On plane algebroid curves

V. Barucci∗  M. D’Anna†  R. Fröberg‡

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1 Introduction

Let \( C \) and \( C' \) be two analytic plane irreducible curves (branches) defined in a
neighbourhood of the origin and having singularities there. The branches are
said to be topologically equivalent if there are neighbourhoods \( U \) and \( U' \) of the
origin such that \( C \) is defined in \( U \), \( C' \) in \( U' \), and there is a homeomorphism
\( T: U \to U' \) such that \( T(C \cap U) = C' \cap U' \).

If \( F(X,Y) \in \mathbb{C}[[X,Y]] \) is an irreducible formal power series, the local ring
\( O = \mathbb{C}[[X,Y]]/(F) \) is called an algebroid branch. Two algebroid branches are
formally equivalent if they have the same multiplicity sequence (see below for the
definition of multiplicity sequence). Every algebroid (analytic resp.) branch is
formally (analytically, resp.) equivalent to an algebraic branch, i.e. a branch
defined by a polynomial [1], and if two analytic branches are formally equiva-

Zariski has shown ([2]) that two branches are formally equivalent if and only
if they have the same semigroup of values (see below for the definition of the
value semigroup of a branch).

The crucial result of Section 2 is Proposition 2.3, which gives the relation
between the value semigroups of an algebroid plane branch \( O \) and its blowup \( O' \).
It is a result contained in [3]. Apéry proved that, in order to show that the value
semigroup \( v(O) \) of an algebroid plane branch \( O \) is symmetric. Subsequently
Kunz proved that, for any analytically irreducible ring \( O \), \( O \) is Gorenstein if
and only if \( v(O) \) is symmetric. So now it is more common to say that the
value semigroup of an algebroid plane branch is symmetric because the ring is
Gorenstein (it is in fact a complete intersection). At any rate we are interested
in Apéry’s result for different reasons. By its use we give an easy proof of the fact
that two plane algebroid branches are formally equivalent if and only if they have
the same semigroup of values. We get also a well known formula of Hironaka
and apply it again in Sections 3 and 4. The material in Section 3 is classical too

∗email barucci@mat.uniroma1.it
†email mdanna@dmi.unict.it
‡email ralf@matematik.su.se
and essentially contained in Enriques-Chisini’s work, but what is new, is the use of Apéry’s Lemma in this context. After characterizing all possible multiplicity sequences for plane branches, we give a criterion to check if a semigroup is the value semigroup of a plane branch. In Section 4, we determine the semigroup of a plane branch from its parametrization, here also using results from [3]. This result is well known, but the proof is new as far as we know. Finally in Section 5 we show that the semigroup ring of the semigroup of a plane curve is a complete intersection.

2 Plane branches

Starting from Apéry’s article [3], we will proceed to explicate and expand various elements that are presented in the original arguments in a summary or not totally developed manner.

Let $\mathcal{O} = \mathbb{C}[[X,Y]]/(F) = \mathbb{C}[[x,y]]$, where $F$ is irreducible in $\mathbb{C}[[X,Y]]$ be an algebroid plane branch. Since $F(X,Y)$ is irreducible, then $F(X,Y)$ must contain some term $X^i$ and some term $Y^j$ (otherwise $F$ is not irreducible since we could factor out $X$ or $Y$). Denote the minimal such powers by $n$ and $m$ respectively. Then, by the Weierstrass Preparation Theorem, the same ideal $(X^n, Y^m)$ we could factor out $X$ and $Y$.

The Puiseux Theorem gives that the branch has a parametric representation $x = t^n, y = \sum a_i t^i$ (or $x = \sum_{i \geq n} b_i t_i, y = t_i^i$, where $\mathbb{C}[[t]] = \mathbb{C}[[t_i]]$). Thus $\mathcal{O} = \mathbb{C}[[x,y]] \subseteq \mathbb{C}[[t]] = \mathcal{O}$, which is a discrete valuation ring. Denote by $v$ the valuation of such ring that consists in associating to any formal power series in $\mathbb{C}[[t]]$ its order. In particular $v(x) = m$ and $v(y) = n$. Since the fraction field of $\mathcal{O}$ equals the fraction field of $\mathcal{O}$, there exist $f_1(t), f_2(t) \in \mathcal{O}$, such that $f_1(t)/f_2(t) = t$, so $f_1(t) = tf_2(t)$ and $v(f_1) = v(f_2) + 1$. Since $\gcd(v(f_1), v(f_2)) = 1$, all sufficiently large integers belong to $v(\mathcal{O}) = \{v(z); z \in \mathcal{O}\setminus\{0\}\}$. Thus $v(\mathcal{O})$ is a numerical semigroup, i.e., a subsemigroup of $\mathbb{N}$ with finite complement to $\mathbb{N}$.

In the sequel we use the following terminology. If $S$ is a subsemigroup of $\mathbb{N}$ and $T$ is a subset of $\mathbb{Z}$, we call $T$ an $S$-module if $s \in S, t \in T$ implies $s + t \in T$. We call $T$ a free $S$-module if $T = \bigcup_{i=1}^{k} T_i$ with $T_i \cap T_j = \emptyset$ if $i \neq j$ and $T_i = n_i + S$ for some $n_i \in \mathbb{Z}$. We call $n_1, \ldots, n_k$ a basis of $T$.

With the hypotheses and notation above, we will construct a new basis $y_0, \ldots, y_{m-1}$ for $\mathcal{O}$ as a $\mathbb{C}[[x]]$-module, such that, for each $i$, $y_0, \ldots, y_i$ is a basis for $\mathcal{O}_i = \mathbb{C}[[x]] + y \mathbb{C}[[x]] + \cdots + y^i \mathbb{C}[[x]]$, and furthermore such that $v(\mathcal{O}_i) = \{v(z); z \in \mathcal{O}\setminus\{0\}\}$ is a free module over $v(\mathbb{C}[[x]]) = m\mathbb{N}$ with basis $\omega_0, \ldots, \omega_i$, where each $\omega_j = v(y_j), j = 0, \ldots, i$ is the smallest value in $v(\mathcal{O})$ in its congruence class (mod $m$). Let $y_0 = 1$, thus $\omega_0 = v(y_0) = 0$ and $v(\mathcal{O}_0) = v(\mathbb{C}[[x]]) = m\mathbb{N}$.
Suppose that \( y_0, \ldots, y_{k-1}, k < m \) have been defined such that \( v(\mathcal{O}_{k-1}) \) is a free \( m\mathbb{N} \)-module with basis \( \omega_0, \ldots, \omega_{k-1} \). We claim that there exists a \( \phi(x, y) \in \mathcal{O}_{k-1} \) such that \( y_k = y^k + \phi(x, y) \) has a value which does not belong to \( v(\mathcal{O}_{k-1}) \). If \( v(y^k) \not\in v(\mathcal{O}_{k-1}) \), we are ready. Otherwise \( v(y^k) = v(z_1) \) for some \( z_1 \in \mathcal{O}_{k-1} \). Then \( v(y^k - c_1 z_1) > v(y^k) \) for some \( c_1 \in \mathbb{C} \). If \( v(y^k - c_1 z_1) \not\in v(\mathcal{O}_{k-1}) \), we are ready. Otherwise take \( z_2 \in \mathcal{O}_{k-1} \) with \( v(z_2) = v(y^k - c_1 z_1) \). Then \( v(y^k - c_1 z_1 - c_2 z_2) > v(y^k - c_1 z_1) \) for some \( c_2 \in \mathbb{C} \) a.s.o. Thus we see that the expansion of \( y^k \) as a power series in \( t \) must contain a term \( a_i t^i \) with \( a_i \neq 0 \) and \( i \not\in v(\mathcal{O}_{k-1}) \), since otherwise \( y^k \in \mathcal{O}_{k-1} \).

