=1}^{L-1} \frac{1}{k!} \int d^D \mathbf{r}_1 \cdots \int d^D \mathbf{r}_k \psi^\dagger(\mathbf{r}_1) \cdots \psi^\dagger(\mathbf{r}_k) V_k(\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_k) \psi(\mathbf{r}_k) \cdots \psi(\mathbf{r}_1) \]
By applying Weyl-Wigner and its inverse Weyl-Wigner transform to different terms of equation for $\langle r|\hat{Q}|r'\rangle$ we obtain

**Equation of Evolution for the Reduced Wigner function**

\[
i\hbar \frac{\partial}{\partial t} F_W = \frac{1}{(2\pi \hbar)^{2D}} \int \int \int \int d^D r' d^D p' d^D \tau d^D \phi e^{\frac{i}{\hbar} \tau \cdot (p' - p)} e^{\frac{i}{\hbar} \phi \cdot (r - r')} \\
\times \left\{ \tilde{H} \left( r' + \frac{\tau}{2}, p' + \frac{\phi}{2} \right) - \tilde{H} \left( r' - \frac{\tau}{2}, p' - \frac{\phi}{2} \right) \right\} F_W(r', p')
\]

$\tilde{H}(r, p) = \text{phase function of single particle operator } \hat{H} = \langle \hat{H} \rangle$.

**Gradient expansion of kinetic equation**

By expanding $\tilde{H}(r' + \tau/2, p' + \phi/2) - \tilde{H}(r' - \tau/2, p' - \phi/2)$ around $\tau = 0$ and by using the Fourier integral theorem

\[
\frac{\partial F_W}{\partial t} + \frac{p_k}{m} \frac{\partial F_W}{\partial x_k} = \sum_{l=0}^{\infty} \frac{(i \hbar/2)^{2l}}{(2l + 1)!} \left[ \frac{\partial^{2l+1} V_{\text{eff}}}{\partial x_{k_1} \cdots \partial x_{k_{2l+1}}} \right] \left[ \frac{\partial^{2l+1} F_W}{\partial p_{k_1} \cdots \partial p_{k_{2l+1}}} \right]
\]
The Quantum Kinetic equation

By applying Weyl-Wigner and its inverse Weyl-Wigner transform to different terms of equation for $\langle r | \hat{Q} | r' \rangle$ we obtain

**Equation of Evolution for the Reduced Wigner function**

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$$\times \left\{ \tilde{H} \left( r' + \frac{\tau}{2}, p' + \frac{\phi}{2} \right) - \tilde{H} \left( r' - \frac{\tau}{2}, p' - \frac{\phi}{2} \right) \right\} F_W(r', p')$$

$\tilde{H}(r, p) =$ phase function of single particle operator $\hat{H} = \langle \hat{H} \rangle$.

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By applying Weyl-Wigner and its inverse Weyl-Wigner transform to different terms of equation for $\langle \boldsymbol{r} | \hat{\varrho} | \boldsymbol{r}' \rangle$ we obtain

- **Equation of Evolution for the Reduced Wigner function**

  $$i\hbar \frac{\partial}{\partial t} \mathcal{F}_W = \frac{1}{(2\pi\hbar)^{2D}} \int \int \int \int d^D \boldsymbol{r}' d^D \boldsymbol{p}' d^D \tau d^D \phi e^{i\frac{\hbar}{\hbar} \tau \cdot (\boldsymbol{p}' - \boldsymbol{p})} e^{i\frac{\hbar}{\hbar} \phi \cdot (\boldsymbol{r}' - \boldsymbol{r})}$$

  $$\times \left\{ \tilde{\mathcal{H}} \left( \boldsymbol{r}' + \frac{\tau}{2}, \boldsymbol{p}' + \frac{\phi}{2} \right) - \tilde{\mathcal{H}} \left( \boldsymbol{r}' - \frac{\tau}{2}, \boldsymbol{p}' - \frac{\phi}{2} \right) \right\} \mathcal{F}_W (\boldsymbol{r}', \boldsymbol{p}')$$

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The effects of interactions are entirely contained in $V_{\text{eff}}(r)$

\[
V_{\text{eff}}(r) = U(r) + \sum_{k=1}^{L-1} \frac{1}{k!} \int d^D r_1 \cdots \int d^D r_k \, g^{(k)}(r_1, \ldots, r_k)
\]

\[
\times n(r_1) \cdots n(r_k) \, V_{k+1}(r, r_1, \ldots, r_k)
\]

being $g^{(k)}(r_1, \ldots r_k)$ the k-order correlation function

\[
g^{(k)}(r_1, \ldots r_k) = \frac{\langle \psi^\dagger(r_1) \cdots \psi^\dagger(r_k) \psi(r_k) \cdots \psi(r_1) \rangle}{n(r_1) \cdots n(r_k)}
\]

