An admissibility and asymptotic-preserving scheme on 2D unstructured meshes

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1. General context and examples

2. Development of an admissibility & asymptotic preserving FV scheme

3. Conclusion and perspectives
Outline

1. General context and examples
2. Development of an admissibility & asymptotic preserving FV scheme
3. Conclusion and perspectives
Problematic

Hyperbolic systems of conservation laws with source terms:

\[ \partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W) \] (1)

- \( A \): set of admissible states,
- \( W \in A \subset \mathbb{R}^N \),
- \( F \): physical flux,
- \( \gamma > 0 \): controls the stiffness,
- \( R: A \to A \); smooth function with some compatibility conditions (cf. [Berthon, LeFloch, and Turpault, 2013]).
Problematic

Hyperbolic systems of conservation laws with source terms:

\[ \partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W) \]  \hspace{1cm} (1)

Under compatibility conditions on \( R \), when \( \gamma t \to \infty \), (1) degenerates into a diffusion equation:

\[ \partial_t w - \text{div}(D(w)\nabla w) = 0 \]  \hspace{1cm} (2)

- \( w \in \mathbb{R} \), linked to \( W \),
- \( D \): positive and definite matrix, or positive function.
Example #1: isentropic Euler with friction

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v &= 0 \\
\frac{\partial}{\partial t} \rho u + \frac{\partial}{\partial x} (\rho u^2 + p(\rho)) + \frac{\partial}{\partial y} \rho uv &= -\kappa \rho u \quad \text{, with: } p'(\rho) > 0, \kappa > 0 \\
\frac{\partial}{\partial t} \rho v + \frac{\partial}{\partial x} \rho uv + \frac{\partial}{\partial y} (\rho v^2 + p(\rho)) &= -\kappa \rho v
\end{align*}
\]

\[\mathcal{A} = \{ (\rho, \rho u, \rho v)^T \in \mathbb{R}^3 / \rho > 0 \}\]

Formalism of (1)

- \(W = (\rho, \rho u, \rho v)^T\)
- \(R(W) = (\rho, 0, 0)^T\)
- \(F(W) = \begin{pmatrix} \rho u, & \rho u^2 + p, & \rho uv \\ \rho v, & \rho uv, & \rho v^2 + p \end{pmatrix}^T\)
- \(\gamma(W) = \kappa\)

Limit diffusion equation

\[
\frac{\partial}{\partial t} \rho - \text{div} \left( \frac{1}{\kappa} \nabla p(\rho) \right) = 0
\]
Example #2: $M_1$ model for radiative transfer

\[
\begin{aligned}
\partial_t E + \partial_x F_{R,x} + \partial_y F_{R,y} &= c\sigma^e aT^4 - c\sigma^a E \\
\partial_t F_{R,x} + c^2 \partial_x P_{xx}(E, F_R) + c^2 \partial_y P_{xy}(E, F_R) &= -c\sigma^f F_{R,x} \\
\partial_t F_{R,y} + c^2 \partial_x P_{yx}(E, F_R) + c^2 \partial_y P_{yy}(E, F_R) &= -c\sigma^f F_{R,y} \\
\rho C_v \partial_t T &= c\sigma^a E - c\sigma^e aT^4 \\
\sigma &= \sigma(E, F_{R,x}, F_{R,y}, T) \\
\mathcal{A} &= \{(E, F_{R,x}, F_{R,y}, T) \in \mathbb{R}^4/E > 0, T > 0, \|F_R\| < cE\}
\end{aligned}
\]

Formalism of (1):

- $W = (E, F_{R,x}, F_{R,y}, T)^T$
- $F(W) = \begin{pmatrix} F_{R,x} & c^2 P_{xx} & c^2 P_{yx} & 0 \\ F_{R,y} & c^2 P_{xy} & c^2 P_{yy} & 0 \end{pmatrix}^T$
- $\gamma(W) = c\sigma^m(W)$
- Limit diffusion equation: *equilibrium diffusion equation*

$$
\partial_t (\rho C_v T + aT^4) - \text{div} \left( \frac{c}{3\sigma r} \nabla (aT^4) \right) = 0
$$
Aim of an AP scheme

