Conservative, skew symmetric compact finite difference schemes for the compressible Navier Stokes equations

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What’s it about?

Supersonic Jet Noise
Screech - spectrum

1. mixing noise
2. shock noise
3. screech

experiment (Seiner 1984)
Jet – noise

- Schlieren visualization of an overexpanded jet.

Van Dyke
Jet – noise

- Schlieren visualization of an overexpanded jet.

shock – diamonds
Jet – noise

- Schlieren visualization of an overexpanded jet.

1. noise source

vortex-pairing
Jet – noise

- Schlieren visualization of an overexpanded jet.
Jet – noise

- Schlieren visualization of an overexpanded jet.

\[ \text{screech} \Rightarrow \text{noise up to 160 dB!} \]
Jet – noise

- Schlieren visualization of an overexpanded jet.

\[ \text{screech} \Rightarrow \text{noise up to 160 dB!} \]
Project – objective
Shape optimization – Screech minimization

- Shape optimization of a jet-nozzle to reduce jet noise.
Objective function – Minimization of screech

- Objective function in frequency space.

\[ J = \lambda_1 \int_{k_u}^{k_0} \left\{ \int_{-\infty}^{\infty} e^{-ikt} p'(t) dt \right\}^2 dk \]

\[ + \lambda_2 \int_{k_u}^{k_0} \left\{ \frac{\partial}{\partial k} \int_{-\infty}^{\infty} e^{-ikt} p'(t) dt \right\}^2 dk \]
Adjoint shape optimization.
Computing Supersonic Jet Noise

DNS/LES of the Flow Field

- High order schemes, preserving the dispersion relation
- Truly nonreflecting boundary condition
- Proper shock treatment / Shock turbulence interaction
Computing Supersonic Jet Noise

DNS/LES of the Flow Field
- High order schemes, preserving the dispersion relation
- *Truely* nonreflecting boundary condition
- Proper shock treatment / Shock turbulence interaction

Half-Way Solution:
Computing Supersonic Jet Noise

DNS/LES of the Flow Field
- High order schemes, preserving the dispersion relation
- \textit{Truly} nonreflecting boundary condition
- Proper shock treatment / Shock turbulence interaction

Understanding/Optimization/Control of Jet Noise:
- Modal analysis
- Adjoint equations (with proper shock treatment)
- Skew-symmetric formulation
- Optimization algorithm
- Feedback Control
Computing Supersonic Jet Noise

**DNS/LES of the Flow Field**
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- *Truely* nonreflecting boundary condition
- Proper shock treatment / Shock turbulence interaction

**Understanding/Optimization/Control of Jet Noise:**
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- **Adjoint equations** (with proper shock treatment)
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- Feedback Control
Overview

1. Objectives

2. Some DNS Results
   - Supersonic Jet Noise
   - Adjoint Optimization

3. Fully Conservative, Skew Symmetric FD Schemes
   - Need for skew-symmetric, conservative formulations
   - What does Skew Symmetry mean?
   - Motivating Example: Burgers’ Equation
   - Euler/Navier-Stokes Equations

4. Conclusion
Overview

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4 Conclusion
Direct Numerical Simulation of a supersonic jet

Nozzle: velocity profile

- planar jet
- $6^{th}$ order compact in $x$ and $y$; $z$ (periodic)
- $4^{th}$ order Runge Kutta or exponential-Krylov in time.

Nozzle: velocity profile

- Inlet 2 Joukowsky profile
- Inlet 1 velocity profile
- grid-stretching
- filter
- sponge
- boundary wave acceleration
Direct Numerical Simulation of a supersonic jet

Nozzle: velocity profile

- $M_{\text{Jet}} = 1.35$
- $p_{\text{Jet}} / p_\infty = 0.9$ (over-expanded)
- $Re_D = 30000$

- $2040 \times 1020 \times 144 \approx 300 \cdot 10^6$ points @ 1020 CPU’s
- 100 000 CPUh
- SGI-Altix 4700 @ LRZ

(example)
Direct Numerical Simulation of a supersonic jet
Nozzle: velocity profile

Spectrum

- SPL [dB]
- $f / f_0$
- •: semi-empirical law; frequency only (Powell 1953)

Time history

- $p'/p_{max}$
- $t/T_s$

Sesterhenn (TU Berlin)  Skew symmetric compact FD schemes  June, 2010  13 / 92
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4. Conclusion
Why use Adjoint methods?

- Solution of $\mathcal{J} = \langle g, u \rangle$ subject to

  $$\mathcal{L} u = f$$

  for $p$ RHS $f$ will cost $p$ solutions + 1 scalar product

- Alternatively solve

  $$\mathcal{L}^* v = g$$

  with $\langle u, \mathcal{L}^* v \rangle := \langle \mathcal{L} u, v \rangle$

- Then

  $$\langle g, u \rangle = \langle \mathcal{L}^* v, u \rangle = \langle v, \mathcal{L} u \rangle = \langle v, f \rangle$$

  for $p$ RHS $f$ will cost 1 solution + $p$ scalar products
Objective Function

\[ J = \lambda_1 \int_{k_u}^{k_o} \left\{ \int_{-\infty}^{\infty} e^{-ikt} u(t) dt \right\}^2 dk + \lambda_2 \int_{k_u}^{k_o} \left\{ \frac{\partial}{\partial k} \int_{-\infty}^{\infty} e^{-ikt} u(t) dt \right\}^2 dk \]
Adjoint Methods for Optimization

Problem Formulation

Cost function

\[ J = \int q^T Q q \, dt \]

\(q\): state-vector

- need to satisfy constraints like: governing equations (e.g. Navier–Stokes) and boundary conditions.

Constraint: Full nonlinear state equations

\[ \mathcal{N}(q, \Phi) = 0 \]

\(\mathcal{N}\): nonlinear operator; e.g. Navier–Stokes

\(\Phi\): control
Lagrange function (enforce variables)

\[ \mathcal{L} = J - \langle \mathcal{N}(q, \Phi), q^+ \rangle \]

\( q^+ \): adjoint variable (Lagrange multiplier)

\[ \langle p, q \rangle = \int \int p q d\Omega dt \]

Optimality condition

\[ \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \]

\[ \frac{\partial J}{\partial \Phi} = \left\langle \frac{\partial}{\partial \Phi} \mathcal{N}(q, \Phi), q^+ \right\rangle \]

gradient depending on adjoint state \( q^+ \)!
Adjoint Methods for Optimization

Adjoint Equation

\[ \frac{\partial L}{\partial q} = 0 \]

\[ \Rightarrow \mathcal{N}^+(q, q^+) = 0 \]

Steepest descent

\[ \Phi^{n+1} = \Phi^n - \alpha^n \frac{\partial J}{\partial \Phi} \]
Some Issues

- **Adjoint equations** need to be solved **backwards in time**.

- Adjoint equations depend on forward solution $q (N^+(q, q^+))$.