Notice that \( y_1 y_{k-1} = (y+\phi_1(x))(y^{k-1}+\phi_{k-1}(x, y)) = y^k + \psi(x, y), \psi(x, y) \in \mathcal{O}_{k-1} \), so \( y_k = y_1 y_{k-1} + \phi(x, y) - \psi(x, y) \) and we could equally well have defined \( y_k \) as an element of the form \( y_1 y_{k-1} + \phi(x, y) \) (where \( \phi(x, y) \in \mathcal{O}_{k-1} \)) with a value which does not belong to \( v(\mathcal{O}_{k-1}) \). In such expression of \( y_k, v(\phi(x, y)) > v(y_k) \) since otherwise \( v(y_k) = v(\phi(x, y)) \in v(\mathcal{O}_{k-1}) \). Thus \( \omega_k = v(y_k) \geq v(y_1 y_{k-1}) = v(y_1) + v(y_{k-1}) = \omega_1 + \omega_{k-1} \). In particular the sequence \( \omega_0, \omega_1, \ldots \) is strictly increasing. Since \( v(\mathcal{O}_{k-1}) \) is free over \( m\mathbb{Z} \), this shows that \( \omega_k \not\equiv \omega_j \) if \( j < k \). Any element \( z \in \mathcal{O}_k \) can be written \( z = a_0(x) y_0 + \cdots + a_k(x) y_k \). All terms in this sum have values in different congruence classes \( (\text{mod } m) \). Thus \( v(z) = \min v(a_i(x) y_i) \). This shows that \( v(\mathcal{O}_k) \) is free with basis \( \omega_0, \ldots, \omega_k \). After \( m \) steps, we get that \( \mathcal{O}_{m-1} = \mathcal{O} \) is a \( \mathbb{C}[[x]] \)-module generated by \( y_0, \ldots, y_{m-1} \) with the requested properties.

If \( S \) is a numerical semigroup and \( a \in S \), then the elements \( n_0, n_1, \ldots, n_{a-1}, \) where \( n_i \) is the smallest element in \( S \) congruent to \( i \) (mod \( a \)), is called the Apery set of \( S \) with respect to \( a \). If we order the elements in the Apery set, and then denote them \( \omega_0, \ldots, \omega_{a-1} \), we have the ordered Apery set. We call the elements \( y_0, \ldots, y_{m-1} \in \mathcal{O} \) constructed as above an Apery basis of \( S \) with respect to \( x \). By the construction, \( \omega_0 = v(y_0), \ldots, \omega_{m-1} = v(y_{m-1}) \) is the ordered Apery set of \( v(\mathcal{O}) \).

In a similar way an Apery basis of \( \mathcal{O} \) with respect to \( y \) is defined.

**Example** If in \( \mathcal{O} = \mathbb{C}[x, y] \) we have \( \gcd(m, n) = 1 \), where \( v(x) = m \) and \( v(y) = n \), then \( y_k = y^k, k = 0, \ldots, m-1 \) is an Apery basis of \( \mathcal{O} \), and thus \( \omega_k = kn, k = 0, \ldots, m-1 \) is the ordered Apery set of \( v(\mathcal{O}) \) with respect to \( m \).

**Example** If in \( \mathcal{O} = \mathbb{C}[x, y] \) we have \( x = t^8, y = t^{12} + t^{14} + t^{15} \), then \( y_0 = 1, y_1 = y, y_2 = y^2 - x^4 = 2t^{26} + \cdots, y_3 = y^3 - x^9 = 2t^{38}, y_4 = y^4 - 2x^3 y^2 - 4x^4 y + 3x^6 = 8t^{53} + \cdots, y_5 = y^5 - 2x^3 y^3 + x^6 y - 4x^8 = 8t^{65} + \cdots, y_6 = y^6 - 3x^3 y^4 - 4x^5 y^3 + 3x^6 y^2 + 4x^8 y - x^9 = 16t^{79} + \cdots \), and \( y_7 = y^7 - 3x^3 y^5 + 3x^6 y^3 - 4x^8 y^2 - x^9 y + 4x^{11} = 16t^{91} + \cdots \) is an Apery basis for \( \mathcal{O} \), so the ordered Apery set of \( v(\mathcal{O}) \) with respect to \( 8 \) is \( \{0, 12, 26, 38, 53, 65, 79, 91\} \). Thus \( v(\mathcal{O}) \) is minimally generated by \( 8, 12, 26, 53 \).

If \( S \) is a numerical semigroup, we denote the Frobenius number of \( S \), i.e. \( \max\{x \in \mathbb{Z}; x \not\in S \} \), by \( \gamma(S) \). The conductor of \( S \) is \( c(S) = \gamma(S) + 1 = \min\{x; [x, \infty) \subseteq S\} \).
Lemma 2.1 Let $S$ be a numerical semigroup with Frobenius number $\gamma$ and $a \in S$. If $\omega_0, \ldots, \omega_{n-1}$ is the ordered Apery set of $S$ with respect to $a$, then $
abla = \omega_{n-1} - a$.

Proof. The smallest element of $S$ congruent to $\gamma \pmod{a}$ is $\gamma + a$, so $\gamma + a$ is one of the elements in the Apery set. On the other hand all the other congruence classes (mod $a$) are represented by $\gamma + 1, \ldots, \gamma + (a - 1) \in S$ (and maybe also by smaller elements of $S$). So $\gamma + a$ is the biggest element in the Apery set, i.e. it is $\omega_{n-1}$.

Now we are ready for the crucial lemma from [3]. If $\mathcal{O} = \mathbb{C}[x, y]$ then $v(x) < v(y)$, we denote the quadratic transform (or blowup) $\mathbb{C}[x, y, x']$ by $\mathcal{O}'$.

Lemma 2.2 If an Apery basis of $\mathcal{O}'$ with respect to $x$ is $y_0', \ldots, y_{m-1}'$, then $y_i = y_i, x'$, for $i = 0, \ldots, m - 1$ is an Apery basis of $\mathcal{O}$ with respect to $x$.

Proof. Let $F(x, y/x)$ be the polynomial of degree $i$ in $y/x$ which defines $y_i'$, i.e. let $F(x, y/x) = (y/x)^i + o_i(x, y/x)$, where $\deg(o_i) < i$ in $y/x$. Then $y_i = x^i F(x, y/x) = y_i' + \phi_i(x, y), \phi_i(x, y) \in \mathcal{O}_{i-1}$ is of the requested form and, if $v(y_i') = \omega_i$, then $\omega_i = v(y_i') = \omega_i' + im$, thus $\omega_i \equiv \omega_i' \pmod{m}$. We have to show that $\omega_i' \notin v(O_{i-1})$. This is because $\omega_i'$ is not congruent to any $\omega_j'$, if $j < i$, and so also $\omega_i$ is not congruent to any $\omega_j$, if $j < i$.

As a consequence we get

Proposition 2.3 [3, Lemma 2] If the ordered Apery set of $v(O')$ with respect to $m = v(x)$ is $0 = \omega_0' < \omega_1' < \cdots < \omega_{m-1}'$, then the ordered Apery set of $v(O)$ with respect to $m$ is $0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1} = \omega_{m-1}' + (m - 1)m$.

