Two generalizations of the metamorphosis definition

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Abstract

Let $B(k, \lambda)$ be the spectrum of integers $n$ such that there exists a $S\lambda(2, k, n)$, a balanced incomplete block design of order $n$, block size $k$ and index $\lambda$. Lindner and Rosa [6] introduced the definition of a $S\lambda(2, 4, n)$ having a metamorphosis into a $S\lambda(2, 3, n)$ and proved that the necessary condition $n \in B(3, \lambda) \cap B(4, \lambda)$ is also sufficient.

The aim of this paper is to present two different generalizations of Lindner and Rosa’s idea in order to consider metamorphoses of $S\lambda(2, 4, n)$ for $n \in B(4, \lambda)$ and $n \notin B(3, \lambda)$.

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1 Introduction

A balanced incomplete block design $S\lambda(2, k, n)$ is a pair $(X, B)$, where $X$ is a $n$-set and $B$ is a collection of $k$-subsets of $X$ (blocks) such that any 2-subset of $X$ is contained in exactly $\lambda$ blocks. For $\lambda = 1$ we write $S(2, k, n)$ instead of $S_1(2, k, n)$.

A maximum packing of triples $MPT(n, \lambda)$ is a pair $(X, C)$, where $X$ is an $n$-set and $C$ is a collection of 3-subsets of $X$ (blocks) such that: (i) each 2-subset of $X$ is contained in at most $\lambda$ blocks of $C$, (ii) if $D$ is any collection of 3-subsets of $X$ satisfying (i), then $|C| \geq |D|$. Let $(X, C)$ be a $MPT(n, \lambda)$; the leave of $(X, C)$ is a multigraph $(X, E)$ where an edge $\{x, y\} \in E$ has multiplicity $m$ if and only if the corresponding 2-subset $\{x, y\}$ is contained in exactly $\lambda - m$ blocks of $C$.

Let $(X, B)$ be a $S\lambda(2, 4, n)$. If a star is removed from each block of $B$ the resulting collection of triangles $P(B)$ is a partial $S\lambda(2, 3, n)$ $(X, P(B))$. If the edges belonging to the deleted stars can be arranged into a collection of triangles $T(B)$, then $(X, P(B) \cup T(B))$ is a $S\lambda(2, 3, n)$, called a metamorphosis of the $S\lambda(2, 4, n)$ $(X, B)$. Lindner and Rosa [6] posed the following spectrum problem: “For every positive integer $\lambda$, determine the spectrum of integers $n$ such that there exists a $S\lambda(2, 4, n)$ having a metamorphosis into a $S\lambda(2, 3, n)$”. The necessary condition for the existence of a $S\lambda(2, 4, n)$ having a metamorphosis into a $S\lambda(2, 3, n)$ is $n \in B(3, \lambda) \cap B(4, \lambda)$, where $B(k, \lambda)$ is the set of the integers $n$ such that there is a $S\lambda(2, k, n)$. Lindner and Rosa [6] proved that these necessary conditions are also sufficient. Table 1 summarizes Lindner and Rosa’s results.

<table>
<thead>
<tr>
<th>$\lambda \pmod{6}$</th>
<th>spectrum of $S\lambda(2, 4, n)$ having a metamorphosis into $S\lambda(2, 3, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$n \geq 4$</td>
</tr>
<tr>
<td>1, 5</td>
<td>$n \equiv 1 \pmod{12}$</td>
</tr>
<tr>
<td>2, 4</td>
<td>$n \equiv 1 \pmod{3}$</td>
</tr>
<tr>
<td>3</td>
<td>$n \equiv 1 \pmod{4}$</td>
</tr>
</tbody>
</table>

For $n \in B(4, \lambda)$ and $n \not\in B(3, \lambda)$, the following question is natural: How can we generalize the metamorphosis definition in order to construct a $S\lambda(2, 4, n)$ having a metamorphosis into some design as close as possible to a $S\lambda(2, 3, n)$? The aim of this paper is to present two different answers.
Metamorphosis of a $S_\lambda(2, 4, n)$ into a minimum $S_\lambda(2, 3, v)$. Let 
$(X, \mathcal{B})$ be a $S_\lambda(2, 4, n)$. Let $v$ be the minimum integer such that $v \geq n$ and $v \in B(3, \lambda)$, and let $V = X \cup Y$ where $|Y| = v - n$. If a star is removed from each block of $\mathcal{B}$ the resulting collection of triangles $P(\mathcal{B})$ is a partial $S_\lambda(2, 3, n) (X, P(\mathcal{B}))$. If the edges belonging to the deleted stars and to graphs $K_Y$ and $K_{X,Y}$, can be arranged into a collection of triples $T(\mathcal{B})$, then $(X, P(\mathcal{B}) \cup T(\mathcal{B}))$ is a $S_\lambda(2, 3, v)$, called a metamorphosis of the $S_\lambda(2, 4, n) (X, \mathcal{B})$ into the minimum $S_\lambda(2, 3, v)$.