- Implementation of adjoint equation can be three times as expensive as direct equation.
Adjoint Methods for Optimization
Example: Burgers Equation

Burgers–equation

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{1}{Re} \frac{\partial^2 u}{\partial x^2} = 0 \]

Adjoint Burgers–equation

\[ - \frac{\partial u^+}{\partial t} - u \frac{\partial u^+}{\partial x} - \frac{1}{Re} \frac{\partial^2 u^+}{\partial x^2} = 0 \]
Optimization method
Conjugate Gradient

- increase convergence for optimization areas with narrow and flat valleys.

- e.g.: Polak–Ribiere Algorithm
Adjoint Methods for Optimization

Optimization Loop

- Adjoint shape optimization.
Adjoint Shape Optimization

Optimization Iter: $l = 0$

Optimization Iter: $l = 4$
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4. Conclusion
Aeroacoustic Radiation by Shock Turbulence Interaction

\[
\begin{align*}
\frac{y}{\delta_\omega(0)} & \quad 78.6 \\
16.9 & \quad 0.0 \\
0.0 & \quad 0.0 \\
& \quad \frac{x}{\delta_\omega(0)} \\
& \quad 70.7 \\
& \quad 127.1
\end{align*}
\]
**RATIO OF KOLMOGOROV SCALE TO SHOCK THICKNESS for weak shocks:**

\[
\delta_u \approx \frac{\nu}{c} \frac{6.89}{M - 1}
\]

approximation for isotropic turbulence:

\[
\varepsilon \approx 15 \frac{\nu u_{rms}^2}{\lambda^2}
\]

\[
\frac{\eta}{\delta_u} \approx \frac{(\nu^3/\varepsilon)^{1/4}}{\nu/c} \frac{(6.89/(M_1 - 1))}{1 M_1 - 1 M_t \sqrt{Re_\lambda}}
\]
Skew Symmetric discretisation

Aim: Calculation and optimisation of flows with shocks and sound

Needed:
- High precision/High order schemes
- Numerical efficient code
- Precise adjoints
→ Finite difference schemes

But: Correct shock treatment relies on conservation laws.

Therefore a conserving high order FD scheme is needed.
→ Skew symmetric schemes
Talking *not* about Gibbs phenomenon
Skew Symmetric discretisation

The core ideas

Skew symmetry . . .

- is defined with respect to a scalar product:

\[(u, Av) = (A^\dagger u, v) \equiv -(Au, v)\]

- is a property of many operators, e.g.

\[\int u(x) \partial_x v(x) \, dx = - \int (\partial_x u(x)) \, v(x) \, dx + [B. T.]\]

- is defined for linear and non-linear operators, e.g.

\[A = [v(x) \, \partial_x + \partial_x \, v(x)]\]

- is the key to conservation properties
"$\partial_x$ is skew-symmetric"

$$(\partial_x u, v) = - (u, \partial_x v) + B.T.$$ 

"$\partial_{xx}$ is symmetric"

$$(\partial_{xx} u, v) = - (\partial_x u, \partial_x v) = (u, \partial_{xx} v)$$
Implications for $u_t + \lambda u_x = 0$

\[
\frac{1}{2} \partial_t (u, u) = \frac{1}{2} \{(\partial_t u, u) + (u, \partial_t u)\} \\
= \frac{1}{2} \{(-\lambda \partial_x u, u) + (u, -\lambda \partial_x u)\} \\
= -\frac{\lambda}{2} \{(\partial_x u, u) + (u, \partial_x u)\} = 0
\]

\[\rightarrow\] Conservation of energy!
Discretized equation

\[
\dot{u} + \lambda Du = 0
\]

\[(u, v) := u^T v\]

\[
\frac{1}{2} \frac{d}{dt} (u, u) = \frac{1}{2} \{(\dot{u}, u) + (u, \dot{u})\}
\]

\[
= \frac{1}{2} \{(-\lambda Du, u) + (u, -\lambda Du)\}
\]

\[
= -\frac{\lambda}{2} \{(Du)^T u + u^T Du\}
\]

\[
= -\frac{\lambda}{2} \{u^T (D^T + D) u\}
\]

\[\longrightarrow \text{Conservation of energy if } D^T = -D!\]
Skew Symmetric discretisation

Skew symmetry in the discrete case . . .
- corresponds to a skew symmetric matrix
- is often violated, e.g.:

\[
D^u = \frac{1}{\Delta x} \begin{pmatrix}
1 & -1 & & \\
0 & 1 & -1 & \\
& \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad D^c = \frac{1}{\Delta x} \begin{pmatrix}
0 & -1 & & \\
1 & 0 & -1 & \\
& 1 & \ddots & \ddots \\
\end{pmatrix}
\]

- is often violated in for non-linear terms, as product rule is violated
- implies to the conservation of a discrete energy
- can be constructed by a Galerkin ansatz for the linear and non-linear case, were analytic properties imply discrete structures
How about $\partial_t + \partial_x u^2/2 = 0$? Is $(\partial_x u)$ skew-symmetric?

$$(v, (\partial_x u)w) =: \int v(\partial_x u)w \, dx = - \int \partial_x v(uw) \, dx$$

$$= (u\partial_x, w)$$

$\rightarrow$ No!

$$(\partial_x u)^\dagger = -(u\partial_x)$$
Skew Symmetric discretisation

Example: The Burgers’ Equation

\[
0 = \partial_t u + \partial_x \left( \frac{u^2}{2} \right) = \partial_t u + u \partial_x u \quad \Rightarrow \quad \partial_t u + \frac{1}{3} [u \partial_x + \partial_x u] u = 0
\]

The (kinetic) energy is \( E = \frac{1}{2} (u, u) = \frac{1}{2} \int u^2 \, dx \)

\[
\partial_t E = (u, \partial_t u) \\
\partial_t E = -\frac{1}{3} (u, [u \partial_x + \partial_x u] u) \\
\partial_t E = +\frac{1}{3} ([u \partial_x + \partial_x u] u, u) - \frac{2}{3} [u^3]_{x_0}^{x_1}
\]

Thus, we get \( \partial_t E = (u, \partial_t u) = -\frac{1}{3} [u^3]_{x_0}^{x_1} \)

Skew symmetry implies conservation
Discrete Version $\frac{1}{2}(DU)u$

define $U := \text{diag}(u)$

$$(DU)^T = U^T D^T = -UD^T$$

NOT $-DU$!!!
Energy Conservation for $\dot{u} + \frac{1}{3}(UD + DU)u = 0$

\[
\frac{d}{dt} u^T u = \dot{u}^T u + u^T \dot{u} = -\frac{1}{3} \left\{ ((UD + DU)u)^T u + u^T (UD + DU)u \right\} = -\frac{1}{3} \left\{ u^T (-DU - UD + UD + DU) u \right\} = 0
\]
Momentum Conservation for $\dot{u} + \frac{1}{3}(UD + DU)u = 0$

$$\frac{d}{dt}1^Tu = -\frac{1}{3}\left\{((1^T(UD + DU)u)\right\}$$

$$= -\frac{1}{3}\left\{\begin{array}{c}
  u^TDu \\
  = 0 \text{ (skew-symmetry)} \\
  1^TD \\
  = 0 \text{ (telescope-sum)} \\
  Uu
\end{array}\right\}$$

$$= 0$$
Example: Finite Volume Discretization

\[
\frac{1}{\Delta x} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}
\]