Recall that the multiplicity of the ring $\mathcal{O} = \mathbb{C}[x, y]$, where $x = a_n t^m + a_{m+1} t^{m+1} + \cdots, a_m \neq 0$ and $y = b_n t^m + b_{n+1} t^{n+1} + \cdots, b_n \neq 0$, is given by min$(m, n)$ i.e. the multiplicity of $\mathcal{O}$ is the smallest positive value in $v(O)$.

Set $\mathcal{O} = \mathcal{O}(0)$, denote by $\mathcal{O}(i+1)$ the blowup of $\mathcal{O}(i)$ and by $\varepsilon_i$ the multiplicity of $\mathcal{O}(i)$. The multiplicity sequence of $\mathcal{O}$ is by definition the sequence of natural numbers $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$. Let $k$ be the minimal index such that $\varepsilon_k = 1$, i.e. such that $v(O(k)) = \mathbb{N}$. Two algebroid branches are formally equivalent if they have the same multiplicity sequence.

As a consequence of Proposition 2.3, we get easily a wellknown formula:

Corollary 2.4 [4, Theorem 1] We have $l_\mathcal{O}(\mathcal{O}/(\mathcal{O} : \mathcal{O})) = \sum_{i=0}^{k} e_i (e_i - 1)$ and $l_\mathcal{O}(\mathcal{O}/(\mathcal{O} : \mathcal{O})) = l_\mathcal{O}(\mathcal{O}/\mathcal{O}) = \frac{1}{2} \sum_{i=0}^{k} e_i (e_i - 1)$.

Proof. Let $\omega^{(i)}$, $\omega^{(i+1)}$, resp. be the $i$th element in the ordered Apery set of $v(O^{(i)})$, $v(O^{(i+1)})$, resp., with respect to $e_j$ and let $\mathcal{O}^{(i) : \mathcal{O}} = t^{e_j} C[[t]]$ $(\mathcal{O}^{(i+1) : \mathcal{O}} = t^{e_j} C[[t]],$ resp.). By Lemma 2.1 $e_j = \omega^{(j+1)}_{e_{j-1}} - e_j + 1$ and $e_{j+1} = \omega^{(j+1)}_{e_{j-1}} - e_j + 1$. Proposition 2.3 gives $\omega^{(j)}_{e_{j-1}} = \omega^{(j+1)}_{e_{j-1}} + e_j (e_j - 1)$ and so
$c_j = c_{j+1} + c_j(e_j - 1)$. It follows that $c_0 = l_\mathcal{O}(\mathcal{O}/(\mathcal{O} : \mathcal{O})) = c_1 + c_0(e_0 - 1) = \cdots = e_k + e_{k-1}(e_{k-1} - 1) + \cdots + e_0(e_0 - 1) = \sum_{i=0}^{k} c_i(e_i - 1)$. Since the ring $\mathcal{O}$ is Gorenstein, we get $l_\mathcal{O}(\mathcal{O}/(\mathcal{O} : \mathcal{O})) = l_\mathcal{O}(\mathcal{O}/\mathcal{O}) = \frac{1}{2}l_\mathcal{O}(\mathcal{O}/(\mathcal{O} : \mathcal{O}))$.

**Example** Not every symmetric semigroup is the value semigroup of an algebroid plane branch. The semigroup generated by 4,5,6 is symmetric and has Apery set $0,5,6,11$ with respect to 4. If this were the value semigroup of a plane branch, then the Apery set of its blowup would be $0, 1 = 5-4, -2 = 6-8, -1 = 11-12$ which obviously is impossible.

**Theorem 2.5** [2] Two algebroid plane branches are formally equivalent if and only if they have the same semigroup.

**Proof.** Let $\mathcal{O} = \mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \ldots$ be the sequence of blowups of $\mathcal{O}$, and let $e_0, \ldots, e_k = 1$ be the corresponding multiplicity sequence. Then $v(\mathcal{O}^{(k)}) = \mathbb{N}$ has ordered Apery set $\{0, 1, \ldots, e_{k-1} - 1\}$ with respect to $e_{k-1}$. Proposition 2.3 gives the ordered Apery set, hence the semigroup, of $\mathcal{O}^{(k-1)}$ with respect to $e_{k-1}$ a.s.o. Thus the multiplicity sequence determines the semigroup of $\mathcal{O}$. On the other hand, the semigroup of $\mathcal{O}$ gives the multiplicity $e_0$ of $\mathcal{O}$. Proposition 2.3 gives the Apery set of $v(\mathcal{O}^{(1)})$, hence $v(\mathcal{O}^{(1)})$ and so on. Thus the semigroup $v(\mathcal{O})$ gives the multiplicity sequence.

Let $c_i$ denote the conductor degree of $\mathcal{O}^{(i)}$, i.e. $\mathcal{O}^{(i)} : \tilde{\mathcal{O}} = t^{c_i} \mathbb{C}[t]$, and call $(c_0, c_1, \ldots)$ the conductor degree sequence of $\mathcal{O}$. Let $f_i = l_{\mathcal{O}^{(i)}}(\tilde{\mathcal{O}}/\mathcal{O}^{(i)})$, and call $(f_0, f_1, \ldots)$ the sequence of singularity degrees of $\mathcal{O}$.

**Corollary 2.6** Two algebroid plane branches are formally equivalent if and only if they have the same conductor degree sequence, and if and only if they have the same sequence of singularity degrees.

**Proof.** If $\omega_0 < \omega_1 < \cdots < \omega_{e_i-1}$ is the Apery set of $v(\mathcal{O}^{(i)})$ with respect to $e_i$, then by Lemma 2.1 $c_i = \omega_{e_i-1} - e_i + 1$. Thus the multiplicity sequence of $\mathcal{O}$ determines, and is determined by, the conductor degree sequence. Since each ring $\mathcal{O}^{(i)}$ is Gorenstein, $f_i = c_i/2$ and the same is true for the sequence of singularity degrees.

**Example** The conductor degree of $\mathcal{O}$ does not suffice to give formal equivalence. The branches $\mathbb{C}[t^4, t^5]$ and $\mathbb{C}[t^3, t^7]$ both have conductor $t^{12} \mathbb{C}[t]$, but they are not formally equivalent.

### 3 The multiplicity sequence for a plane branch

A sequence of numbers $e_0 \geq e_1 \geq e_2 \geq \cdots$ is a multiplicity sequence of a (not necessarily plane) branch if and only if $0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \ldots$ constitute a semigroup [5]. We will now determine which multiplicity sequences occur for plane branches. We will also use this result together with Proposition 2.3 and Theorem 2.5 to get an algorithm to determine if a symmetric semigroup is the semigroup of a plane branch.
Let \( O = \mathbb{C}[t^{\delta_0} + \sum_{i \geq N} a_i t^i, \sum_{i > \delta_0} b_i t^i] \) be a branch. Let, for \( i \geq 1 \), \( \delta_i = \min\{j; b_j \neq 0, \gcd(\delta_0, \ldots, \delta_{i-1}, j) < \gcd(\delta_0, \ldots, \delta_{i-1})\} \). Let \( d_0 = \delta_0 \) and \( \gcd(\delta_0, \ldots, \delta_i) = d_i \) for \( i \geq 1 \). Set also \( k = \min\{i; d_i = 1\} \). (There exists such a \( k \) since the integral closure of \( O \) is \( \mathbb{C}[t] \).) We call the parametrization standard if \( N > \delta_k \). The numbers \( \delta_0, \delta_1, \ldots \) are called the characteristic exponents of \( O \).