Metamorphosis of a $S_\lambda(2, 4, n)$ into a MPT$(n, \lambda)$. Let $(X, \mathcal{B})$ be a $S_\lambda(2, 4, n)$. If a star is removed from each block of $\mathcal{B}$ the resulting collection of triangles $P(\mathcal{B})$ is a partial $S_\lambda(2, 3, n) (X, P(\mathcal{B}))$. If the edges belonging to the deleted stars can be arranged into a collection of triangles $T(\mathcal{B})$ and a collection of edges $E$ such that $(X, P(\mathcal{B}) \cup T(\mathcal{B}) \cup E)$ is a MPT$(n, \lambda)$ with leave $(X, E)$, then $(X, P(\mathcal{B}) \cup T(\mathcal{B}) \cup E)$ is called a metamorphosis of the $S_\lambda(2, 4, n) (X, \mathcal{B})$ into a MPT$(n, \lambda)$.

It is straightforward to see that both these definitions coincide with Lindner and Rosa’s metamorphosis whenever $n \in B(3, \lambda)$.

In this paper we solve the spectrum problems related to above definitions, leaving a few open cases in the case of the metamorphosis of a $S_\lambda(2, 4, n)$ into a MPT$(n, \lambda)$.

## 2 Metamorphosis of a $S_\lambda(2, 4, n)$ into a minimum $S_\lambda(2, 3, v)$

In Table 2 we show the sets $B(k, \lambda)$ of integers $n$ for which there exists a $S_\lambda(2, k, n)$ for $k = 3, 4$ [11].

<table>
<thead>
<tr>
<th>$\lambda$ (mod 6)</th>
<th>$B(4, \lambda)$</th>
<th>$B(3, \lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$n \geq 4$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>1, 5</td>
<td>$n \equiv 1, 4$ (mod 12)</td>
<td>$n \equiv 1, 3$ (mod 6)</td>
</tr>
<tr>
<td>2, 4</td>
<td>$n \equiv 1$ (mod 3)</td>
<td>$n \equiv 0, 1$ (mod 3)</td>
</tr>
<tr>
<td>3</td>
<td>$n \equiv 0, 1$ (mod 4)</td>
<td>$n \equiv 1$ (mod 2)</td>
</tr>
</tbody>
</table>

Pairing Tables 1 and 2, we get the necessary conditions for the existence of a $S_\lambda(2, 4, n)$ having a metamorphosis into a minimum $S_\lambda(2, 3, v)$ (see Table...
3). The sufficiency for \( v = n \) is proved in [6]. In this section we prove the sufficiency for \( v > n \).

<table>
<thead>
<tr>
<th>( \lambda ) (mod 6)</th>
<th>( n \geq 4 )</th>
<th>( v - n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 (mod 12)</td>
<td>0</td>
</tr>
<tr>
<td>1, 5</td>
<td>4 (mod 12)</td>
<td>3</td>
</tr>
<tr>
<td>2, 4</td>
<td>1 (mod 3)</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1 (mod 4)</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0 (mod 4)</td>
<td>1</td>
</tr>
</tbody>
</table>

A \((K, \lambda)\)-GDD (group divisible design of index \( \lambda \), block sizes in \( K \) and order \( v \)) is a triple \((V, \mathcal{G}, \mathcal{B})\), where \( V \) is a \( v \)-set, \( \mathcal{G} = \{G_1, G_2, \ldots, G_m\} \) is a partition of \( V \) into subsets (called groups), and \( \mathcal{B} \) is a collection of subsets (blocks) of \( V \) which satisfy the properties:

1. If \( B \in \mathcal{B} \) then \( |B| \in K \).
2. Every pair of distinct elements of \( V \) occurs in exactly \( \lambda \) blocks or one group, but not both.
3. \( |\mathcal{G}| > 1 \).