In general

\[1^T D = 0\]
Taylor Series

\[ u(x \pm h) = u + u'h + u''h^2/2 + u'''h^3/6 + \ldots \]

leads via \( \alpha u_{i-1} + \beta u_i + \gamma u_{i+1} \) to

\[ (\alpha + \beta + \gamma) u_i + h(-\alpha + \gamma)u'_i + \ldots \]

\[ D \cdot 1 = 0 \]

Constency + Skew symmetry implies telescope-property
Time Integration

Integrate

\[ \partial_t u + \frac{1}{3}(DU + UD)u = 0 \]

Use trapezoidal rule

\[ \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3}(DU^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D)u^{n+\frac{1}{2}} = 0 \]

With

\[ u^{n+\frac{1}{2}} := \frac{u^{n+1} + u^n}{2} \]
\[
\left( u^{n+\frac{1}{2}} \right)^T \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3} \left( u^{n+\frac{1}{2}} \right)^T \left( DU^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D \right) u^{n+\frac{1}{2}} = 0
\]

With

\[
\frac{u^{n+1} + u^n}{2} \frac{u^{n+1} - u^n}{\Delta t} = \frac{(u^{n+1})^2 - (u^n)^2}{2\Delta t} = 0
\]
Skew Symmetric discretisation
Galerkin ansatz

\[
f(x) \approx \tilde{f}(x) = \sum_{i=1}^{N} \hat{f}_i \tau_i(x)
\]

where the amplitudes \( \hat{f}_i \) are selected by a consistency condition, in our case the collocation condition

\[
\tilde{f}(x_i) \equiv f(x_i)
\]

This defines a matrix transformation for the coefficients and the function values

\[
f = T\hat{f}_j
\]

with transformation matrix

\[
T_{ij} = \tau_j(x_i)
\]
Skew Symmetric discretisation

Galerkin ansatz

Operators a discretised by mapping on the ansatz-functions (testfunctions=ansatzfunctions), e.g., \( f(x) = \partial_x F(x) \)

\[
\int_{x_0}^{x_1} vf \, dx = \int_{x_0}^{x_1} v \partial_x F \, dx
\]

\[
\sum_{ij} \hat{v}_i \hat{f}_j \int t_i(x) t_j(x) \, dx = \sum_{ij} \hat{v}_i \hat{F}_j \int t_i(x) \partial_x t_j(x)
\]

\[
\hat{S}_{ij} \hat{f} = \hat{A}_{ij} \hat{F}
\]

or \( D = S^{-1} A \). \( S \) defines the scalar product.
Skew Symmetric discretisation
Galerkin ansatz

Operators a discretised by mapping on the ansatz-functions (testfunctions=ansatzfunctions), e.g., \( f(x) = \partial_x F(x) \)

\[
\int_{x_0}^{x_1} v f dx = \int_{x_0}^{x_1} v \partial_x F dx
\]

\[
\sum_{ij} \hat{v}_i \hat{f}_j \hat{S}_{ij} = \sum_{ij} \hat{v}_i \hat{F}_j \hat{A}_{ij}
\]

\[
S \hat{f} = \hat{A} \hat{F}
\]

or \( D = S^{-1} A \). S defines the scalar product.
Skew Symmetric discretisation
Galerkin ansatz

The derivative $S\hat{f} = \hat{A}\hat{F}$

- $A$ is antisymmetric

$$A_{ij} = \int x t_i(x) \partial_x t_j(x) = -\int x t_j(x) \partial_x t_i(x) + [B.T.]$$

- $D = S^{-1}A$ is antisymmetric with respect to $S$

$$(u, Dv)_S = u^t SDv = u^t SS^{-1} Av = u^t Av = -v^t SS^{-1} Au = -(v, Du)_S$$

- Complete freedom in choosing the ansatz functions $\tau_i$
- $S\hat{f} = \hat{A}\hat{F}$ is compact derivative for $\tau(x)$ with compact support
Skew Symmetric discretisation
Galerkin ansatz: B-Splines

b-splines have compact support:

Figure: B-Spline and its first derivative

$A$ and $S$ have band structure: Derivative $\hat{S}f = \hat{A}\hat{F}$ is found to be

$$
(f_{i+1} + f_{i-1}) + \frac{1}{120} (f_{i+2} + f_{i-2}) = \frac{5}{12} (F_{i+1} - F_{i-1}) + \frac{1}{24} (F_{i+2} - F_{i-2})
$$

→ compact derivative, 6th order
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4. Conclusion
Skew Symmetric discretisation

Burgers’ Equation

Same procedure for Burgers equation (in skew symmetric notation)

\[ \partial_t u + \frac{1}{3} [\partial_x u + u \partial_x] u - \nu (\partial_x)^2 u = 0 \]

leading to

\[ \partial_t \hat{u}_j S_{ij} + \frac{1}{3} \hat{u}_j \hat{u}_k A_{ijk} = \nu \hat{u}_j A^2_{ij} \]

with

\[ \hat{S}_{ij} = \int \tau_i(x) \tau_j(x) \, dx, \]

\[ \hat{A}_{ijk} = \int \tau_i(x) \left( \partial_x \tau_j(x) + \tau_j(x) \partial_x \right) \tau_k(x) \, dx \]

\[ \hat{A}^2_{ij} = \int \tau_i(x) (\partial_x)^2 \tau_j(x) \, dx \]
Skew Symmetric discretisation
Burgers’ Equation

\[ \hat{A}_{ijk} = \int \tau_i(x) \left( \partial_x \tau_j(x) + \tau_j(x) \partial_x \right) \tau_k(x) \, dx \]

antisymmetric by definition for given \( j \):

\[
\hat{A}_j = \begin{pmatrix}
0 & \frac{7}{240} & \frac{1}{80} & 0 & 0 \\
-\frac{7}{240} & 0 & \frac{31}{80} & \frac{7}{120} & 0 \\
-\frac{1}{80} & -\frac{31}{80} & 0 & \frac{31}{80} & \frac{1}{80} \\
0 & -\frac{7}{120} & -\frac{31}{80} & 0 & \frac{7}{240} \\
0 & 0 & -\frac{1}{80} & -\frac{7}{240} & 0 \\
\end{pmatrix}
\]
Skew Symmetric discretisation

Burgers’ Equation

Construct $u$-dependent matrix:

$$\hat{A}\hat{u} = \sum_i \hat{u}_i A_i$$

$$\hat{A}\hat{u} = \begin{pmatrix} u_1 & \cdot & \cdot \\ \cdot & u_2 & \cdot \\ \cdot & \cdot & u_3 \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$
Skew Symmetric discretisation