It follows from the proof of Lemma 3.1 below, that we always can get a standard parametrization from a given one.

**Lemma 3.1** Let \( O = \mathbb{C}[t^{\delta_0} + \sum_{i \geq N} a_i t^i, \sum_{i > \delta_0} b_i t^i] \) be a branch with standard parametrization and with characteristic exponents \((\delta_0, \ldots, \delta_k)\). Then the characteristic exponents of \( O' \) are:

a) \((\delta_0, \delta_1 - \delta_0, \ldots, \delta_k - \delta_0)\), if \( \delta_0 < \delta_1 - \delta_0 \).

b) \((\delta_2 - \delta_1, \delta_0, \delta_0 + \delta_2 - \delta_1, \ldots, \delta_0 + \delta_k - \delta_1)\), if \( \delta_k > \delta_1 - \delta_0 \) and \( \delta_0 \) is not a multiple of \( \delta_1 - \delta_0 \).

c) \((\delta_1 - \delta_0, \delta_0 + \delta_2 - \delta_1, \ldots, \delta_0 + \delta_k - \delta_1)\), if \( \delta_0 \) is a multiple of \( \delta_1 - \delta_0 \).

**Proof** We can suppose that \( v(\sum_{i \geq \delta_0} b_i t^i) = \delta_1 \). Then the blowup \( O' \) of \( O \) is \( (t^{\delta_0} + \cdots, t^{\delta_1 - \delta_0} + \cdots) \). One of the following three cases will occur:

a) \( \delta_0 < \delta_1 - \delta_0 \)

b) \( \delta_0 > \delta_1 - \delta_0 \) and \( \delta_0 \) is not a multiple of \( \delta_1 - \delta_0 \)

c) \( \delta_0 \) is a multiple of \( \delta_1 - \delta_0 \).

We will in each case write \( O' \) in standard form and derive its characteristic exponents. In case a) \( O' \) is of standard form. We keep the meaning of \( \delta_i \) and \( d_i \) from above and denote the corresponding entities for \( O' \) with \( \delta'_i \) and \( d'_i \). It follows that \( d'_i = d_i \) for all \( i \) and that \( O' \) has characteristic exponents \((\delta'_0, \delta'_1, \ldots, \delta'_k) = (\delta_0, \delta_1 - \delta_0, \ldots, \delta_k - \delta_0)\). In case b) we first make the coordinate change \( X = y, Y = x \) to get \( (t^{\delta_1 - \delta_0}(1 + \sum_{i \geq 1} c_i t^i), t^{\delta_0} t^{\delta_1 - \delta_0}) \). Let \( i_0 = \min\{i; c_i \neq 0\} \). Then we choose a new parameter \( t_1 \), by \( t = t_1(1 - c_{i_0} x^{-\delta_0} t_1^{\delta_0}) \) to get the parametrization \( (t_1^{\delta_1 - \delta_0}(1 + \sum_{i \geq 1} c'_i t_1^i), t_1^{\delta_0} t_1^{\delta_1 - \delta_0}) \). Now \( v(\sum_{i \geq 1} c'_1 t_1^i) > v(\sum_{i \geq 1} c_i t^i) \). We continue to change parameter in this way. After a finite number of steps we get a parametrization of the branch of the type \( (t^{\delta_1 - \delta_0}(1 + \sum_{i \geq 1} c_i t^i), t^{\delta_0} t^{\delta_1 - \delta_0}) \) with \( d'_i = d_i \) for all \( i \), and with characteristic exponents \((\delta_1 - \delta_0, \delta_0 + \delta_2 - \delta_1, \ldots, \delta_0 + \delta_k - \delta_1)\). In case c) finally, we use a similar reparametrization and get \( d'_i = d_{i+1} \), and a branch with characteristic exponents \((\delta_1 - \delta_0, \delta_0 + \delta_2 - \delta_1, \ldots, \delta_0 + \delta_k - \delta_1)\).

If \( m_0, m_1, \ldots \) and \( h_0, h_1, \ldots \) are natural numbers, denote by \( m_0^{(h_0)}, m_1^{(h_1)}, \ldots \) the sequence of natural numbers given by \( m_0 \) repeated \( h_0 \) times, \( m_1 \) repeated \( h_1 \) times and so on. Suppose that for a couple \( m, n \) of natural numbers, the Euclidean algorithm gives

\[
m = nq_1 + r_1
\]
\[
n = r_1q_2 + r_2
\]
\[
\cdots
\]
Denote by $M(m,n)$ the sequence of natural numbers $r_i q_i + 1, r_{i+1} q_{i+2} + 0$.

Of course such a sequence ends with $r_{i+1} = \gcd(m,n)$ (if $m < n$, and so $q_i = 0$, $n$ appears 0 times, i.e. it does not appear, hence $M(m,n) = M(n,m)$). With this notation:

**Theorem 3.2** A sequence of natural numbers is the multiplicity sequence of an algebroid plane branch if and only if it is of the following form:

$$M(m_0, m_1), M(m_2, m_3), \ldots, M(m_{2k}, m_{2k+1}), 1, 1, \ldots$$

where, for $i \geq 0$, $\gcd(m_{2i}, m_{2i+1}) = m_{2i+2}$ and $m_{2i+3}$ is such that $m_{2i+4} < m_{2i+2}$, and finally $\gcd(m_{2k}, m_{2k+1}) = 1$.

**Proof.** Let $\mathcal{O}$ be an algebroid plane branch with standard parametrization. Then, by Lemma 3.1, its multiplicity sequence is

$$M(\delta_0, \delta_1), M(d_1, \delta_2 - \delta_1), M(d_2, \delta_3 - \delta_2), \ldots, M(d_{k-1}, \delta_k - \delta_{k-1}), 1, 1, \ldots$$

and is a sequence of the requested form. Conversely, given a sequence of natural numbers as in the statement, we can get characteristic exponents $(\delta_0, \delta_1, \ldots, \delta_k)$ and so an $\mathcal{O}$.

We give two concrete examples.

**Example 6, 4, 2, 1, 1, \ldots = M(10,6), 1, 1, \ldots** is an admissible multiplicity sequence (i.e. the multiplicity sequence of an algebroid plane branch), but 6, 4, 2, 1, 1, \ldots is not.

**Example** Let $\mathcal{O} = \mathbb{C}[x,y]$ with

$$x = t^{2^n}, y = t^{3 \cdot 2^n + 2^n - 1} + \cdots + t^{3 \cdot 2^n + 2^n - 1 + \cdots + 2 + 1}.$$ 

The multiplicity sequence is

$$M(3 \cdot 2^n, 2^n + 1), M(2^n, 2^n - 1), M(2^n - 1, 2^n - 2), \ldots, M(2, 1), \ldots$$

Now we are ready to give an algorithm to determine if a symmetric semigroup is the semigroup of values of a plane curve.

**Lemma 3.3** Let $S$ be a symmetric semigroup, $m = \min(S \setminus \{0\})$ and let

$$0 = \omega_0 < \omega_1 < \cdots < \omega_{m-1}$$

be its ordered Apery set with respect to $m$. Suppose that $\omega_0 < \omega_1 - m < \cdots < \omega_{m-1} - (m - 1)m$ is the ordered Apery set of a semigroup $S'$. Then $S'$ is symmetric.
Proof. A semigroup $S$ is symmetric if and only if $\omega_1 + \omega_{n-1-i} = \omega_{n-1-i}$. [3].