- For $L = 2$ we obtain the usual Hartree approximation with two-body interactions $V_{\text{eff}}(r) = U(r) + \int d^D r_1 \, n(r_1) \, V_2(r, r_1)$
- For $L > 2$ we obtain the Hartree approximation plus some corrections due to Fermion-Boson-Exclusion correlations

We remark that, these results can be generalized by including explicitly the spin degrees of freedom. By including others interaction terms [3, 4]
The effects of interactions are entirely contained in $V_{\text{eff}}(\mathbf{r})$

$$V_{\text{eff}}(\mathbf{r}) = U(\mathbf{r}) + \sum_{k=1}^{L-1} \frac{1}{k!} \int d^D r_1 \cdots \int d^D r_k \ g^{(k)}(r_1, \cdots, r_k)$$

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The effects of interactions are entirely contained in $V_{\text{eff}}(r)$:

$$V_{\text{eff}}(r) = U(r) + \sum_{k=1}^{L-1} \frac{1}{k!} \int d^D r_1 \cdots \int d^D r_k \ g^{(k)}(r_1, \cdots, r_k)$$

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## The QMEP for a System of Identical Particles

- **Von Neumann strategy** [1]
  \[ S = -k_B \text{Tr}(\rho \ln \rho) \]
  It does not contain the statistical information.

- **Statistical weight strategy** [2]
  S in terms of \( \langle N_\nu \rangle = \langle N_\nu \rangle/y \)

### Fermi and Bose statistics
\[
S = -k_B \sum_y (\langle N_\nu \rangle \ln (\langle N_\nu \rangle) + (1 - \langle N_\nu \rangle) \ln (1 - \langle N_\nu \rangle))
\]

### Anyonic systems satisfying Fractional Exclusion statistics
\[
S = -k_B \sum_y \left[ \langle N_\nu \rangle \ln (\langle N_\nu \rangle) + (1 - \langle N_\nu \rangle) \ln (1 - \langle N_\nu \rangle) \right] - y \left[ (1 - \langle N_\nu \rangle) \ln (1 - \langle N_\nu \rangle) - (1 - y) \langle N_\nu \rangle \right]
\]

For \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics.
The QMEP for a system of identical particles

- **Von Neumann strategy** [1] $S = -k_B \text{Tr}(\rho \ln \rho)$
  
  It does not contain the statistical information.

- **Statistical weight strategy** [2] $S$ in terms of $\langle N_\nu \rangle = \langle \mathcal{N}_\nu \rangle / y$

  $S = -k_B \sum y [\langle N_\nu \rangle \ln \langle N_\nu \rangle \pm (1 \mp \langle N_\nu \rangle) \ln (1 \mp \langle N_\nu \rangle)]$

  Anyonic systems satisfying Fractional Exclusion statistics

  $S = -k_B \sum y \left\{ \langle N_\nu \rangle \ln \langle N_\nu \rangle + (1 - \kappa \langle N_\nu \rangle) \ln (1 - \kappa \langle N_\nu \rangle) - 
  \right.$

  $\left. [1 + (1 - \kappa \langle N_\nu \rangle)] \ln [1 + (1 - \kappa \langle N_\nu \rangle)] \right\}$

  For $\kappa = 1$ and $\kappa = 0$ we recover Fermi and Bose statistics

- **Present strategy** [3] Reduced Density Operator approach

  $S = -k_B \text{Tr} \{ \hat{\Phi} (\hat{\rho}) \}$
**The QMEP for a system of identical particles**

- **Von Neumann strategy** [1]  
  \[ S = -k_B \text{Tr}(\rho \ln \rho) \]

  *It does not contain the statistical information.*

- **Statistical weight strategy** [2] \( S \) in terms of \( \langle N_\nu \rangle = \langle N_\nu \rangle / y \)
  \[ S = -k_B \sum y \left[ \langle N_\nu \rangle \ln \langle N_\nu \rangle \pm (1 \mp \langle N_\nu \rangle) \ln (1 \mp \langle N_\nu \rangle) \right] \]

Anyonic systems satisfying Fractional Exclusion statistics

\[ S = -k_B \sum y \left\{ \langle N_\nu \rangle \ln \langle N_\nu \rangle + (1 - \kappa \langle N_\nu \rangle) \ln (1 - \kappa \langle N_\nu \rangle) - [1 + (1 - \kappa \langle N_\nu \rangle)] \ln [1 + (1 - \kappa \langle N_\nu \rangle)] \right\} \]