Burgers’ Equation

Construct $u$-dependent matrix:

$$\hat{A}^u = \sum_i \hat{u}_i A_i$$

$$A^u = \begin{pmatrix} 0 & +a_i^+ & +b_i^+ \\ -b_i^- & -a_i^- & 0 & +a_i^+ & +b_i^+ \end{pmatrix} = -(A^u)^T$$

with

$$a_i^\pm = \frac{7}{240} u^{i\pm 2} + \frac{31}{80} u^{i\pm 1} + \frac{31}{80} u^i + \frac{7}{240} u^{i\mp 1}$$

$$b_i^\pm = \frac{1}{80} u^i + \frac{7}{120} u^{i\pm 1} + \frac{1}{80} u^{i\pm 2}$$
Skew Symmetric discretisation

Burgers’ Equation

\[ \hat{S}\partial_t \hat{u} + \frac{1}{3} A\hat{u} \hat{u} = \nu A^2 \hat{u} \]

- Derivative is 6\textsuperscript{th} order for const deriv, 4\textsuperscript{th} for mixed terms
- Still needed: Time Integration scheme → later
- Energy conserving!
Skew Symmetric discretisation

Burgers’ Equation

\[ \hat{S} \partial_t \hat{u} + \frac{1}{3} A \hat{u} \hat{u} = \nu A^2 \hat{u} \]

- Derivative is 6\textsuperscript{th} order for const deriv, 4\textsuperscript{th} for mixed terms
- Still needed: Time Integration scheme → later
- Energy conserving!
- ... but what about momentum conservation?

Remember starting point:

\[ \partial_t u + \frac{1}{3} [\partial_x u + u \partial_x] u - \nu (\partial_x)^2 u = 0 \]

Not in divergence form!
Skew Symmetric discretisation

Burgers’ Equation: Momentum Conservation

Assume

\[ \tilde{1}(x) = \hat{1}_i \tau(x) \equiv 1 \forall x \]

then

\[ \sum_i \hat{1}_i \hat{A}_{ijk} = \int 1 \left( \partial_x \tau_j + \tau_j \partial_x \right) \tau_k dx = \int \tau_j \partial_x \tau_k + [B.T.] = \hat{A}_{jk} + [B.T.] \]

and therefore \( 1^t A^u u = u^t A u + [uu]_x = 2/3[uu]_x \)

\[
0 = (\hat{1}, \hat{S} \partial_t \hat{u}) + \frac{1}{3}(\hat{1}, \hat{A} \hat{u} \hat{u}) - \nu(\hat{1}, A^2 \hat{u})
\]

\[
= \partial_t (\hat{1}, \hat{S} \hat{u}) + \frac{1}{2} \hat{u}_j \hat{u}_k [\tau_j(x) \tau_k(x)] - \nu \hat{u}_j [\partial_x \tau_j(x)]
\]

This is a generalised telescoping sum property.
Skew Symmetric discretisation
Numerical results: Burgers’ Equation

Time integration by midpoint rule\(^1\)

Skew Symmetric discretisation
Numerical results: Burgers’ Equation

Time integration by midpoint rule\textsuperscript{2}

How does conservative FD compare with FV?

Example: Burgers’ Equation:

<table>
<thead>
<tr>
<th>conservation</th>
<th>conservative FD</th>
<th>Finite Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>momentum</td>
<td>by telescoping sum</td>
<td>enforced by use of fluxes</td>
</tr>
<tr>
<td>energy</td>
<td>by skew symmetry</td>
<td>usually not fulfilled</td>
</tr>
</tbody>
</table>

Both schemes have correct shock treatment, but conservative FD has inherent stability!
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Skew Symmetric discretisation
Euler Equations

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x p &= 0 \\
\partial_t (\rho e + \rho u^2 / 2) + \partial_x \left( \rho u (e + p/\rho + u^2 / 2) \right) &= 0
\end{align*}
\]
Skew Symmetric discretisation

Euler Equations

\[
\begin{align*}
\partial_t \varrho + \partial_x (\varrho u) &= 0 \\
\partial_t (\varrho u) + \partial_x (\varrho u^2) + \partial_x p &= 0 \\
\partial_t (\varrho e + \varrho u^2/2) + \partial_x \left( \varrho u (e + p/\varrho + u^2/2) \right) &= 0
\end{align*}
\]

(Anti)-Symmetries:

\[
\begin{align*}
\partial_t \varrho + \partial_x (\varrho u) &= 0 \\
\frac{1}{2} \left( \partial_t \varrho + \varrho \partial_t \right) u + \frac{1}{2} \left( \partial_x u \varrho + u \varrho \partial_x \right) u + \partial_x p &= 0 \\
\frac{1}{\gamma - 1} \partial_t p + \frac{\gamma}{\gamma - 1} \partial_x (up) - u \partial_x p &= 0
\end{align*}
\]
Skew Symmetric discretisation
Euler Equations

\[
\begin{align*}
\frac{1}{2} (\partial_t \rho + \rho \partial_t) u + \frac{1}{2} (\partial_x u \rho + u \rho \partial_x) u + \partial_x p &= 0 \\
\frac{1}{\gamma - 1} \partial_t p + \frac{\gamma}{\gamma - 1} \partial_x (up) - u \partial_x p &= 0
\end{align*}
\]
Skew Symmetric discretisation

Euler Equations

\[
\begin{aligned}
\partial_t \varrho + \partial_x (\varrho u) &= 0 \\
\frac{1}{2} (\partial_t \varrho + \varrho \partial_t) u + \frac{1}{2} (\partial_x u \varrho + u \varrho \partial_x) u + \partial_x p &= 0 \\
\frac{1}{\gamma - 1} \partial_t p + \frac{\gamma}{\gamma - 1} \partial_x (u p) - u \partial_x p &= 0
\end{aligned}
\]

with the Galerkin ansatz

\[
\begin{aligned}
\hat{S} \partial_t \hat{\varrho} + \hat{B} \hat{u} \hat{\varrho} &= 0 \\
[\partial_t \hat{S} \hat{\varrho} + \hat{S} \hat{\varrho} \partial_t] \hat{u} + \hat{A} \hat{u} \hat{u} + 2 \hat{A} \hat{p} &= 0 \\
\frac{1}{\gamma - 1} \hat{S} \partial_t \hat{p} + \frac{\gamma}{\gamma - 1} B \hat{u} \hat{p} - C \hat{u} \hat{p} &= 0
\end{aligned}
\]
Skew Symmetric discretisation