Since $S$ is symmetric then $\omega_1 + \omega_{n-1-i} = \omega_{n-1-i}$ implies that $\omega_i - im + \omega_{n-1-i} - (n - 1 - i)m = \omega_{n-1} - (n - 1)m$, hence the semigroup with Apery set $0 = \omega_0, \omega_1 - m, \ldots, \omega_{n-1-i} - (n - 1 - i)m$ is symmetric.

Given a symmetric semigroup $S$ satisfying the hypotheses of Lemma 3.3, one could repeat the process for the ordered Apery set of $S'$ with respect to its minimal non zero element, and so on until we find either a semigroup which does not satisfy these hypotheses or we find $\mathbb{N}$. But even if, after a finite number of steps, we get $\mathbb{N}$, it is not true that $S$ is a value semigroup of a plane branch, as the following example shows.

Example Let $S = \{6, 10, 29\}$; its ordered Apery set with respect to 6 is $\{0, 10, 20, 29, 39, 49\}$. The set obtained applying Lemma 3.3 is $\{0, 4 = 10 - 6, 8 = 20 - 12, 11 = 29 - 18, 15 = 39 - 24, 19 = 49 - 30\}$, hence it is the ordered Apery set of $S'$ with respect to 6. Hence $S' = \{0, 4, 6, 8, 10, 11, 12, 14, \ldots\} = \{4, 6, 11\}$. The ordered Apery set of $S'$ with respect to 4 is $0, 6, 11, 17$. Hence we get the new set $\{0, 2 = 6 - 4, 3 = 11 - 8, 5 = 17 - 12\}$ which is still ordered and determines the semigroup $S'' = \{2, 3\}$. Its ordered Apery set with respect to 2 is 0, 3. Thus we get the set $\{0, 1 = 3 - 2\}$, which is the ordered Apery set with respect to 2 of $\mathbb{N}$. On the other hand the semigroup $S$ is not the value semigroup of a plane branch $\mathcal{O}$ since the multiplicity sequence of $\mathcal{O}$ should be 6, 4, 2, 1, 1, ... which is not admissible, since the subsequence 6, 4 can be obtained only by $M(10, 6)$ but $M(10, 6) = 6, 4, 2, 2$.

Let $S$ be the value semigroup of a plane branch $\mathcal{O}$. By Proposition 2.3 we get that $S'$ (defined as in Lemma 3.3) is again a symmetric semigroup and $S' = v(\mathcal{O'})$. Repeating the process, if $S^{(0)} = S$ and $S^{(j+1)} = (S^{(j)})'$, and denoting by $m_j$ the minimal non zero element of $S^{(j)}$ and by $\omega_0^{(j)}, \omega_1^{(j)}, \ldots, \omega_{m_j-1}^{(j)}$ its ordered Apery set with respect to $m_j$, we get that $\omega_0^{(j)}, \omega_1^{(j)} - m_j, \ldots, \omega_{m_j-1}^{(j)} - (m_j - 1)m_j$ is the ordered Apery set of a symmetric semigroup $S^{(j+1)}$, and $S^{(j+1)} = v(\mathcal{O}^{(j+1)})$. Since there exists an $n \geq 1$ such that $\mathcal{O}^{(n)} = \mathbb{C}[t]$, then $S^{(n)} = \mathbb{N}$. Moreover the sequence $m_0, m_1, m_2, \ldots$ is the multiplicity sequence of $\mathcal{O}$, hence is an admissible multiplicity sequence.

Conversely if $S = S_0$ is a symmetric semigroup, let $S^{(j)}, m_j, \omega_i^{(j)}$ be defined as above. If the sets $\{0 = \omega_0^{(j)}, \omega_1^{(j)} - m_j, \ldots, \omega_{m_j-1}^{(j)} - (m_j - 1)m_j\}$ are ordered Apery sets for every $j = 0, \ldots, n - 1$ and the sequence $m_0, m_1, \ldots, m_{n-1}, 1, 1, \ldots$ is an admissible multiplicity sequence, then $S$ is the value semigroup of a plane branch. In fact, since the sequence $m_0, m_1, \ldots, m_{n-1}, 1, 1, \ldots$ is an admissible multiplicity sequence, then there exists a plane branch $\mathcal{O}$ having this sequence as multiplicity sequence. Now, by Theorem 2.5, the multiplicity sequence determines the value semigroups $v(\mathcal{O}(k)), k = 0, \ldots, n - 1$, and these semigroups, by Proposition 2.3 and Lemma 3.3 have the same ordered Apery sets of the semigroups $S^{(k)}$, hence they are the same semigroups.

This discussion gives a criterion to check if $S$ is the value semigroup of a plane branch, since we can apply repeatedly the process described in Lemma
3.3 until we find either a semigroup which does not satisfy the hypotheses in Lemma 3.3 or we find \( \mathbb{N} \). If the last case occurs, then it is enough to check if the sequence \( m_0, \ldots, m_{n-1}, 1, 1, \ldots \) is admissible.

The condition that at each step the sequence \( 0 = \omega^{(i)}_0, \omega^{(i)}_1 - m_j, \ldots, \omega^{(i)}_{m_j - 1} - (m_j - 1)m_j \) is an ordered Apery set (and not only an Apery set) is necessary as the following example shows.

**Example** Let \( S = \{0, 4, 8, 9, 10, 12, 13, 14, 16, \ldots \} \) be the semigroup with ordered Apery set \( \{0, 9, 10, 19\} \) with respect to 4. The sequence \( 0, 5 = 9 - 4, 2 = 10 - 8, 7 = 19 - 12 \) is not increasing. If we consider the semigroup \( S' \) with ordered Apery set \( \{0, 2, 5, 7\} \) with respect to 4 it is the symmetric semigroup \( \{0, 2, 4, \ldots \} \) and then in two more steps we get \( \mathbb{N} \).

Notice that the sequence \( m_0, m_1, \ldots \) is in this case \( 4, 2, 1, 1, \ldots \); it is admissible as multiplicity sequence since it is \( M(6,4), M(2,1), 1,1, \ldots \). However, applying Theorem 2.5, we get the semigroup \( \{0,4,6,8,10,12,13,14,\ldots\} \) with ordered Apery set \( \{0,6,13,19\} \) and applying Theorem 3.2 we get the parametrization \( O = \mathbb{C}[t^4, t^6 + t^7] \).

### 4 The semigroup of values for a plane branch

The following theorem is proved in different ways in e.g. [2], [6], [7], [8], [9].

**Theorem 4.1** Let \( O = \mathbb{C}[t^0 + \sum_{i>N} a_i t^i, \sum_{i>\delta_0} b_i t^i] \) be a branch with standard parametrization. Denote the minimal generators of \( v(O) \) by \( \delta_0 < \cdots < \delta_s \).

Then \( s = k \), \( \delta_0 = \delta_k \), \( \delta_1 = \delta_1 \) and \( \delta_i = \delta_{i-1} \frac{d_i}{d_{i-1}} + \delta_i - \delta_i-1 \) if \( i = 2, \ldots, k \).

We will divide the proof into several steps. From now on we will, for a plane branch with characteristic exponents (\( \delta_0, \delta_1, \ldots \)), let \( \delta_i \) denote the numbers defined in Theorem 4.1. It is clear that \( d_i = \gcd(\delta_0, \ldots, \delta_i) = \gcd(\delta_0, \ldots, \delta_i) \).

We keep also this notation in the sequel.