Conservation laws:
\[ \partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W) \]

Diffusion equation:
\[ \partial_t w - \text{div}(D(w)\nabla w) = 0 \]

Numerical scheme consistent:
\[ \Delta t, \Delta x \to 0 \]

Limit scheme consistent?
\[ \gamma t \to \infty \]
Example of a non AP scheme in 1D

\[ \partial_t \mathbf{W} + \partial_x (F(\mathbf{W})) = \gamma(\mathbf{W})(R(\mathbf{W}) - \mathbf{W}) \]  

(1)

\[
\mathbf{W} = (\rho, \rho u)^T \quad F(\mathbf{W}) = (\rho u, \rho u^2 + p)^T
\]

\[
\gamma(\mathbf{W}) = \kappa \quad R(\mathbf{W}) = (\rho, 0)^T
\]

\[
\frac{\mathbf{W}_{i+1}^n - \mathbf{W}_i^n}{\Delta t} = - \frac{1}{\Delta x} \left( F_{i+1/2} - F_{i-1/2} \right) + \gamma(\mathbf{W}_i^n)(R(\mathbf{W}_i^n) - \mathbf{W}_i^n)
\]

Limit

\[
\frac{\rho_{i+1}^{n+1} - \rho_{i}^{n}}{\Delta t} = \frac{b_{i+1/2}\Delta x(\rho_{i+1}^{n} - \rho_{i}^{n}) - b_{i-1/2}\Delta x(\rho_{i}^{n} - \rho_{i-1}^{n})}{2\Delta x^2}
\]
control of numerical diffusion:
- telegraph equations: [Gosse and Toscani, 2002],
- M1 model: [Buet and Desprès, 2006], [Buet and Cordier, 2007], [Berthon, Charrier, and Dubroca, 2007], ...
- Euler with gravity and friction:
  [Chalons, Coquel, Godlewski, Raviart, and Seguin, 2010],

ideas of hydrostatic reconstruction used in ‘well-balanced’ scheme used to have AP properties:
- Euler with friction: [Bouchut, Ounaissa, and Perthame, 2007],

using convergence speed and finite differences:
- [Aregba-Driollet, Briani, and Natalini, 2012],

generalization of Gosse and Toscani:
- [Berthon and Turpault, 2011],
- [Berthon, LeFloch, and Turpault, 2013].
State-of-the-art for AP schemes in 2D

- Cartesian and admissible meshes $\Rightarrow$ 1D
- Unstructured meshes:
  1. MPFA based scheme:
     - [Buet, Després, and Franck, 2012],
  2. Using the diamond scheme (Coudière, Vila, and Villedieu) for the limit scheme:
     - [Berthon, Moebs, and Turpault, 2014],
  3. SW with Manning-type friction:
     - [Duran, Marche, Turpault, and Berthon, 2015].
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Aim of the development

- for any 2D unstructured meshes,
- for any system of conservation laws which could be written as (1),
- under a ‘hyperbolic’ CFL condition:
  - stability,
  - preservation of $\mathcal{A}$,
  - preservation of the asymptotic behaviour,

$$\max_{\substack{K \in \mathcal{M} \\ i \in \mathcal{E}_K}} \left( b_{K,i} \frac{\Delta t}{\Delta x} \right) \leq \frac{1}{2}.$$
1 General context and examples

2 Development of an admissibility & asymptotic preserving FV scheme
   - Choice of a limit scheme
   - Hyperbolic part
   - Numerical results for the hyperbolic part
   - Scheme for the complete system
   - Results for the complete system

3 Conclusion and perspectives
Aim of an AP scheme

Conservation laws:
\[ \partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W) \]

Diffusion equation:
\[ \partial_t w - \text{div}(D(w) \nabla w) = 0 \]

Numerical scheme
\[ \Delta t, \Delta x \to 0 \]
consistent?