Navier-Stokes Equations

Navier-Stokes Equation is similar

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\frac{1}{2} (\partial_t \rho + \rho \partial_t) u + \frac{1}{2} (\partial_x u \rho + u \rho \partial_x) u + \partial_x p &= \mu \partial_x^2 u \\
\frac{1}{\gamma - 1} \partial_t p + \frac{\gamma}{\gamma - 1} \partial_x (up) - u \partial_x p &= \mu \partial_x u \partial_x u + \lambda \partial_x T 
\end{align*}
\]

with the Galerkin ansatz

\[
\begin{align*}
\hat{S} \partial_t \hat{\rho} + \hat{B} \hat{u} \hat{\rho} &= 0 \\
[\partial_t \hat{S} \hat{\rho} + \hat{S} \partial_t] \hat{u} + \hat{A} \hat{\rho} \hat{u} + 2 \hat{A} \hat{p} &= \mu \hat{A}^2 \hat{u} \\
\frac{1}{\gamma - 1} \hat{S} \partial_t \hat{p} + \frac{\gamma}{\gamma - 1} \hat{B} \hat{u} \hat{p} - \hat{C} \hat{u} \hat{p} &= \mu \hat{C} \hat{u} \hat{u} + \lambda \hat{A} T (\hat{\rho}, \hat{p})
\end{align*}
\]
Skew Symmetric discretisation

Euler Equations

\[
\hat{S} \partial_t \hat{\rho} + \hat{B} \hat{u} \rho = 0
\]

\[
\left[ \partial_t \hat{S} \hat{\rho} + \hat{S} \partial_t \hat{\rho} \right] \hat{u} + \hat{A} \hat{u} \rho + 2 \hat{A} \hat{p} = 0
\]

\[
\frac{1}{\gamma - 1} \hat{S} \partial_t \hat{p} + \frac{\gamma}{\gamma - 1} \hat{B} \hat{u} \rho - \hat{C} \hat{u} \rho = 0
\]

What is new

- set of equations: conservation of energy depends on momentum
depends on mass equation

- non-constant time derivative
Skew Symmetric discretisation
Euler Equations: Time integration

Galerkin ansatz in time $u_i(t) = \sum_n u_i^n \omega^n(t)$
with linear splines in time, $u_i(t_n) = \frac{1}{2}(u_i^{n-1/2} + u_i^{n-1/2})$

$$\hat{S}\partial_t \hat{\rho} + \hat{B} \hat{u} \hat{\rho} = 0$$

becomes

$$\mathbf{A}^{pq} \hat{S}^{q} + S^{pqr} \hat{B}^{q} \hat{\rho}^{r} = 0$$
Skew Symmetric discretisation

Euler Equations: Time integration

Galerkin ansatz in time $u_i(t) = \sum_n u^n_i \omega^n(t)$

with linear splines in time, $u_i(t_n) = \frac{1}{2}(u_i^{n-1/2} + u_i^{n-1/2})$

\[
\hat{S}\partial_t \hat{\rho} + \hat{B} \hat{u} \hat{\rho} = 0
\]

becomes

\[
\frac{1}{2\Delta t} \hat{S} \left( \hat{\rho}^{n-1} - \hat{\rho}^{n+1} \right) = \\
\frac{1}{12} \left( \hat{B}^{\hat{u}^{n-1}} + \hat{B}^{\hat{u}^n} \right) \hat{\rho}^{n-1} + \frac{1}{12} \left( \hat{B}^{\hat{u}^{n-1}} + 6\hat{B}^{\hat{u}^n} + \hat{B}^{\hat{u}^{n+1}} \right) \hat{\rho}^n + \frac{1}{12} \left( \hat{B}^{\hat{u}^n} + \hat{B}^{\hat{u}^{n+1}} \right) \hat{\rho}^{n+1}
\]

Procedure is 2 Step, symmetric, conserving and implicit.
Skew Symmetric discretisation

Euler Equations: Conserved quantities

Mass

\[ \frac{1}{2} \hat{S} \left( (\hat{\rho}^{N+1} + \hat{\rho}^{N}) - (\hat{\rho}^{0} + \hat{\rho}^{1}) \right) = 0 \]

Momentum

\[ \frac{1}{4} \left( \hat{\rho}^{N-2} S + \hat{\rho}^{N-1} S \right) \hat{u}^{N-1} + \frac{1}{4} \left( \hat{\rho}^{N-1} S + \hat{\rho}^{N} S \right) \hat{u}^{N} - LB + dt \cdot R(N, N - 1) \]

and Energy

\[ \frac{1}{\gamma - 1} \hat{S} \left( \hat{\rho}^{N} + \hat{\rho}^{N} \right) + \frac{1}{4} \hat{u}^{N} \left( \hat{S}^{\hat{\rho}^{N-1}} + \hat{S}^{\hat{\rho}^{N-1}} \right) \hat{u}^{N} - LB + dt \cdot R(N, N - 1) \]
Skew Symmetric discretisation

Euler Equations: Numerical results
Skew Symmetric discretisation
Euler Equations: Numerical results

Energy conserved to $10^{-15}$
Skew Symmetric discretisation

Euler Equations: Time integration

\[
\frac{1}{2\Delta t} \hat{S} \left( \hat{\rho}^{n-1} - \hat{\rho}^{n+1} \right) = \\
\frac{1}{12} \left( \hat{B}^{\hat{u}^{n-1}} + \hat{B}^{\hat{u}^{n}} \right) \hat{\rho}^{n-1} + \frac{1}{12} \left( \hat{B}^{\hat{u}^{n-1}} + 6\hat{B}^{\hat{u}^{n}} + \hat{B}^{\hat{u}^{n+1}} \right) \hat{\rho}^{n} \\
+ \frac{1}{12} \left( \hat{B}^{\hat{u}^{n}} + \hat{B}^{\hat{u}^{n+1}} \right) \hat{\rho}^{n+1}
\]

The energy equation is similar \((D^{u} = \gamma B^{\hat{u}} + (1 - \gamma) C^{\hat{u}})\)

\[
\frac{1}{2\Delta t} \hat{S} \left( \hat{\rho}^{n-1} - \hat{\rho}^{n+1} \right) = \\
\frac{1}{12} \left( \hat{D}^{\hat{u}^{n-1}} + \hat{D}^{\hat{u}^{n}} \right) \hat{\rho}^{n-1} + \frac{1}{12} \left( \hat{D}^{\hat{u}^{n-1}} + 6\hat{D}^{\hat{u}^{n}} + \hat{D}^{\hat{u}^{n+1}} \right) \hat{\rho}^{n} \\
+ \frac{1}{12} \left( \hat{D}^{\hat{u}^{n}} + \hat{D}^{\hat{u}^{n+1}} \right) \hat{\rho}^{n+1}
\]
Skew Symmetric discretisation