We say that the \((K, \lambda)\)-GDD is of type \( v_1^{h_1}v_2^{h_2} \ldots v_t^{h_t} \), if there are \( h_i \) groups of size \( v_i \), \( i = 1, 2, \ldots, t \). We write \((k, \lambda)\)-GDD instead of \((\{k\}, \lambda)\)-GDD.

Let \((V, \mathcal{G}, \mathcal{B})\) be a \((4, \lambda)\)-GDD. If a star is removed from each block of \( \mathcal{B} \) the resulting collection of triangles \( P(\mathcal{B}) \) is a partial \((3, \lambda)\)-GDD \((V, \mathcal{G}, P(\mathcal{B}))\).

If the edges belonging to the deleted stars can be arranged into a collection of triangles \( T(\mathcal{B}) \), then \((V, \mathcal{G}, P(\mathcal{B}) \cup T(\mathcal{B}))\) is a \((3, \lambda)\)-GDD, called a metamorphosis of the \((4, \lambda)\)-GDD \((V, \mathcal{G}, \mathcal{B})\).

The following result is given by Lindner and Rosa [6].

**Lemma 2.1.** For every integer \( h \geq 5 \), there is a \((4, 1)\)-GDD of type \( 12^h \) having a metamorphosis into a \((3, 1)\)-GDD of type \( 12^h \).

Obviously, only the cases \( \lambda = 1, 3 \) must be considered. Starting cases are collected in the following lemma. See [9] for a proof.

**Lemma 2.2.** 1. A \( S(2, 4, n) \) having a metamorphosis into a \( S(2, 3, n + 3) \) exists for \( n = 4, 16, 28, 40, 52 \).
2. A $S_3(2, 4, n)$ having a metamorphosis into a $S_3(2, 3, n + 1)$ exists for $n = 4, 8, 12, 16, 28, 32$.

3. There exists a $S(2, 4, 16)$ with one hole $H$ of size 4 having a metamorphosis into a partial $S(2, 3, 16)$ whose leave is given by three 1-factors on vertex set $X \setminus H$.

**Theorem 2.3.** A $S(2, 4, n)$ having a metamorphosis into a $S(2, 3, n + 3)$ exists for every integer $n \equiv 4 \pmod{12}$, $n \geq 4$.

**Proof** For $n = 4, 16, 28, 40, 52$, see Lemma 2.2. Let $n = 4 + 12h$, $h \geq 5$. By Lemma 2.1, there is a $(4, 1)$-GDD of type $12^h$ having a metamorphosis into a $(3, 1)$-GDD of type $12^h$. Denote the groups by $G_i$, $i = 1, 2, \ldots, h$. Let $H = \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Produce a $S(2, 4, 16)$ on vertex set $G_1 \cup H$ having a metamorphosis into a $S(2, 3, 19)$ on vertex set $G_1 \cup H \cup \{a_1, a_2, a_3\}$. For every $i = 2, 3, \ldots, h$, produce a copy of the $S(2, 4, 16)$, given in 3 of Lemma 2.2, on vertex set $G_i \cup H$ and hole $H$. This design has a metamorphosis into a partial $S(2, 3, 16)$ with leave $F^i_j$, $j = 1, 2, 3$. Form the triples $\{a_j, x, y\}$, $\{x, y\} \in F^i_j$. □

**Theorem 2.4.** A $S_3(2, 4, n)$ having a metamorphosis into a $S_3(2, 3, n + 1)$ exists for every integer $n \equiv 0 \pmod{4}$, $n \geq 4$.

**Proof** For $n = 4, 8, 12, 16, 28, 32$, see Lemma 2.2. A PBD of order $m$ with block sizes 5, 9 and 13 exists for all $m \equiv 1 \pmod{4}$ except for $m = 17, 29, 33$ [11]. Remove a point to obtain a $(\{5, 9, 13\}, 1)$-GDD of order $m - 1$ with groups whose sizes lie in $\{4, 8, 12\}$. Place a solution on each block (see [6]) and on each group.