Limit scheme
\[ \gamma t \to \infty \]
Choice of the limit scheme

FV scheme to discretize diffusion equations:

\[ \partial_t w - \text{div}(D(w) \nabla w) = 0. \]  \hspace{2cm} (2)

Choice: scheme developed in [Droniou and Le Potier, 2011] (DLP)

- conservative and consistent,
- preserves $\mathcal{A}$,
- nonlinear,

\[
(D(w_K) \nabla_i w_K) \cdot n_{K,i} = \sum_{J \in S_{K,i}} \nu_{K,i}^J(w)(w_J - w_K),
\]

- $S_{K,i}$ the set of points used for the reconstruction on edges $i$ of cell $K$,
- $\nu_{K,i}^J(w) \geq 0$. 

Presentation of the DLP scheme

\[ M_{K,i} = \sum_{J \in S_{K,i}} \omega_{K,i}^J \times J \]
\[ M_{L,i} = \sum_{J \in S_{L,i}} \omega_{L,i}^J \times J \]
\[ w_{M_{K,i}} = \sum_{J \in S_{K,i}} \omega_{K,i}^J \times w_J \]
\[ w_{M_{L,i}} = \sum_{J \in S_{L,i}} \omega_{L,i}^J \times w_J \]
Presentation of the DLP scheme

Two approximations

\[ \nabla_i w_K \cdot n_{K,i} = \frac{w_{M_K,i} - w_K}{|K_{M_K,i}|} \]
\[ \nabla_i w_L \cdot n_{L,i} = \frac{w_{M_L,i} - w_L}{|L_{M_L,i}|} \]

Convex combination: \( \gamma_{K,i} + \gamma_{L,i} = 1, \gamma_{K,i} \geq 0, \gamma_{L,i} \geq 0 \)

\[ \nabla_i w_K \cdot n_{K,i} = \gamma_{K,i}(w) \nabla_i w_K \cdot n_{K,i} + \gamma_{L,i}(w) \nabla_i w_L \cdot n_{L,i} \]
\[ = \sum_{J \in S_{K,i}} \nu_{K,i}(w)(w_J - w_K), \text{ with: } \nu_{K,i}(w) \geq 0 \]

Properties of the DLP scheme

- consistent with the diffusion equation on any mesh,
- satisfies the maximum principle.
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Aim of an AP scheme

Conservation laws:
\[ \partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W) \]

Consistent: \( \Delta t, \Delta x \to 0 \)

Diffusion equation:
\[ \partial_t w - \text{div}(D(w)\nabla w) = 0 \]

\( \gamma t \to \infty \)

Numerical scheme \( \gamma t \to \infty \) → Limit scheme

\( \gamma t \to \infty \)

Consistent?
Scheme for the hyperbolic part

\[ W_{K}^{n+1} = W_{K}^{n} - \frac{\Delta t}{|K|} \sum_{i \in E_{K}} F_{i}(W_{K}, W_{L}, \ldots) \cdot n_{K,i} \]  

(3)

**Theorem**

We assume that the conservative flux $F_{i}$ has the following properties:

1. **Consistency:** if $W_{K}^{n} \equiv W$ then $F_{i} \cdot n_{K,i} = F(W) \cdot n_{K,i}$,

2. **Admissibility:**
   
   a. $\exists \nu_{K,i}^{J} \geq 0$, $F_{i} \cdot n_{K,i} = \sum_{J \in S_{K,i}} \nu_{K,i}^{J} F_{KJ} \cdot \eta_{KJ}$
   
   b. $\sum_{i \in E_{K}} |e_{i}| \sum_{J \in S_{K,i}} \nu_{K,i}^{J} \cdot \eta_{KJ} = 0$.

Then the scheme (3) is stable, and preserves $A$ under the classical following CFL condition:

\[ \max_{K \in M} \left( b_{KJ} \frac{\Delta t}{\delta_{KJ}} \right) \leq 1. \]  

(4)
Example of fluxes (with Rusanov)

1. TP flux:

\[ \mathcal{F}_i(W_K, W_L) \cdot n_{K,i} = \frac{F(W_K) + F(W_L)}{2} \cdot n_{K,i} - b_{KL}(W_L - W_K) \]

2. HLL-DLP flux:

\[ \mathcal{F}_i(W_K, W_L, W_J) \cdot n_{K,i} = \sum_{J \in S_{K,i}} \nu_{K,i}^J \left( \frac{F(W_K) + F(W_J)}{2} \cdot \eta_{KJ} - b_{KJ}(W_J - W_K) \right) \]

But...