For \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics

- **Present strategy** [3] \( \text{Reduced Density Operator approach} \)
  \[ S = -k_B \text{Tr} (\hat{\Phi}(\hat{\varrho})) \]
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- **Von Neumann strategy** [1] \( S = -k_B \text{Tr}(\rho \ln \rho) \)
  
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  Fermi and Bose statistics

  \[
  S = -k_B \sum y \left[ \langle \overline{N}_{\nu} \rangle \ln \langle \overline{N}_{\nu} \rangle \pm \left( 1 \mp \langle \overline{N}_{\nu} \rangle \right) \ln \left( 1 \mp \langle \overline{N}_{\nu} \rangle \right) \right]
  \]

  Anyonic systems satisfying Fractional Exclusion statistics

  \[
  S = -k_B \sum y \left\{ \langle \overline{N}_{\nu} \rangle \ln \langle \overline{N}_{\nu} \rangle + \left( 1 - \kappa \langle \overline{N}_{\nu} \rangle \right) \ln \left( 1 - \kappa \langle \overline{N}_{\nu} \rangle \right) - \left[ 1 + (1 - \kappa) \langle \overline{N}_{\nu} \rangle \right] \ln \left[ 1 + (1 - \kappa) \langle \overline{N}_{\nu} \rangle \right] \right\}
  \]

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Anyonic systems satisfying Fractional Exclusion statistics

\[ S = -k_B \sum y \left\{ \langle N_\nu \rangle \ln \langle N_\nu \rangle + (1 - \kappa \langle N_\nu \rangle) \ln (1 - \kappa \langle N_\nu \rangle) - \right. \]

\[ \left. [1 + (1 - \kappa) \langle N_\nu \rangle] \ln [1 + (1 - \kappa) \langle N_\nu \rangle]\right\} \]

For \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics.

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The QMEP for a system of identical particles

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  Anyonic systems satisfying Fractional Exclusion statistics
  \[ S = -k_B \sum_y \{ \langle N_\nu \rangle \ln \langle N_\nu \rangle + (1 - \kappa \langle N_\nu \rangle) \ln (1 - \kappa \langle N_\nu \rangle) - \\
  [1 + (1 - \kappa \langle N_\nu \rangle)] \ln [1 + (1 - \kappa \langle N_\nu \rangle)] \} \]
  For \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics

- **Present strategy** [3] Reduced Density Operator approach
  \[ S = -k_B \text{Tr}\{\hat{\Phi}(\Omega)\} \]
The QMEP for a System of Identical Particles

- Von Neumann strategy [1]  \[ S = -k_B \text{Tr}(\rho \ln \rho) \]
  
  It does not contain the statistical information.

- Statistical weight strategy [2]  \[ S \text{ in terms of } \langle N_\nu \rangle = \langle N_\nu \rangle/y \]
  
  Fermi and Bose statistics
  \[ S = -k_B \sum y [\langle N_\nu \rangle \ln \langle N_\nu \rangle \pm (1 \mp \langle N_\nu \rangle) \ln (1 \mp \langle N_\nu \rangle)] \]

  Anyonic systems satisfying Fractional Exclusion statistics
  \[ S = -k_B \sum y \{\langle N_\nu \rangle \ln \langle N_\nu \rangle + (1 - \kappa \langle N_\nu \rangle) \ln (1 - \kappa \langle N_\nu \rangle) - \]
  \[ [1 + (1 - \kappa \langle N_\nu \rangle) \ln [1 + (1 - \kappa \langle N_\nu \rangle)]\} \]
  
  For \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics

- Present strategy [3]  \[ \text{Reduced Density Operator approach} \]
  \[ S = -k_B \text{Tr} \left\{ \hat{\Phi}(\hat{\rho}) \right\} \]
We consider for the Bose and Fermi Statistics the function

\[ \hat{\Phi}(\hat{\varrho}) = \hat{\varrho} \ln \left( \frac{\hat{\varrho}}{y} \right) \pm \left( \hat{I} \mp \frac{\hat{\varrho}}{y} \right) \ln \left( \hat{I} \mp \frac{\hat{\varrho}}{y} \right) \]

We consider for the Fractional Exclusion Statistics the function

\[ \hat{\Phi}(\hat{\varrho}) = \hat{\varrho} \ln \left( \frac{\hat{\varrho}}{y} \right) + \left( \hat{I} - \kappa \frac{\hat{\varrho}}{y} \right) \ln \left( \hat{I} - \kappa \frac{\hat{\varrho}}{y} \right) - \left[ \hat{I} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \ln \left[ \hat{I} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \]

where for \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics.