Euler Equations: Time integration

Momentum equation similar, just some more terms

\[
\frac{1}{4\Delta t} \left( (\hat{S}^{\hat{n}^{-1}} + \hat{S}^{\hat{n}}) \hat{u}^{n-1} - (\hat{S}^{\hat{n}} + \hat{S}^{\hat{n}+1}) \hat{u}^{n+1} \right) \\
= \frac{1}{120} \left( 2\hat{A}^{\hat{o}^{n-1}} u^{n-1} + 2\hat{A}^{\hat{o}^{n}} u^{n-1} + 3\hat{A}^{\hat{o}^{n-1}} u^{n-1} + 3\hat{A}^{\hat{o}^{n}} u^{n} \right) \hat{u}^{n-1} \\
+ \frac{1}{120} \left( 2\hat{A}^{\hat{o}^{n+1}} u^{n+1} + 2\hat{A}^{\hat{o}^{n-1}} u^{n-1} + 3\hat{A}^{\hat{o}^{n+1}} u^{n} + 3\hat{A}^{\hat{o}^{n}} u^{n-1} + 24\hat{A}^{\hat{o}^{n}} u^{n} \right) \hat{u}^{n} \\
+ \frac{1}{120} \left( 2\hat{A}^{\hat{o}^{n+1}} u^{n} + 2\hat{A}^{\hat{o}^{n}} u^{n+1} + 3\hat{A}^{\hat{o}^{n+1}} u^{n+1} + 3\hat{A}^{\hat{o}^{n}} u^{n} \right) \hat{u}^{n+1} \\
+ \hat{\Delta} \left( \frac{1}{6} \hat{\rho}^{n-1} + \frac{2}{3} \hat{\rho}^{n} + \frac{1}{6} \hat{\rho}^{n+1} \right)
\]

(still everything sparse!)
with all matrices with the structure

\[
\frac{1}{f} \begin{pmatrix}
\vdots & \vdots & \vdots \\
- b_i & a_i & d_i & a_i^+ & b_i^+ \\
\vdots & \vdots & \vdots 
\end{pmatrix}
\] (1)

were for \( \hat{B}^\hat{\upsilon} \) we have \( f = 240 \) and

\[
\begin{align*}
  d_i &= +31(u_{i+1} - u_{i-1}) + u_{i+2} - u_{i-2}, \\
  a_i^{\pm} &= \pm(31u_i + 62u_{i\pm1} + 7u_{i\pm2}), \\
  b_i^{\pm} &= \pm(u_i + 7u_{i\pm1} + 2u_{i\pm2}),
\end{align*}
\] (0)

were for \( \hat{S}^\hat{\upsilon} \) we have \( f = 840 \) and

\[
\begin{align*}
  d_i &= 288\varrho_i + 86(\varrho_{i+1} + \varrho_{i-1}) + \varrho_{i+2} + \varrho_{i-2}, \\
  a_i^{\pm} &= 5\varrho_{i\pm1} + 86\varrho_i + 86\varrho_{i\pm1} + 5\varrho_{i\pm2}, \\
  b_i^{\pm} &= \varrho_i + 5\varrho_{i\pm1} + \varrho_{i\pm2},
\end{align*}
\] (1)
with all matrices with the structure

\[
\frac{1}{f} \begin{pmatrix}
\ddots & \ddots & \ddots \\
\dot{} & b_i^- & a_i^- & d_i & a_i^+ & b_i^+ \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]  

(2)

were for \( \hat{A} \hat{\hat{u}} \) we have \( f = 3360 \) and

\[
d_i = 0 \\
a_i^\pm = \pm \left( 11 \left( \varrho_i u_{i\mp 1} + \varrho_i u_i + \varrho_i u_{i\pm 1} + \varrho_i u_{i\pm 2} \right) \\
+ 18 \left( \varrho_i u_{i\pm 2} + \varrho_i u_{i\mp 1} \right) \\
+ 69 \left( \varrho_i u_{i\pm 2} + \varrho_i u_{i\mp 1} + \varrho_i u_i + \varrho_i u_{i\mp 2} \right) \\
+ 474 \left( \varrho_i u_i + \varrho_i u_{i\pm 1} \right) + 748 \left( \varrho_i u_{i\pm 1} + \varrho_i u_i \right) \right) \\
\]  

(3)

\[
b_i^\pm = \pm \left( 3 \left( \varrho_i u_i + \varrho_i u_{i\pm 2} \right) + 10 \left( \varrho_i u_{i\pm 2} + \varrho_i u_i \right) \\
+ 29 \left( \varrho_i u_{i\pm 2} + \varrho_i u_i + \varrho_i u_{i\pm 1} + \varrho_i u_{i\pm 1} \right) \\
+ 138 \varrho_i u_{i\pm 1} \right),
\]  

(4)
\[ \frac{1}{f} \begin{pmatrix} 
\ddots & \ddots & \ddots & \ddots & \ddots \\
 b_i^- & a_i^- & d_i & a_i^+ & b_i^+ \\
 & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix} \]  

(6)

and were for \( \hat{C} \hat{u} \) we have \( f = 240 \) and

\[
\begin{align*}
d_i &= u_{i-2} - u_{i+2} - 31u_{i+1} + 31u_{i-1} \\
a_i^\pm &= \pm(7u_{i\mp 1} + 62u_i + 31u_{i\pm 1}) \\
b_i^\pm &= \pm(2u_i + 7u_{i\pm 1} + u_{i\pm 2})
\end{align*}
\]  

(7)
Overview

1 Objectives

2 Some DNS Results
   • Supersonic Jet Noise
   • Adjoint Optimization

3 Fully Conservative, Skew Symmetric FD Schemes
   • Need for skew-symmetric, conservative formulations
   • What does Skew Symmetry mean?
   • Motivating Example: Burgers’ Equation
   • Euler/Navier-Stokes Equations

4 Conclusion
Our Goal in Computation of Supersonic Jet Noise includes

- **Simulation of supersonic Jet-Noise**
- **Adjoint** Simulation of the Navier-Stokes Equations
- **skew-symmetric** formulation of the Navier-Stokes Equations (with shock treatment)

enables

- **Optimisation** of Supersonic Flow

needed

- skew symmetric filter for the computation of non-resolved shocks
Wave decomposition for isentropic flow

Continuity eq.

\[ \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0 \]

Momentum eq.

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \]
Wave decomposition for isentropic flow

Continuity eq.

\[ \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0 \]

Momentum eq.

\[ \pm \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \]

\[ \left( \frac{\partial u}{\partial t} \pm \frac{1}{\rho c} \frac{\partial p}{\partial t} \right) + (u \pm c) \left( \frac{\partial u}{\partial x} \pm \frac{1}{\rho c} \frac{\partial p}{\partial x} \right) = 0 \]
Acoustic Waves

\[
\left( \frac{\partial u}{\partial t} \pm \frac{1}{\varrho c} \frac{\partial p}{\partial t} \right) + (u \pm c) \left( \frac{\partial u}{\partial x} \pm \frac{1}{\varrho c} \frac{\partial p}{\partial x} \right) = 0
\]
Acoustic Waves

\[
\left( \frac{\partial u}{\partial t} \pm \frac{1}{\varrho c} \frac{\partial p}{\partial t} \right) + (u \pm c) \left( \frac{\partial u}{\partial x} \pm \frac{1}{\varrho c} \frac{\partial p}{\partial x} \right) = 0
\]

homentropic flow

\[
\frac{\partial R^\pm}{\partial t} + \lambda \frac{\partial R^\pm}{\partial x} = 0
\]
Application to the Euler Equations