### 3 Metamorphosis of a $S_\lambda(2, 4, n)$ into a $MPT(n, \lambda)$

Let $(X, \mathcal{C})$ be a $MPT(n, \lambda)$ with leave $(X, \mathcal{E})$. If $\mathcal{E} = \emptyset$, then $(X, \mathcal{C})$ is a $S_\lambda(2, 3, n)$ and Lindner and Rosa’s metamorphosis works. So we have to find a solution for $n \equiv 4 \pmod{12}$, if $\lambda \equiv 1, 5 \pmod{6}$, and for $n \equiv 0 \pmod{4}$, if $\lambda \equiv 3 \pmod{6}$. So, only $\lambda = 1, 3$ must be considered. Note that there are different graphs which can be leaves of a $MPT(n, 3)$, $n \equiv 8 \pmod{12}$ [10]. In this paper we don’t consider all possible leaves but only one, as shown in Table 4.
Table 4

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$n$</th>
<th>leave</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\equiv 1, 5 \pmod{6}$</td>
<td>$\equiv 4 \pmod{12}$</td>
<td>1FY</td>
</tr>
<tr>
<td>$\equiv 3 \pmod{6}$</td>
<td>$\equiv 0 \pmod{12}$</td>
<td>1F</td>
</tr>
<tr>
<td>$\equiv 3 \pmod{6}$</td>
<td>$\equiv 4 \pmod{12}$</td>
<td>1FY</td>
</tr>
<tr>
<td>$\equiv 3 \pmod{6}$</td>
<td>$\equiv 8 \pmod{12}$</td>
<td>1F$_3$</td>
</tr>
</tbody>
</table>

Here 1F, 1FY and 1F$_3$ are the following graphs.

1F  a matching on $n$ vertices;

1FY  a tripole (matching on $n - 4$ vertices and a tree on 4 vertices with one vertex of degree 3);

1F$_3$  a matching on $n - 2$ vertices and a triple edge $\{a,b\}, \{a,b\}, \{a,b\}$.

Starting cases are collected in the following lemma. See [9] for a proof.

Lemma 3.1.  
1. There exists a $S(2,4,n)$ having a metamorphosis into a $MPT(16,1)$.

2. There exists a $S(2,4,16) (X,B)$ with one hole $H$ of size 4 having a metamorphosis into a partial $S(2,3,16) (X,C)$ whose leave is one 1-factor on vertex set $X \setminus H$.

3. A $S_3(2,4,n)$ having a metamorphosis into a $MPT(n,3)$ (with leave shown in Table 4) there is for $n = 8, 12, 20, 24, 32$.

4. There exists a $(4,1)$-GDD of type $4^4$ having a metamorphosis into a $(3,1)$-GDD of type $4^4$.

5. There exists a $S_3(2,4,32) (X,B)$ such that: (i) $(X,B)$ embeds a $S_3(2,4,8) (A,A)$ having a metamorphosis into a $MPT(8,3) (A,P(A) \cup T(A))$; (ii) $(X,B)$ has a metamorphosis into a $MPT(32,3) (X,P(B) \cup T(B))$; (iii) $(A,P(A) \cup T(A))$ is embedded into $(X,P(B) \cup T(B))$; (iv) the leave of $(A,P(A) \cup T(A))$ is a subgraph of the leave of $(X,P(B) \cup T(B))$.

Theorem 3.2. A $S(2,4,n)$ having a metamorphosis into a $MPT(n,1)$ exists for every integer $n \equiv 4 \pmod{12}$, $n \geq 4$, except possibly for $n = 28, 40, 52$. 

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Proof The proof for \( n = 4 \) is trivial. For \( n = 16 \), Lemma 3.1 gives a \( S(2, 4, 16) \) having a metamorphosis into a \( MPT(16, 1) \) which embeds a \( MPT(4, 1) \). Let \( n = 4 + 12h, h \geq 5 \). By Lemma 2.1, there is a \((4, 1)^2\)-GDD of type \( 12^h \) having a metamorphosis into a \((3, 1)^2\)-GDD of type \( 12^h \). Denote the groups by \( G_i, i = 1, 2, \ldots, h \). Let \( H = \{\infty_1, \infty_2, \infty_3, \infty_4\} \). As in Lemma 3.1, produce a \( S(2, 4, 16) \) on vertex set \( G_1 \cup H \). For \( i = 2, 3, \ldots, h \), produce a copy of the \( S(2, 4, 16) \) on vertex set \( G_i \cup H \), having the hole \( H \).

**Theorem 3.3.** A \( S_3(2, 4, n) \) having a metamorphosis into a \( MPT(n, 3) \) (with leave shown in Table 4) exists for every integer \( n \equiv 0 \pmod{4}, n \geq 4 \), except possibly for \( n = 28, 36, 44, 48, 52, 56, 68, 80, 92, 104 \).