- We prove that
  \[ S = -k_B \text{Tr} \{ \hat{\Phi}(\hat{\varrho}) \} \]

- The macroscopic local moments

By considering an arbitrary set of single-particle observables \( \{\hat{M}_A\} \) and the corresponding space-phase functions \( \{\tilde{\psi}_A\} \)

\[ M_A(r, t) = \int d^D p \, \tilde{\psi}_A(r, p) \mathcal{F}_W(r, p, t) \]

We use \( S = -k_B \text{Tr}\{ \hat{\Phi}(\hat{\varrho}) \} \) as informational entropy.
We consider for the Bose and Fermi Statistics the function

\[ \hat{\Phi}(\hat{\varrho}) = \hat{\varrho} \ln \left( \frac{\hat{\varrho}}{y} \right) \pm \left( \hat{l} \mp \frac{\hat{\varrho}}{y} \right) \ln \left( \hat{l} \mp \frac{\hat{\varrho}}{y} \right) \]

We consider for the Fractional Exclusion Statistics the function \([\dagger]\)

\[ \hat{\Phi}(\hat{\varrho}) = \hat{\varrho} \ln \left( \frac{\hat{\varrho}}{y} \right) + \left( \hat{l} - \kappa \frac{\hat{\varrho}}{y} \right) \ln \left( \hat{l} - \kappa \frac{\hat{\varrho}}{y} \right) - \left[ \hat{l} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \ln \left[ \hat{l} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \]

where for \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics.

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\[ S = -k_B \text{Tr} \left\{ \hat{\Phi}(\hat{\varrho}) \right\} \]

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\[
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\]

We consider for the Fractional Exclusion Statistics the function [†]
\[
\hat{\Phi}(\hat{\varrho}) = \hat{\varrho} \ln \left( \frac{\hat{\varrho}}{y} \right) + \left( \hat{I} - \kappa \frac{\hat{\varrho}}{y} \right) \ln \left( \hat{I} - \kappa \frac{\hat{\varrho}}{y} \right) - \left[ \hat{I} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \ln \left[ \hat{I} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right]
\]

where for $\kappa = 1$ and $\kappa = 0$ we recover Fermi and Bose statistics.

• We prove that
\[
S = -k_B \text{Tr} \{\hat{\Phi}(\hat{\varrho})\}
\]

• The macroscopic local moments

By considering an arbitrary set of single-particle observables $\{\hat{M}_A\}$ and the corresponding space-phase functions $\{\tilde{\psi}_A\}$

\[
M_A(r, t) = \int d^Dp \, \tilde{\psi}_A(r, p) \mathcal{F}_W(r, p, t)
\]

We use
\[
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• We consider for the Bose and Fermi Statistics the function
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We consider for the Fractional Exclusion Statistics the function [†]
\[ \hat{\Phi}(\hat{\varrho}) = \hat{\varrho} \ln \left( \frac{\hat{\varrho}}{y} \right) + \left( \hat{I} - \kappa \frac{\hat{\varrho}}{y} \right) \ln \left( \hat{I} - \kappa \frac{\hat{\varrho}}{y} \right) - \left[ \hat{I} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \ln \left[ \hat{I} + (1 - \kappa) \frac{\hat{\varrho}}{y} \right] \]

where for \( \kappa = 1 \) and \( \kappa = 0 \) we recover Fermi and Bose statistics.

• We prove that
\[ S = -k_B \text{ Tr} \left\{ \hat{\Phi}(\hat{\varrho}) \right\} \]

The macroscopic local moments

By considering an arbitrary set of single-particle observables \( \{\hat{M}_A\} \) and the corresponding space-phase functions \( \{\tilde{\psi}_A\} \)
\[ M_A(r, t) = \int d^D p \, \tilde{\psi}_A(r, p) \, \mathcal{F}(r, p, t) \]

We use \( S = -k_B \text{ Tr} \{\hat{\Phi}(\hat{\varrho})\} \) as informational entropy.
We consider for the Bose and Fermi Statistics the function

$$\Phi(\rho) = \rho \ln \left( \frac{\rho}{y} \right) \pm \left( I \mp \frac{\rho}{y} \right) \ln \left( I \mp \frac{\rho}{y} \right)$$

We consider for the Fractional Exclusion Statistics the function \[†\]

$$\Phi(\rho) = \rho \ln \left( \frac{\rho}{y} \right) + \left( I - \kappa \frac{\rho}{y} \right) \ln \left( I - \kappa \frac{\rho}{y} \right) - \left[ I + (1 - \kappa) \frac{\rho}{y} \right] \ln \left[ I + (1 - \kappa) \frac{\rho}{y} \right]$$

where for \(\kappa = 1\) and \(\kappa = 0\) we recover Fermi and Bose statistics.