**Lemma 4.2** The conductor of \( S = (\delta_0, \ldots, \delta_k) \) is

\[
\left( \frac{d_0}{d_1} - 1 \right)(\delta_1 - d_1) + \left( \frac{d_1}{d_2} - 1 \right)(\delta_2 - d_2) + \cdots + \left( \frac{d_{k-1}}{d_k} - 1 \right)(\delta_k - d_k),
\]

and \( S \) is symmetric.

**Proof.** Since \( \gcd(\delta_0, \delta_1) = d_1 \), we have that \( i \delta_1, 0 \leq i \leq \frac{d_0}{d_1} - 1 \), are all different \( \pmod{\delta_0} \). They are also all smaller than \( \delta_2 \), since \( \delta_2 > \frac{d_0}{d_1} \delta_1 \). In the same way all \( i \delta_1 + j \delta_2, 0 \leq i \leq \frac{d_1}{d_2} - 1, 0 \leq j \leq \frac{d_1}{d_2} - 1 \) are all different \( \pmod{\delta_0} \), and they are all smaller than \( \delta_3 \), since \( \delta_3 > \frac{d_1}{d_2} \delta_2 > \left( \frac{d_2}{d_3} - 1 \right) \delta_2 + \left( \frac{d_2}{d_3} - 1 \right) \delta_1 \) a.s.o. In this way we see that the Apery set of \( S \) with respect to \( \delta_0 \) is \( \{ j_1 \delta_1 + j_2 \delta_2 + \cdots + j_k \delta_k ; 0 \leq j_i < \frac{d_i}{d_{i-1}} \}, i = 1, \ldots, k \) and \( i_1 \delta_1 + i_2 \delta_2 + \cdots + i_k \delta_k \) if and only if \( i_k = j_k, \ldots, i_s = j_s, i_{s-1} > j_{s-1} \) for some \( s \), i.e., if the last nonzero coordinate
of \((i_1 - j_1, \ldots, i_k - j_k)\) is positive. (We have found \(\frac{d_0}{d_1} \frac{d_1}{d_2} \cdots \frac{d_{k-1}}{d_k} = \frac{d_k}{d_0} = d_0 = \delta_0\) elements which are smallest in their congruence classes \((\text{mod } \delta_0)\).) Hence, the largest number in the Apéry set is \(\omega_{\delta_0-1} = (\frac{d_0}{d_1} - 1)\delta_1 + (\frac{d_1}{d_2} - 1)\delta_2 + \cdots + (\frac{d_{k-1}}{d_k} - 1)\delta_k\). Since the conductor equals \(\omega_{\delta_0-1} - (\delta_0 - 1)\) (cf. Lemma 2.1, we get the first statement after a small calculation. If \(\omega_i = i_1 \delta_1 + \cdots + i_k \delta_k\), it is easy to see that \(\omega_{\delta_0-1-i} = (\frac{d_0}{d_1} - 1 - i_1)\delta_1 + \cdots + (\frac{d_{k-1}}{d_k} - 1 - i_k)\delta_k\). Thus \(\omega_i + \omega_{\delta_0-1-i} = \omega_{\delta_0-1}\), which gives that \(S\) is symmetric (cf. [3]).

For a semigroup \(S\) and an integer \(d > 0\), we define the \(d\)-conductor of \(S\) to be \(c_d(S) = \min\{nt; nd \in S \text{ if } m \geq n\}\). Thus \(c_1(S)\) is the usual conductor of \(S\).

**Corollary 4.3** Let \(S = (\delta_0, \ldots, \delta_k)\) and let \(d_i = \gcd(\delta_0, \ldots, \delta_i). Then\)

\[
c_{d_i}(S) = \left(\frac{d_0}{d_1} - 1\right)(\delta_1 - d_1) + \left(\frac{d_1}{d_2} - 1\right)(\delta_2 - d_2) + \cdots + \left(\frac{d_{k-1}}{d_k} - 1\right)(\delta_k - d_k)
\]

for every \(i \leq k\).

**Proof.** By the proof of Lemma 4.2, the semigroup \(S_i = (\frac{\delta_0}{d_1}, \ldots, \frac{\delta_i}{d_i})\) has conductor \(c_d(S_i) = \left(\frac{d_0}{d_1} - 1\right)(\delta_1 - d_1) + \left(\frac{d_1}{d_2} - 1\right)(\delta_2 - d_2) + \cdots + \left(\frac{d_{i-1}}{d_i} - 1\right)(\delta_i - d_i).\)

Then \(c_{d_i}(\delta_0, \ldots, \delta_k) = c_d(S_i).\) A calculation gives that \(\delta_{i+1} > c_{d_i}(\delta_0, \ldots, \delta_k),\)

hence \(\delta_j > c_{d_j}(\delta_0, \ldots, \delta_k)\) if \(j > i\). Thus \(c_{d_i}(\delta_0, \ldots, \delta_k)\).

**Lemma 4.4** For \(i = 2, \ldots, k\) we have \(\delta_i = \frac{1}{d_i} \sum_{j=1}^{i-1} (d_j - d_1) \delta_j + \delta_i\). Thus the conductor of \(S = (\delta_0, \ldots, \delta_k)\) is \(\sum_{i=1}^k (d_i - d_1) \delta_i + (1 - d_0)\). Furthermore \(c_d(S) = \frac{1}{d_i} \sum_{j=1}^{i-1} \delta_j (d_j - d_1) + d_i - d_0.\)

**Proof.** By a calculation, replacing in Lemma 4.2 and in Corollary 4.3 \(\delta_i\) with \(\frac{1}{d_i} \sum_{j=1}^{i-1} (d_j - d_1) \delta_j + \delta_i\), we get the claim.

For the next proposition, we need a technical lemma. Let \(g(t) = \sum_{i \geq 0} a_k t^i, a_0 \neq 0\) be a power series such that \(\gcd\{i; a_i \neq 0\} = 1\). Let, for \(i = 1, \ldots, k - 1, \ d_i = (d_1, \ldots, d_{k-1}),\) and let \(d_i(g(t)) = (e_1(g), \ldots, e_{k-1}(g)),\) where \(e_s(g) = \min\{j; a_j \neq 0\}, \ d_s\) does not divide \(j\). The easy proof of the next lemma is left to the reader.

**Lemma 4.5** Let \(g(t) = \sum_{i \geq 0} a_k t^i, a_0 \neq 0, h(t) = \sum_{i \geq 0} b_k t^i, b_0 \neq 0,\) be powerseries such that \(\gcd\{i; a_i \neq 0\} = \gcd\{i; b_i \neq 0\} = 1.\) Then

(a) \(d_i(gh) \geq \min(d_i(g), d_i(h)) (\text{coefficientwise}).\)

(b) If \(g = h\) there is equality in (a).

(c) If \(d_i(g(t)) = (e_1, \ldots, e_{k-1}),\) then \(d_{i+1}(\sum_{i \geq e_i} a_k t^i) = (e_2(g) - e_1(g), \ldots, e_s(g) - e_1(g)).\)

We will call a power series **monic** if its least nonzero coefficient is 1.
Proposition 4.6 Let $\mathcal{O} = \mathbb{C}[t^{d_0} + \sum_{i \geq N} a_i t^i, \sum_{i > d_0} b_i t^i]$ be a branch of standard parametrization and with characteristic exponents $(\delta_0, \ldots, \delta_k)$. Let $\delta_i$ be defined as in Theorem 4.1. Then we have $\langle \delta_0, \ldots, \delta_k \rangle \subseteq v(\mathcal{O})$, i.e. $\delta_i \in v(\mathcal{O})$ for $i = 0, \ldots, k$.