**Proof** For \( n = 8, 12, 20, 24, 32 \), see Lemma 3.1. For \( n \equiv 0 \pmod{12}, n \geq 60 \), place a copy of the \( S_3(2, 4, 12) \), given in Lemma 3.1, into each group of the \((4, 1)^2\)-GDD of Lemma 2.1.

For \( n \equiv 4 \pmod{12}, n \geq 4 \) and \( n \neq 28, 40, 52 \), paste together a solution of \( \lambda = 1 \) (Theorem 3.2) and a solution of \( \lambda = 2 \) [6].

Let \( n \equiv 8 \pmod{12}, n \geq 116 \). The Handbook of Combinatorial Designs [11] gives a \((4, 1)^2\)-GDD of type \( 6^u \) for every \( u \geq 5 \), and of type \( 6^{2u} \) for every \( u \geq 4 \). Giving weight 4 to all points, we get a \((4, 1)^2\)-GDD \((X, G_1, B_1)\) of type \( 24^u \) and a \((4, 1)^2\)-GDD \((X, G_2, B_2)\) of type \( 24^{2u+12} \), respectively. Let \( A = \{a_1, a_2, \ldots, a_8\} \). If \( n \equiv 8 \pmod{24} \), then

- On each block of \( B_1 \) place a copy of the \((4, 1)^2\)-GDD given in 4 of Lemma 3.1.
- For each group \( G \in G_1 \), produce a copy of the \( S_3(2, 4, 32) \), given in 5 of Lemma 3.1, having vertex set \( G \cup A \) and hole \( A \).
- On the hole \( A \) place a \( S_3(2, 4, 8) \) having metamorphosis into a \( MPT(8, 3) \).

If \( n \equiv 20 \pmod{24} \), then

- On each block of \( B_2 \) place a copy of the \((4, 1)^2\)-GDD given in 4 of Lemma 3.1.
- For each group \( G \in G_2 \) such that \( |G| = 24 \), produce a copy of the \( S_3(2, 4, 32) \), given in 5 of Lemma 3.1, having vertex set \( G \cup A \) and hole \( A \).
- On the group of size 20 place a \( S_3(2, 4, 20) \) having metamorphosis into a \( MPT(20, 3) \). □
4 Open Questions and Remarks

1. Remove the possible exceptions in Theorems 3.2 and 3.3.

2. For \( \lambda = 3 \) and \( n \equiv 8 \quad (\text{mod} \quad 12) \), find a metamorphosis of a \( S_\lambda(2, 4, n) \) into a \( MPT(n, \lambda) \) with any possible leave [10].

3. Let \( G_1 \) be a subgraph of \( G \). Then Lindner and Rosa’s metamorphosis can be easily generalized in the following way. Let \( (X, \mathcal{B}) \) be a \( G \)-decomposition of the multigraph \( \lambda K_n \) [11]. If a graph isomorphic to \( G \setminus G_1 \) is removed from each \( G \)-block of \( \mathcal{B} \), the resulting collection of \( G_1 \)-blocks \( P(\mathcal{B}) \) is a partial \( G_1 \)-decomposition of \( \lambda K_n \) \( (X, P(\mathcal{B})) \). If the edges belonging to the deleted subgraphs can be arranged into a collection of \( G_1 \)-blocks \( T(\mathcal{B}) \), then \( (X, P(\mathcal{B}) \cup T(\mathcal{B})) \) is a \( G_1 \)-decomposition of \( \lambda K_n \), called a metamorphosis of the \( G \)-decomposition \( (X, \mathcal{B}) \). The related spectrum problem has been solved for many pairs of graphs \( G \) and \( G_1 \) [1, 2, 3, 4, 5, 7, 8].

Extend both metamorphosis definitions, given in this paper for \( S_\lambda(2, 4, n) \), to \( G \)-decompositions of \( \lambda K_n \) and solve the related spectrum problems.

4. During the meeting ISGDA (Messina, October 2003) we learnt that the generalization of the metamorphosis definition given in Section 3 is not new. The problem of the metamorphosis of some graph designs of order \( v \) and index \( \lambda \) into a \( MPT(v, \lambda) \) is considered by other authors and their papers are not published yet. But, as we know, no other paper studies the same problem of Section 3.

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References