We prove that

$$S = -k_B \text{Tr} \left\{ \Phi(\rho) \right\}$$

The macroscopic local moments

By considering an arbitrary set of single-particle observables \(\{\hat{M}_A\}\) and the corresponding space-phase functions \(\{\tilde{\psi}_A\}\)

$$M_A(r, t) = \int d^Dp \, \tilde{\psi}_A(r, p) \, F_{\mathcal{W}}(r, p, t)$$

We use \(S = -k_B \text{Tr}\{\hat{\Phi}(\hat{\rho})\}\) as informational entropy.
We search the extremal value of entropy subject to the constrain that the information of the physical system is described by $M_A(r, t)$

$$\tilde{S} = S - \int d^D r \left\{ \sum_{A=1}^{N} \tilde{\lambda}_A(r, t) \left[ \int d^D p \tilde{\psi}_A F_W - M_A(r, t) \right] \right\}$$

For the Fractional Exclusion statistics we obtain

$$\hat{\rho} = y \left\{ \hat{w}(\hat{\xi}) + \kappa \hat{I} \right\}^{-1} [\hat{w}(\hat{\xi})]^\kappa \hat{I} + \hat{w}(\hat{\xi})]^{1-\kappa} = \hat{\xi}$$

$$\hat{\xi} = \exp \left[ W^{-1} \left( \sum_{A=1}^{N} \lambda_A \tilde{\psi}_A \right) \right]$$

For $\kappa = 1, 0$ we obtain the Fermi and Bose statistics

$$\hat{\rho} = y \left\{ \exp \left[ W^{-1} \left( \sum_{A=1}^{N} \lambda_A(r, t) \tilde{\psi}_A \right) \right] \pm \hat{I} \right\}^{-1}$$

Boltzmann statistics

$$\hat{\rho} = y \exp \left\{ W^{-1} \left( - \sum_{A=1}^{N} \lambda_A(r, t) \tilde{M}_A \right) \right\}$$
We search the extremal value of entropy subject to the constrain that the information of the physical system is described by $M_A(r, t)$

$$\widetilde{S} = S - \int d^D r \left\{ \sum_{A=1}^{N} \tilde{\lambda}_A(r, t) \left[ \int d^D p \tilde{\psi}_A F_W - M_A(r, t) \right] \right\}$$

For the Fractional Exclusion statistics we obtain

$$\hat{\varrho} = y \left\{ \hat{w}(\hat{\xi}) + \kappa \hat{l} \right\}^{-1}$$

$$[\hat{w}(\hat{\xi})]^\kappa [\hat{l} + \hat{w}(\hat{\xi})]^{1-\kappa} = \hat{\xi}$$

$$\hat{\xi} = \exp \left[ \gamma^{-1} \left( \sum_{A=1}^{N} \lambda_A \tilde{\psi}_A \right) \right]$$

For $\kappa = 1, 0$ we obtain the Fermi and Bose statistics

$$\hat{\varrho} = y \left\{ \exp \left[ \gamma^{-1} \left( \sum_{A=1}^{N} \lambda_A(r, t) \tilde{\psi}_A \right) \right] \pm \hat{l} \right\}^{-1}$$

Boltzmann statistics

$$\hat{\varrho} = y \exp \left\{ \gamma^{-1} \left( - \sum_{A=1}^{N} \lambda_A(r, t) \tilde{M}_A \right) \right\}$$
We search the extremal value of entropy subject to the constrain that the information of the physical system is described by $M_A(r, t)$

$$\tilde{S} = S - \int d^D r \left\{ \sum_{A=1}^{N} \tilde{\lambda}_A(r, t) \left[ \int d^D p \tilde{\psi}_A F_W - M_A(r, t) \right] \right\}$$

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$$\hat{\varrho} = y \exp \left\{ \gamma W^{-1} \left( - \sum_{A=1}^{N} \lambda_A(r, t) \tilde{M}_A \right) \right\}$$
We search the extremal value of entropy subject to the constrain that the information of the physical system is described by $M_A(r, t)$

$$\tilde{S} = S - \int d^D r \left\{ \sum_{A=1}^{N} \tilde{\lambda}_A(r, t) \left[ \int d^D p \tilde{\psi}_A \mathcal{F}_W - M_A(r, t) \right] \right\}$$

For the Fractional Exclusion statistics we obtain

$$\hat{\varrho} = y \left\{ \hat{w}(\hat{\xi}) + \kappa \hat{l} \right\}^{-1}$$

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For $\kappa = 1, 0$ we obtain the Fermi and Bose statistics

$$\hat{\varrho} = y \left\{ \exp \left[ \mathcal{W}^{-1} \left( \sum_{A=1}^{N} \lambda_A(r, t) \tilde{\psi}_A \right) \right] \pm \hat{l} \right\}^{-1}$$