\[
\left( \frac{\partial u}{\partial t} + \frac{1}{\varrho c} \frac{\partial \varrho}{\partial t} \right) + (u + c) \left( \frac{\partial u}{\partial x} + \frac{1}{\varrho c} \frac{\partial \varrho}{\partial x} \right) = 0
\]

\[
\left( \frac{\partial u}{\partial t} - \frac{1}{\varrho c} \frac{\partial \varrho}{\partial t} \right) + (u - c) \left( \frac{\partial u}{\partial x} - \frac{1}{\varrho c} \frac{\partial \varrho}{\partial x} \right) = 0
\]
Application to the Euler Equations

\[ X^+ := (u + c) \left( \frac{\partial u}{\partial x} + \frac{1}{\varrho c} \frac{\partial p}{\partial x} \right) \]

\[ \left( \frac{\partial u}{\partial t} + \frac{1}{\varrho c} \frac{\partial p}{\partial t} \right) + (u+c) \left( \frac{\partial u}{\partial x} + \frac{1}{\varrho c} \frac{\partial p}{\partial x} \right) = 0 \]

\[ X^- := (u - c) \left( \frac{\partial u}{\partial x} - \frac{1}{\varrho c} \frac{\partial p}{\partial x} \right) \]

\[ \left( \frac{\partial u}{\partial t} - \frac{1}{\varrho c} \frac{\partial p}{\partial t} \right) + (u-c) \left( \frac{\partial u}{\partial x} - \frac{1}{\varrho c} \frac{\partial p}{\partial x} \right) = 0 \]
Euler Equations in Wave Form

\[ \frac{\partial p}{\partial t} = -\frac{\rho c}{2} (X^+ + X^-) \]

\[ \frac{\partial u}{\partial t} = -\frac{1}{2} (X^+ - X^-) \]
Wave-Expression of the Navier–Stokes Eqn.

\[
\frac{\partial p}{\partial t} = -\frac{\varrho c}{2} \left( (X^+ + X^-) + (Y^+ + Y^-) + (Z^+ + Z^-) \right) + \frac{p}{C_v} \left( \frac{\partial s}{\partial t} + X^s + Y^s + Z^s \right)
\]

\[
\frac{\partial u}{\partial t} = -\left( \frac{1}{2} (X^+ - X^-) + Y^u + Z^u \right) + \frac{1}{\varrho} \frac{\partial \tau_{1j}}{\partial x_j}
\]

\[
\frac{\partial v}{\partial t} = -\left( X^v + \frac{1}{2} (Y^+ - Y^-) + Z^v \right) + \frac{1}{\varrho} \frac{\partial \tau_{2j}}{\partial x_j}
\]

\[
\frac{\partial w}{\partial t} = -\left( X^w + Y^w + \frac{1}{2} (Z^+ - Z^-) \right) + \frac{1}{\varrho} \frac{\partial \tau_{3j}}{\partial x_j}
\]

\[
\frac{\partial s}{\partial t} = -\left( X^s + Y^s + Z^s \right) + \frac{R}{p} \left( -\frac{\partial q_i}{\partial x_i} + \phi \right)
\]
Adjoint Equations in Characteristic Wave Formulation

DNS-Code in characteristic wave formulation

Euler equations \((u)\)

\[
\frac{\partial u}{\partial t} = -\frac{1}{2} \left( X^+ + X^- \right)
\]

direct wave

\[
X^\pm = (u \pm c) \left( \frac{1}{\rho c} \frac{\partial p}{\partial x} \pm \frac{\partial u}{\partial x} \right)
\]
Adjoint Equations in Characteristic Wave Formulation

DNS-Code in characteristic wave formulation

adjoint DNS-Code in adjoint characteristic wave formulation

**Euler equations (u)**

\[
\frac{\partial u}{\partial t} = -\frac{1}{2} \left[ X^+ + X^- \right]
\]

direct wave

\[
X^\pm = (u \pm c) \left( \frac{1}{\varrho c} \frac{\partial p}{\partial x} \pm \frac{\partial u}{\partial x} \right)
\]

**adjoint Euler equations (u*)**

\[
-\frac{\partial u^*}{\partial t} = -\frac{1}{2} \left[ X^{++} - X^{--} \right]
\]

adjoint wave

\[
X^{\pm *} = (-u \pm c) \left( -\varrho c \frac{\partial p^*}{\partial x} \pm \frac{\partial u^*}{\partial x} \right)
\]
### Adjoint Equations in Characteristic Wave Formulation

DNS-Code in characteristic wave formulation

### adjoint DNS-Code in adjoint characteristic wave formulation

#### Euler equations \((u)\)

\[
\frac{\partial u}{\partial t} = -\frac{1}{2} \left[ X^+ + X^- \right]
\]

direct wave

\[
X^\pm = (u \pm c) \left( \frac{1}{\varrho c} \frac{\partial p}{\partial x} \pm \frac{\partial u}{\partial x} \right)
\]

#### adjoint Euler equations \((u^*)\)

\[
-\frac{\partial u^*}{\partial t} = -\frac{1}{2} \left[ X^{++} - X^{--} \right]
\]

adjoint wave

\[
X^{\pm*} = (-u \pm c) \left( -\varrho c \frac{\partial p^*}{\partial x} \pm \frac{\partial u^*}{\partial x} \right)
\]

analogous for adjoint compressible Navier–Stokes equations
adds tons of source–terms (continuous approach)
Adjoint Methods for Optimization

Adjoint equation for $u$

\[
\frac{\partial}{\partial t} u^+ = -\gamma p \frac{\partial}{\partial x} p^+ + 2/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - \frac{\partial}{\partial x} \frac{\partial}{\partial z} w^+ + 2/3 \frac{\partial}{\partial z} \frac{\partial}{\partial x} w^+ - \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ - \frac{\partial}{\partial x} \frac{\partial}{\partial z} v^+ - u + 4/3 \frac{\partial}{\partial x} \mu \frac{\partial}{\partial x} u^+ - 2/3 \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ - 21/3 \frac{\partial}{\partial x} \mu \frac{\partial}{\partial x} w^+ - 21/3 \frac{\partial}{\partial x} \frac{\partial}{\partial z} w^+ - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} e^2 - \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} e^2 - \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ -
\]

\[
5/3 \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ - 2/3 \frac{\partial}{\partial x} \frac{\partial}{\partial z} e^2 - \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ - 24/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} u^+ - 4/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - 8/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} w^+ - 8/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - 3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ +
\]

\[
\frac{\partial}{\partial z} w + 2 \frac{\partial}{\partial z} \mu - 3 \frac{\partial}{\partial z} \frac{\partial}{\partial x} e^2 + 8/3 \frac{\partial}{\partial x} \frac{\partial}{\partial z} \mu - 4/3 \frac{\partial}{\partial x} \frac{\partial}{\partial z} \mu + 2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} \mu - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} \mu + 2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} e^2 - \frac{\partial}{\partial x} \frac{\partial}{\partial z} u^+ -
\]

\[
2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} u + 2/3 \frac{\partial}{\partial y} v - 2/3 \frac{\partial}{\partial z} w + 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} u^+ - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} w^+ - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu \frac{\partial}{\partial y} v^+ +
\]

\[
2 \frac{\partial}{\partial y} \frac{\partial}{\partial z} u + 2 \frac{\partial}{\partial y} \frac{\partial}{\partial z} v + \frac{\partial}{\partial y} \frac{\partial}{\partial z} e^2 + 2 \frac{\partial}{\partial y} \frac{\partial}{\partial z} e^2 - \frac{\partial}{\partial x} \frac{\partial}{\partial y} v^+ -
\]

\[
2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} e^2 - 3 \frac{\partial}{\partial z} \frac{\partial}{\partial y} e^2 + 8/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu - 4/3 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu + 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu - 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} \mu + 2 \frac{\partial}{\partial y} \frac{\partial}{\partial x} e^2 - \frac{\partial}{\partial y} \frac{\partial}{\partial x} u^+ -
\]
Sponge: efficient boundary condition for aeroacoustic applications