Proof. Let, for $i = 1, \ldots, k - 1$, $d_i = (d_i, \ldots, d_{k-1})$, where $d_i = \gcd(\delta_0, \ldots, \delta_i)$ as above. We will, by induction, construct monic elements $f_i \in \mathcal{O}$ such that $v(f_i) = \delta_i$ and such that $d_i(f_i/t^{d_i}) = (\delta_{i+1} - \delta_i, \delta_{i+2} - \delta_i, \ldots, \delta_k - \delta_i)$ if $1 \leq i < k$. We let $f_0 = t^{d_0} + \sum_{i \geq N} a_i t^i$. If $v(\sum_{i > d_0} b_i t^i)$ is not a multiple of $\delta_0$, then $\omega(\sum_{i > d_0} b_i t^i) = \delta_0$ and we let $f_1 = b_0^{\delta_0} \sum_{i > d_0} b_i t^i$. If $v(\sum_{i > d_0} b_i t^i) = m_0 \delta_0$, let $f'_1 = \sum_{i > d_0} b_i t^i - cf_0^{m_0}$, where $c \neq 0$ is chosen so that $v(f'_1) > \omega(\sum_{i > d_0} b_i t^i)$. Repeat this until $v(f'_1(n)) = \delta_1$, and let $f_1 = c f'_1(n)$, where $c'$ is chosen so that $f_1$ is monic. It is clear that $d_1(f_1/t^{d_1}) = (\delta_2 - \delta_1, \delta_3 - \delta_2, \ldots, \delta_k - \delta_1)$. Suppose we have constructed $f_0, f_1, \ldots, f_i \in \mathcal{O}$ so that the conditions in the proposition are fulfilled. Then $f_i^{d_i-1/d_i}$ has value $\gamma_i = \delta_i d_i^{-1/d_i}$, which is a multiple of $d_i - 1$. A simple calculation, using Lemma 4.4, shows that $\gamma_i - c_d((\delta_0, \ldots, \delta_i)) = \delta_i - d_i + d_0 > 0$. Thus of course $\gamma_i > c_{d-1}((\delta_0, \ldots, \delta_i-1))$. This last means that $\gamma_i = \sum_{j=0}^{i-1} n_j \delta_j$ for some $n_j \geq 0$. We choose $f''_i = f_i^{d_i-1/d_i} - f_0 \cdots f_{i-1}$. It follows from Lemma 4.5(b) it follows that $d_i(f''_i/t^{d_i^{d_i-1/d_i}}) = d_i(f_i/t^{d_i})$. Since, for $j < i$, $d_i(f_j/t^{d_j}) = (\delta_{j+1} - \delta_j, \ldots, \delta_k - \delta_j)$, we have $d_i(f_j/t^{d_j}) = (\delta_{j+1} - \delta_j, \ldots, \delta_k - \delta_j)$ (coefficientwise). Lemma 4.5(a) and (b) shows that $d_i(f_0^{m_0} \cdots f_{i-1}^{m_{i-1}}/t^{d_i^{d_i-1/d_i}}) > (\delta_{i+1} - \delta_i, \ldots, \delta_k - \delta_i)$. Thus the smallest power in $f''_{i+1}$ which is not a multiple of $d_i$ and has nonzero coefficient is $\delta_{i+1}$. If $v(f''_{i+1})$ is not a multiple of $d_i$, we choose $f_{i+1} = c f''_i$ (c chosen so that $f_{i+1}$ is monic). If $v(f''_{i+1})$ is a multiple of $d_i$, then $\gamma_i > c_d((\delta_0, \ldots, \delta_i))$ shows that $v(f''_{i+1} - f_0^{m_0} \cdots f_{i-1}^{m_{i-1}}) = v(f''_{i+1}) > v(f''_{i+1})$ for some $m_0, \ldots, m_i \geq 0$. We repeat until $v(f''_i(n)) = \delta_{i+1}$, and let $f_{i+1} = c' f''_i(n)$, where $c'$ is chosen so that $f_{i+1}$ is monic. It follows from Lemma 4.5(c) that $d_{i+1}(f_{i+1}/t^{d_{i+1}}) = (\delta_{i+2} - \delta_{i+1}, \ldots, \delta_k - \delta_{i+1})$.

Lemma 4.7 Let $\mathcal{O}$ be a branch with characteristic exponents $(\delta_0, \ldots, \delta_k)$. Then the semigroup $v(\mathcal{O})$ has conductor $\sum_{i=1}^k (d_{i-1} - d_i) \delta_i + (1 - d_0)$.

Proof. We make induction over the number $l$ of blowups we need to get a regular branch. If $l = 1$, then $\mathcal{O} = \mathbb{C}[t^{d_0}, t^{d_0+1} + \ldots]$. It follows from Proposition 2.3 that $v(\mathcal{O}) = \langle \delta_0, \delta_0+1 \rangle$, which has conductor $\langle \delta_0-1 \rangle\delta_0 = (\delta_0-1)(\delta_0+1)+1 - \delta_0 = (d_0 - d_1)\delta_1 + 1 - d_0$. Suppose the claim is proved for $l - 1$. Let $c$ and $c'$ denote the conductors of $v(\mathcal{O})$ and $v(\mathcal{O}')$, respectively. In case a) of Lemma 3.1, a calculation using Lemma 4.4 gives $c = c' - \sum_{i=1}^k (d_{i-1} - d_i)\delta_{i-1} = \delta_0^2 - \delta_0$. By induction the statement is true for $v(\mathcal{O}')$. Proposition 2.3 shows it is true for $v(\mathcal{O})$. A similar calculation in case b) of Lemma 3.1 shows that $c = c' = \delta_0^2 - \delta_0$ also in this case. In case c) of Lemma 3.1 finally, we get, by using $\delta_1 - \delta_0 = d_1, \delta_0 = kd_1, \delta_1 = (k+1)d_1$ for some $k$, that $c = c' = k^2d_1^2 + kd_1 = \delta_0^2 - \delta_0$ also in case c).
Proposition 4.8. Let \( \langle \delta_0, \ldots, \delta_k \rangle \subseteq v(\mathcal{O}) \) and that by Lemmas 4.4 and 4.7 these two semigroups have the same conductor. Since \( \langle \delta_0, \ldots, \delta_k \rangle \) is symmetric, all strictly larger semigroups have smaller conductor. This gives that the two semigroups are in fact the same.

We get an easy criterion for a semigroup \( \langle a_0, \ldots, a_k \rangle \) to be a semigroup for a plane branch. The following seems to be a simpler characterization of the semigroup of a plane branch, with respect to equivalent characterizations found in [2] or [10].

Proposition 4.8 Let \( S \) be a semigroup which is minimally generated by \( a_0 < a_1 < \cdots < a_k \) and let \( d_i = \gcd(a_0, \ldots, a_i), i = 0, \ldots, k \). Then \( S \) is the semigroup of a plane branch if and only if the following conditions are satisfied.

(a) \( d_0 > d_1 > \cdots > d_k = 1 \).
(b) \( a_i > \operatorname{lcm}(d_{i-2}, a_{i-1}) \) for \( i = 2, \ldots, k \).

Proof. The necessity follows from Theorem 4.1, the sufficiency from the branch \( \mathbb{C}[[t^{a_0}, t^{a_1} + t^{a_2} - \operatorname{lcm}(d_{0}, a_1) + \cdots + t^{a_1} + \cdots + a_k - \operatorname{lcm}(d_{k-2}, a_{k-1})]] \).

We give two concrete examples.