1. Fully respect the theorem, but not consistent in the diffusion limit
2. Does not respect the second property of admissibility, but consistent in the limit
Procedure to preserve $\mathcal{A}$

1. $\tilde{W}^{n+1}$ is computed with the HLL-DLP flux and with the CFL condition (4),

2. Physical Admissibility Detection (PAD): if $\tilde{W}^{n+1} \in \mathcal{A}$ then the time iterations can continue, else:
   - property 2b is enforced by using the TP flux on all not-admissible cells,
   - $\Delta t$ and $\tilde{W}^{n+1}$ are re-computed with the TP flux.
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### Advection equation

\[
\partial_t W + \text{div}(aW) = 0, \text{ with: } a = (1, 1)^T.
\]

- \( W_0(x, y) = \sin(2\pi x) \sin(2\pi y) \)
- \( W(x, y, t) = \sin(2\pi(x - a_x t)) \sin(2\pi(y - a_y t)) \)

### Convergence results

<table>
<thead>
<tr>
<th>Mesh</th>
<th>HLL-DLP flux</th>
<th>TP flux</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nb. cells</td>
<td>Size</td>
<td>Error</td>
</tr>
<tr>
<td>24 705</td>
<td>6.68 × 10⁻⁴</td>
<td>6.68 × 10⁻²</td>
</tr>
<tr>
<td>98 561</td>
<td>3.34 × 10⁻⁴</td>
<td>3.58 × 10⁻²</td>
</tr>
<tr>
<td>393 729</td>
<td>1.67 × 10⁻⁴</td>
<td>1.86 × 10⁻²</td>
</tr>
<tr>
<td>1 573 889</td>
<td>8.35 × 10⁻⁵</td>
<td>9.49 × 10⁻³</td>
</tr>
<tr>
<td>6 293 505</td>
<td>4.17 × 10⁻⁵</td>
<td>4.81 × 10⁻³</td>
</tr>
</tbody>
</table>
Wind tunnel with step [Woodward and Colella, 1984]

- fluid at Mach 3
- TP flux correction < 1%
2D Riemann problems [Kurganov and Tadmor, 2002]

- TP flux correction < 1%
- No TP flux correction
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Scheme for the complete model

Complete system

\[
\partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W)
\]  

(1)

\[
W^{n+1}_K = W^n_K - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{F}_{K,i} \cdot n_{K,i},
\]

(5)

Construction of \(\overline{F}_{K,i}\)

\[
\overline{F}_{K,i} \cdot n_{K,i} = \sum_{J \in S_{K,i}} \nu_{K,i}^J \overline{F}_{KJ} \cdot \eta_{KJ}, \quad \alpha_{KJ} = \frac{b_{KJ}}{b_{KJ} + \gamma_K \delta_{KJ}},
\]

\[
\overline{F}_{KJ} \cdot \eta_{KJ} = \alpha_{KJ} \overline{F}_{KJ} \cdot \eta_{KJ} - (\alpha_{KJ} - \alpha_{KK}) F(W^n_K) \cdot \eta_{KJ} - (1 - \alpha_{KJ}) b_{KJ}(R(W^n_K) - W^n_K).
\]
Scheme for the complete model

- Is the scheme with the source term AP?
  \[\Rightarrow \text{generally not} \ldots\]

Equivalent formulation

Rewrite (1) into:

\[
\partial_t W + \text{div}(F(W)) = \gamma(W)(R(W) - W),
\]

\[
= \gamma(W)(R(W) - W) + (\bar{\gamma} - \gamma)W,
\]

\[
\partial_t W + \text{div}(F(W)) = (\gamma(W) + \bar{\gamma})(\bar{R}(W) - W).
\]

with:

- \(\gamma(W) + \bar{\gamma} > 0\)
- \(\bar{R}(W) = \frac{\gamma R(W) + \gamma W}{\gamma + \bar{\gamma}}\)
Limit scheme

Introduction of $\bar{\gamma}$

$$\partial_t W + \text{div}(F(W)) = (\gamma(W) + \bar{\gamma}(W))(R(W) - W)$$

Rescaling

$$\begin{cases} 
\gamma &\leftarrow \frac{\gamma}{\varepsilon} \\
\Delta t &\leftarrow \frac{\Delta t}{\varepsilon}
\end{cases}$$