Boltzmann statistics

$$\hat{\varrho} = y \exp \left\{ \mathcal{W}^{-1} \left( - \sum_{A=1}^{N} \lambda_A(r, t) \tilde{M}_A \right) \right\}$$
We search the extremal value of entropy subject to the constrain that the information of the physical system is described by $M_A(r, t)$:

$$\tilde{S} = S - \int d^Dr \left\{ \sum_{A=1}^{N} \tilde{\lambda}_A(r, t) \left[ \int d^Dp \tilde{\psi}_A F_W - M_A(r, t) \right] \right\}$$

For the Fractional Exclusion statistics we obtain

$$\hat{\varrho} = y \left\{ \hat{w}(\hat{\xi}) + \kappa \hat{l} \right\}^{-1} \left[ \hat{w}(\hat{\xi})^\kappa [\hat{l} + \hat{w}(\hat{\xi})]^{1-\kappa} = \hat{\xi} \right.$$

$$\hat{\xi} = \exp \left[ W^{-1} \left( \sum_{A=1}^{N} \lambda_A \tilde{\psi}_A \right) \right]$$

For $\kappa = 1, 0$ we obtain the Fermi and Bose statistics

$$\hat{\varrho} = y \left\{ \exp \left[ W^{-1} \left( \sum_{A=1}^{N} \lambda_A(r, t) \tilde{\psi}_A \right) \right] \pm \hat{l} \right\}^{-1}$$

Boltzmann statistics

$$\hat{\varrho} = y \exp \left\{ W^{-1} \left( - \sum_{A=1}^{N} \lambda_A(r, t) \tilde{M}_A \right) \right\}$$
We search the extremal value of entropy subject to the constraint that the information of the physical system is described by $M_A(r, t)$

$$\tilde{S} = S - \int d^D r \left\{ \sum_{A=1}^{N} \tilde{\lambda}_A(r, t) \left[ \int d^D p \tilde{\psi}_A \mathcal{F}_W - M_A(r, t) \right] \right\}$$

For the Fractional Exclusion statistics we obtain

$$\hat{\varrho} = y \left\{ \hat{w}(\hat{\xi}) + \kappa \hat{l} \right\}^{-1} \left[ \hat{w}(\hat{\xi})^\kappa \hat{l} + \hat{w}(\hat{\xi})^{1-\kappa} \right] = \hat{\xi}$$

$$\hat{\xi} = \exp \left[ \mathcal{W}^{-1} \left( \sum_{A=1}^{N} \lambda_A \tilde{\psi}_A \right) \right]$$

For $\kappa = 1, 0$ we obtain the Fermi and Bose statistics

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Boltzmann statistics

$$\hat{\varrho} = y \exp \left\{ \mathcal{W}^{-1} \left( - \sum_{A=1}^{N} \lambda_A(r, t) \tilde{M}_A \right) \right\}$$
Bosons (−) and Fermions (+) for $D = 1, 2, 3$

$[\Psi(r), \Psi(r')]_\mp = [\Psi^\dagger(r), \Psi^\dagger(r')]_\mp = 0 \quad [\Psi(r), \Psi^\dagger(r')]_\mp = \delta^D(r-r')$

Anyons for $D = 1, 2 \Rightarrow q$-deformed bracket $[A, B]_q = AB - q BA$

$[\Psi(r), \Psi(r')]_q = [\Psi^\dagger(r), \Psi^\dagger(r')]_q = 0 \quad [\Psi(r), \Psi^\dagger(r')]_{q-1} = \delta^D(r-r')$

- $q(r, r') = q^{-1}(r', r)$ with $q(r, r) = \pm 1$
- $q(r, r')$ correspond to a phase factor that denotes the statistics
  \[ q(r, r') = \pm e^{i\nu \pi S(r, r')} \]
  - $(\pm \Rightarrow$ bosonic and fermionic representations$)$
  - $S(r, r')$ the anyonic phase
  - $\nu$ denotes the exchange statistics (for $\nu = 0, 1 \Rightarrow$ Fermi, Bose)

For $D = 2 \Rightarrow S(r, r) = 0$ and $S(r, r') = \pm 1$ (Braid group)

[Braid group

For $D = 1 \Rightarrow S(r, r') = \text{sgn}(r-r') = \begin{cases} 1 & \text{if } r > r' \\ 0 & \text{if } r = r' \\ -1 & \text{if } r < r' \end{cases}$]
[\star] Bosons (-) and Fermions (+) for \( D = 1, 2, 3 \)

\[
[\psi(r), \psi(r')]_\mp = [\psi^\dagger(r), \psi^{\dagger}(r')]_\mp = 0 \quad [\psi(r), \psi^{\dagger}(r')]_\mp = \delta^D(r - r')
\]