Example Let \( S = \langle 30, 42, 280, 855 \rangle \). Then \( S \) satisfies the conditions in Proposition 4.8, so \( S = v(\mathcal{O}) \) for some \( \mathcal{O} \). We can choose e.g. \( \mathcal{O} = [[t^{30}, t^{42} + t^{112} + t^{127}]] \). The conductor equals \( t^{1554} \mathbb{C}[[t]] \). With the notation of the previous section, the multiplicity sequence is \( M(30, 42), M(6, 70), M(2, 15), \ldots, \), which is \( 30, 12^{(2)}, 6^{(3)}, 4, 2^{(9)}, 1^{(2)}, \ldots \).

Example Let \( \mathcal{O} = \mathbb{C}[[x, y]] \) with
\[
x = t^2 2^n, \quad y = t^3 2^n + t^3 2^n + 2^2 + \cdots + t^3 2^n + 2^{2i-1} + 2^{i+1}.
\]

The generators of \( v(\mathcal{O}) \) are \( \delta_0 = 2^{n+1}, \delta_i = 2^{n-i+1}(3 \cdot 2^{2i-2} + (4^{i-1} - 1)/3) \) if \( i = 1, \ldots, n+1 \).

5 Complete intersection rings arising from the semigroup of a plane branch

Let \( S = \langle \delta_0, \ldots, \delta_k \rangle = v(\mathcal{O}) \) be the semigroup of a plane branch, where \( \delta_0 < \delta_1 < \cdots < \delta_k \) is a minimal set of generators of \( S \), and let \( \mathbb{C}[S] = \mathbb{C}[t^{\delta_0}, \ldots, t^{\delta_k}] = \mathbb{C}[Y_0, \ldots, Y_k]/I = T \). We will show that \( T \) has an associated graded ring (in the \( \langle Y_0, \ldots, Y_k \rangle \)-filtration), which is a complete intersection. In particular this implies that \( T \) is a complete intersection [11]. We will use [12, Theorem 1] which states that if all elements in \( \operatorname{Ap}(S, \delta_0) \), the Apery set of \( S \) with respect to \( \delta_0 \), have unique expressions as linear combinations of the generators of \( S \), then the relations are determined by the minimal elements above the Apery set. In the following results, we suppose \( S = v(\mathcal{O}) \), where \( \mathcal{O} \) is a plane branch. We also keep the notation of the previous sections.

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Lemma 5.1 All elements in \( \text{Ap}(S, \delta_0) \) have unique expressions.

Proof. The elements in \( \text{Ap}(S, \delta_0) \) are of the form \( i_1 \delta_1 + \cdots + i_k \delta_k \), with \( 0 \leq i_j < d_{j-1}/d_j \) (cf. proof of Lemma 4.2). Suppose \( i_1 \delta_1 + \cdots + i_k \delta_k = j_0 \delta_0 + \cdots + j_k \delta_k \). Then \( i_k \delta_k \equiv j_k \delta_k \pmod{d_k-1} \). Since \( i_1 \delta_1 + \cdots + i_{k-1} \delta_{k-1} < \delta_k \), this implies that \( i_k = j_k \). If \( k > 1 \) we get \( i_{k-1} \delta_{k-1} = j_{k-1} \delta_{k-1} \pmod{d_{k-2}} \), which gives \( i_{k-1} = j_{k-1} \) a.s.o. Finally \( 0 = j_0 \delta_0 \), so \( j_0 = 0 \).

Next we determine the “minimals” (cf. [12]), i.e. the minimal elements \((n_1, \ldots, n_k) \in \mathbb{N}^k \) such that \( n_1 \delta_1 + \cdots + n_k \delta_k \notin \text{Ap}(S, \delta_0) \) (the order in \( \mathbb{N}^k \) is the usual one). Some \( n_j \) must be at least \( d_{j-1}/d_j \), otherwise the element belongs to \( \text{Ap}(S, \delta_0) \). On the other hand at most one \( n_j \geq d_{j-1}/d_j \) and there must be equality, if the element is minimal outside \( \text{Ap}(S, \delta_0) \). Thus the mininals are

\[
\{(d_0/d_1, 0, \ldots, O), (0, d_1/d_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_{k-1}/d_k)\}.
\]

Thus the following theorem follows from [12, Theorem 1].

Theorem 5.2 A minimal presentation for \( \mathbb{C}[S] \) is

\[
\mathbb{C}[S] = \mathbb{C}[Y_0, \ldots, Y_k]/(Y_1^{d_0/d_1} - m_1, \ldots, Y_k^{d_{k-1}/d_k} - m_k)
\]

where \( m_j \) is a monomial in \( Y_0, \ldots, Y_j \) for \( j = 1, \ldots, k \). Thus \( \mathbb{C}[S] \) is a complete intersection.

Corollary 5.3 The associated graded ring of \( \mathbb{C}[S] \) with respect to the filtration given by powers of \( (Y_0, \ldots, Y_k) \) is \( \mathbb{C}[Y_0, \ldots, Y_k]/(Y_1^{d_0/d_1}, \ldots, Y_k^{d_{k-1}/d_k}) \). Thus it is a complete intersection.

Proof. Since \( m_j = Y_0^{n_0} \ldots Y_{j-1}^{n_{j-1}} \) and \( n_0 \delta_0 + \cdots + n_{j-1} \delta_{j-1} = (d_{j-1}/d_j) \delta_j \), it is clear that \( n_0 + \cdots + n_{j-1} > (d_{j-1}/d_j) \), so \( \text{in}(Y_1^{d_0/d_1} - m_1) = Y_j^{d_{j-1}/d_j} \). Since \( Y_1^{d_0/d_1}, \ldots, Y_k^{d_{k-1}/d_k} \) is a regular sequence, we get the result, cf. [11].

Remark. Notice that not only for semigroups of plane branches the two result above hold. For example, if \( S = (4, 6, 7) \), then \( S \) is not the semigroup of a plane branch, but \( \mathbb{C}[S] = \mathbb{C}[X, Y, Z]/(Y^2 - X^3, Z^2 - X^2 Y) \) is a complete intersection and also its associated graded ring is a complete intersection.

Corollary 5.4 The generating function for \( S \), i.e. \( \sum_{i \in S} t^i \), equals

\[
(1 - t^{(d_0/d_1) \delta_1}) \cdots (1 - t^{(d_{k-1}/d_k) \delta_k})/((1 - t^{\delta_0}) \cdots (1 - t^{\delta_k})�
\]

Proof. As graded algebra \( \mathbb{C}[S] \) is generated by \( k + 1 \) elements of degrees \( \delta_i \), \( i = 0, \ldots, k \) and has \( k \) minimal relations of degrees \( (d_{i-1}/d_i) \delta_i \), \( i = 1, \ldots, k \), which constitute a regular sequence.

Examples. If \( \mathcal{O} = \mathbb{C}[t^{8}, t^{12} + t^{14} + t^{15}] \), then \( v(\mathcal{O}) = (8, 12, 26, 53) \) so the generating function is \( (1 - t^{24})(1 - t^{38})(1 - t^{106})/((1 - t^{8})(1 - t^{12})(1 - t^{26})(1 - t^{53})) \).

If \( \mathcal{O} = \mathbb{C}[t^{30}, t^{42} + t^{12} + t^{127}] \), then \( v(\mathcal{O}) = (30, 42, 280, 855) \) so the generating function is \( (1 - t^{210})(1 - t^{840})(1 - t^{1710})/((1 - t^{30})(1 - t^{42})(1 - t^{280})(1 - t^{855})) \).
References