- **nonreflecting acoustic**
- **reduction of spurious backscattering** for leaving hydodynamic waves

**Sponge function (1D)**

\[
\frac{\partial u}{\partial t} = RHS + \sigma(x) (u_{ref} - u)
\]
Sponge: efficient boundary condition for aeroacoustic applications

- nonreflecting acoustic
- reduction of spurious backscattering for leaving hydrodynamic waves

**Sponge function (1D)**

\[
\frac{\partial u}{\partial t} = \text{RHS} + \sigma(x) (u_{\text{ref}} - u)
\]
## Problem:

- **linear ODE:** \[ \frac{dw(t)}{dt} = Aw(t); \quad w(0) = v \]

- **Solution:** \[ w(t) = e^{tA}v \]

A is a stiff matrix i.e. \( \lambda_{|u+c|} \gg \lambda_u \)
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Solution:

increase time-step (\( CFL = \frac{u \cdot dt}{dx} \))
Krylov exponential time integration

Motivation

Problem:
- Linear ODE: \[ \frac{dw(t)}{dt} = Aw(t); \quad w(0) = v \]
- Solution: \[ w(t) = e^{tA}v \]

A is a stiff matrix i.e. \( \lambda |u+c| >> \lambda u \)

Solution:
- Increase time-step (CFL = \( \frac{u \cdot dt}{dx} \))

Numerically unstable

Sesterhenn (TU Berlin)  Skew symmetric compact FD schemes  June, 2010  81 / 92
## Krylov exponential time integration

Exponential time integration

<table>
<thead>
<tr>
<th>method (explicit)</th>
<th>max. CFL-number (time-step)</th>
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Example

- linear ODE: \( \frac{dw(t)}{dt} = Aw(t); \quad w(0) = v \)

- Solution: \( w(t) = e^{tA}v \)
Problem:

- computing the exponential:
  \[ w(t) = e^{tA}v = v + \frac{(tA)}{1!}v + \frac{(tA)^2}{2!}v + \ldots \]

- \( A \) is a huge and sparse matrix
Problem:

- computing the exponential:
  
  \[ w(t) = e^{tA}v = v + \frac{(tA)}{1!}v + \frac{(tA)^2}{2!}v + \ldots \]

- \( A \) is a huge and sparse matrix

- \( w(t) \): elements of a **Krylov subspace**

  \[ K_m = \text{span}\{v, Av, \ldots, A^{m-1}v\}. \]
Reformulating the Problem:

- find an element of the Krylov subspace!
  \[ K_m = \text{span}\{v, Av, ..., A^{m-1}v\} \]

or more precisely

- project \( w(t) = e^{tA}v \) on \( K_m \).
Reformulating the Problem:

- find an element of the Krylov subspace!

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⇒ Arnoldi
Krylov exponential time integration

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Krylov exponential time integration

Arnoldi

orthonormal basis $V_m$

Hessenberg matrix $H_m$
**Krylov exponential time integration**

Arnoldi

**orthonormal basis** $V_m$

**Hessenberg matrix** $H_m$

$$H_m = V_m^T A V_m$$

$H_m$ is a projection of $A$ on $K_m$ with respect to $V_m$
Krylov exponential time integration
Arnoldi – matrix

\[ A \, V_m = V_m \, H_m + h_{m+1} \, v_{m+1} \, e_1^T \]
Approximation of the matrix exponential:

\[ V_m^T e^{tA} V_m \approx e^{tH_m} \]

\[ w(t) = e^{tA} v \quad \text{exact} \quad \implies \quad w(t) \approx V_m e^{tH_m} e_1 \quad \text{approximation} \]
Approximation of the matrix exponential:

\[ V_m^T e^{tA} V_m \approx e^{tH_m} \]

**exact**

\[ w(t) = e^{tA}v \]

\[ w(t) \approx V_m e^{tH_m} e_1 \]

**approximation**

The trick:

\[ H_m \in \mathbb{R}^{m \times m} \leftrightarrow A \in \mathbb{R}^{n \times n} \]

\[ m \ll n \]

e.g.

\[ 3 \ll 10^6 \]
Krylov exponential time integration
Application to nonlinear equations

What happens to nonlinear equations?

\[ \frac{dw(t)}{dt} = f(w(t)) \]
Krylov exponential time integration
Application to nonlinear equations

Linearized equations:

\[
\frac{dw(t)}{dt} \approx f(w(t_n)) + A(w(t) - w(t_n))
\]

solution of the linearized equations:

\[
w(t) \approx w(t_n) + t\varphi(tA)f(w(t_n))
\]

with \(\varphi(z) = \frac{e^z - 1}{z}\)
Krylov exponential time integration
Application to nonlinear equations

Linearized equations:

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\frac{dw(t)}{dt} \approx f(w(t_n)) + A(w(t) - w(t_n))
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with \( \varphi(z) = \frac{e^z - 1}{z} \)

Krylov approximation:

\[
w(t) \approx w(t_n) + tV_m \varphi(tH_m)e_1 f(w(t_n))
\]
Krylov performance

**Performance**

- proc.-time $[\text{h}]$ vs. CFL
- $t/\tau_{RK3}$
- CFL range: 0 to 20
- proc.-time range: 0 to 2.5

**Error**

- err vs. CFL
- CFL range: 0.5 to 50
- err range: $10^{-8}$ to $10^{-4}$

- Krylov and Runge Kutta schemes

- Sesterhenn (TU Berlin)
- Skew symmetric compact FD schemes
- June, 2010
Krylov Ritz-Values

Ritz-Values of $H_m$

![Graph showing Ritz-Values of $H_m$.]
Conclusion

- **Krylov** exponential time integration up to **twice as fast** as standard explicit methods.

- **Error** in the order of **standard explicit methods**.

- Advantageous for fluid flow with a **wide range of temporal scales**.
  - Low Mach number flows
  - Reactive flows

- ⇒ Good for **aeroacoustic applications**.

_Schulze et al. 2008. Int. J. Num. Meth. in Fluids_