Anyons for \( D = 1, 2 \) \( \Rightarrow \) q-deformed bracket \([A, B]_q = AB - q BA\)

\[
[\psi(r), \psi(r')]_q = [\psi^{\dagger}(r), \psi^{\dagger}(r')]_q = 0 \quad [\psi(r), \psi^{\dagger}(r')]_{q-1} = \delta^D(r - r')
\]

- \( q(r, r') = q^{-1}(r', r) \) with \( q(r, r) = \pm 1 \)
- \( q(r, r') \) correspond to a phase factor that denotes the statistics

\[
q(r, r') = \pm e^{i\nu \pi S(r, r')}
\]

- \( (\pm \Rightarrow \) bosonic and fermionic representations\)
- \( S(r, r') \) the anyonic phase
- \( \nu \) denotes the exchange statistics (for \( \nu = 0, 1 \) \( \Rightarrow \) Fermi, Bose)
- For \( D = 2 \) \( \Rightarrow S(r, r) = 0 \) and \( S(r, r') = \pm 1 \) (Braid group) [+1 (-1) if \( r' \) is moved counterclockwise (clockwise) around \( r \)]
- For \( D = 1 \) \( \Rightarrow S(r, r') = \text{sgn}(r-r') = \left\{ \begin{array}{cl} 1 & \text{if } r > r' \\ 0 & \text{if } r = r' \\ -1 & \text{if } r < r' \end{array} \right. \)
Bosons \((-\rangle\) and Fermions \((+\rangle\) for \(D = 1, 2, 3\)

\[ [\psi(r), \psi(r')]_\mp = [\psi^\dagger(r), \psi^\dagger(r')]_\mp = 0 \quad [\psi(r), \psi^\dagger(r')]_\mp = \delta^D(r - r') \]

Anyons for \(D = 1, 2 \Rightarrow q\)-deformed bracket \([A, B]_q = AB - q BA\)

\[ [\psi(r), \psi(r')]_q = [\psi^\dagger(r), \psi^\dagger(r')]_q = 0 \quad [\psi(r), \psi^\dagger(r')]_{q-1} = \delta^D(r - r') \]

- \(q(r, r') = q^{-1}(r', r)\) with \(q(r, r) = \pm 1\)

- \(q(r, r')\) correspond to a phase factor that denotes the statistics

\[ q(r, r') = \pm e^{i\nu\pi S(r, r')} \]

- \((\pm \Rightarrow \text{bosonic and fermionic representations})\)

- \(S(r, r')\) the anyonic phase

- \(\nu\) denotes the exchange statistics (for \(\nu = 0, 1 \Rightarrow \text{Fermi, Bose}\))

- For \(D = 2 \Rightarrow S(r, r) = 0 \quad \text{and} \quad S(r, r') = \pm 1 \quad (\text{Braid group})\]

- \([+1 (-1) \text{ if } r' \text{ is moved counterclockwise (clockwise) around } r]\) 

- For \(D = 1 \Rightarrow S(r, r') = \text{sgn}(r-r') = \begin{cases} 1 & \text{if } r > r' \\ 0 & \text{if } r = r' \\ -1 & \text{if } r < r' \end{cases} \)
Bosons (−) and Fermions (+) for $D = 1, 2, 3$

$$[\psi(r), \psi(r')]_\mp = [\psi^\dagger(r), \psi^\dagger(r')]_\mp = 0 \quad [\psi(r), \psi^\dagger(r')]_\mp = \delta^D(r - r')$$

Anyons for $D = 1, 2 \Rightarrow q$-deformed bracket $[A, B]_q = AB - q BA$

$$[\psi(r), \psi(r')]_q = [\psi^\dagger(r), \psi^\dagger(r')]_q = 0 \quad [\psi(r), \psi^\dagger(r')]_q^{-1} = \delta^D(r - r')$$

- $q(r, r') = q^{-1}(r', r)$ with $q(r, r) = \pm 1$
- $q(r, r')$ correspond to a phase factor that denotes the statistics

$$q(r, r') = \pm e^{i\nu\pi S(r, r')}$$

- $\pm$ ⇒ bosonic and fermionic representations
- $S(r, r')$ the anyonic phase
- $\nu$ denotes the exchange statistics (for $\nu = 0, 1 \Rightarrow$ Fermi, Bose)

- For $D = 2 \Rightarrow S(r, r) = 0$ and $S(r, r') = \pm 1$ (Braid group) $[+1 (-1)$ if $r'$ is moved counterclockwise (clockwise) around $r)$]

- For $D = 1 \Rightarrow S(r, r') = \text{sgn}(r - r') = \begin{cases} 1 & \text{if } r > r' \\ 0 & \text{if } r = r' \\ -1 & \text{if } r < r' \end{cases}$
Bosons (−) and Fermions (+) for $D = 1, 2, 3$