- $\varepsilon^{-1}$: $R(W) = W$
- $\varepsilon^0$:

$$W^{n+1}_K = W^n_K - \sum_{i \in \varepsilon_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in S_K,i} \frac{\nu^{J,i}_K}{\gamma_K + \bar{\gamma}^{J,i}_K} \left[ \frac{b_{KJ}}{\delta_{KJ}} F_{KJ} - \left( \frac{b_{KJ}}{\delta_{KJ}} - \frac{b_{KK}}{\delta_{KK}} \right) F(W^n_K) \right] \cdot \eta_{KJ} |_{R(W)=W}$$
Limit scheme for the Euler equations

- $R(U) = U \Rightarrow \rho u = \rho v = 0$
- $\gamma + \bar{\gamma} := \kappa + \bar{\kappa}$

Actual limit scheme

$$\rho_k^{n+1} = \rho_k^n + \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \frac{b_{KJ}^2}{2(\kappa_K + \bar{\kappa}_{K,i}^J)} \delta_{KJ} (\rho_J - \rho_K).$$

AP correction $\bar{\kappa}$

$$\frac{\nu_{K,i}^J b_{KJ}^2}{2(\kappa_K + \bar{\kappa}_{K,i}^J) \delta_{KJ}} (\rho_J - \rho_K) = \frac{\nu_{K,i}^J}{\kappa} (p_J - p_K) \Rightarrow \kappa_K + \bar{\kappa}_{K,i}^J = \kappa_K \frac{\nu_{K,i}^J b_{KJ}^2}{2 \nu_{K,i}^J \delta_{KJ} p_J - p_K} (\rho_J - \rho_K)$$

$$\rho_k^{n+1} = \rho_k^n + \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \frac{\nu_{K,i}^J}{\kappa} (p_J - p_K) \longrightarrow \partial_t \rho - \text{div} \left( \frac{1}{\kappa} \nabla p(\rho) \right) = 0.$$
Theorem

\[ W_{K}^{n+1} = W_{K}^{n} - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \bar{F}_{K,i} \cdot n_{K,i} \]

\[ = \sum_{J \in \mathcal{E}_K} \omega_{KJ} \left( W_{K}^{n} - \frac{\Delta t}{\delta_{KJ}} [\tilde{F}_{KJ} - \tilde{F}_{KK}] \cdot \eta_{KJ} \right) \]  

The scheme (5) is consistent with the system of conservation laws (1), under the same assumptions of the previous theorem. Moreover, it preserves the set of admissible states \( \mathcal{A} \) under the CFL condition:

\[ \max_{K \in \mathcal{M}} \left( \frac{b_{KJ}}{\delta_{KJ}} \frac{\Delta t}{\eta_{KJ}} \right) \leq 1. \]
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\[
\rho_0(x, y) = 0.1 \exp \left( \left( \frac{x-0.5}{0.01} \right)^2 \right) + 0.1, \quad u = 0, \quad \kappa = 2000, \quad t_f = 10
\]
Comparison in the diffusion limit (2D)

\[ \rho_0(x, y) = \begin{cases} 
1 & \text{if } (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < 0.1^2 \\
0.1 & \text{otherwise}
\end{cases}, \quad u = 0, \quad \kappa = 2000, \quad t_f = 10 \]
Comparison in the diffusion limit (2D)

(a) HLL-DLP-AP
(b) DLP
(c) HLL-DLP-NoAP
(d) HLL-TP
Space probe test case (Euler-$M_1$)

Figure: Density: $\rho$

Figure: Pressure: $p$
Space probe test case (Euler-\(M_1\))

**Figure:** Opacity: \(\sigma^f\)

**Figure:** Anistropy factor: \(f = \frac{\|F_R\|}{cE}\)
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Conclusion

- generic theory for various hyperbolic problems with asymptotic behaviours,
- first order scheme that preserve $A$ and the asymptotic limit.

Perspectives

- extend the limit scheme to take care of diffusion systems and nonlinear diffusion equation,
- high-order schemes.
Thanks for your attention.