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Anyons for $D = 2$

- we consider the Bose or Fermi fields $\tilde{\Psi}(r)$ and $\tilde{\Psi}^\dagger(r)$
- we introduce the new Anyonic (global) fields $\Psi(r)$ and $\Psi^\dagger(r)$

\[
\Psi(r) = e^{-i\frac{e}{\hbar}\Lambda(r)}\tilde{\Psi}(r) \quad \quad \Psi^\dagger(r) = \tilde{\Psi}^\dagger(r) e^{i\frac{e}{\hbar}\Lambda(r)}
\]

\[
\Lambda(r) = \gamma \int d^2 r' \varphi(r - r') \tilde{\Psi}^\dagger(r) \tilde{\Psi}(r')
\]

$\varphi(r - r') \Rightarrow$ azimuthal angle of the vector from $r'$ to $r$ in the plane.

- By defining the (Chern-Simons) gauge potential $A(r) = \nabla \Lambda(r)$ and the corresponding Chern-Simons Magnetic field $B = \nabla \wedge A$

\[
H = \frac{1}{2m} \int d^2 r \left( \left(-i\hbar \frac{\partial}{\partial x_j} - \frac{e}{c} A_j \right) \tilde{\Psi}(r) \right)^\dagger \left( \left(-i\hbar \frac{\partial}{\partial x_j} - \frac{e}{c} A_j \right) \tilde{\Psi}(r) \right) + \\
\sum_{k=1}^{L} \frac{1}{k!} \int d^2 r_1 \cdots \int d^2 r_k \tilde{\Psi}^\dagger(r_1) \cdots \tilde{\Psi}^\dagger(r_k) V_k(r_1, \ldots, r_k) \tilde{\Psi}(r_k) \cdots \tilde{\Psi}(r_1)
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\[
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[\Psi(r), \Psi(r')]_q &= [\Psi^\dagger(r), \Psi^\dagger(r')]_q = 0 \\
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\end{align*}
\]

\[ q(r, r') = \pm e^{i\nu \pi S(r,r')} \]

**Conclusions**

- The net effect of the Chern-Simons dynamics is to produce a nonlocal vector potential \( A \) for each particle. This determines a magnetic field \( B \) which vanishes everywhere except at the particle locations. Therefore, the particles effectively carry both a charge \( e \) and a magnetic flux \( \Phi \).
- (Braid group structure) The exotic statistics are then produced by the Aharonov-Bohm effect: when two particles are exchanged, they pick up a phase because the charge of one particle moves around the flux of the other and **vice-versa**.
\[
\left[ \tilde{\Psi}(r), \tilde{\Psi}(r') \right]_\mp = \left[ \tilde{\Psi}^\dagger(r), \tilde{\Psi}^\dagger(r') \right]_\mp = 0
\]

\[
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\]

\[
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&\uparrow & &\uparrow & &\uparrow \\
[\Psi(r), \Psi(r')]_q &= [\psi^+(r), \psi^+(r')]_q = 0 & [\psi(r), \psi^+(r')]_{q-1} &= \delta^2(r - r') \\
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Bosons (−) and Fermions (+) for $D = 1, 2, 3$

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- By considering the equation of motion for the operator $\Psi(r)$

$$i\hbar \frac{\partial}{\partial t} \Psi(r) = \mathcal{H}(r)\Psi(r), \quad -i\hbar \frac{\partial}{\partial t} \Psi^\dagger(r) = \Psi^\dagger(r)\mathcal{H}(r)$$

$$\mathcal{H}(r) = -\frac{\hbar^2}{2m} \nabla^2 + U(r) + \sum_{k=1}^{L-1} \frac{1}{k!} \int d^3 r_1 \cdots \int d^3 r_k \Psi^\dagger(r_1) \cdots \Psi^\dagger(r_k) \times V_{k+1}(r, r_1, \ldots, r_k)\Psi(r_k) \cdots \Psi(r_1)$$

- We determine the equation of motion for the quantity $\Psi^\dagger(r')\Psi(r)$

$$i\hbar \frac{\partial}{\partial t} \Psi^\dagger(r')\Psi(r) = [\mathcal{H}(r), \Psi^\dagger(r')\Psi(r)]$$

By performing its statistical average in the Hartree approximation $[]$
Bosons (−) and Fermions (+) for $D = 1, 2, 3$

\[[\psi(r), \psi(r')]_{\mp} = [\psi^\dagger(r), \psi^\dagger(r')]_{\mp} = 0 \quad [\psi(r), \psi^\dagger(r')]_{\mp} = \delta^D(r - r')\]

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