Grooming for Two-Period Optical Networks

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Abstract

Minimizing the number of add-drop multiplexers (ADMs) in a unidirectional SONET ring can be formulated as a graph decomposition problem. When traffic requirements are uniform and all-to-all, groomings that minimize the number of ADMs (equivalently, the drop cost) have been characterized for grooming ratio at most six. However, when two different traffic requirements are supported, these solutions do not ensure optimality. In two-period optical networks, \( n \) vertices are required to support a grooming ratio of \( C \) in the first time period, while in the second time period a grooming ratio of \( C' \), \( C' < C \), is required for \( v \leq n \) vertices. This allows the two-period grooming problem to be expressed as an optimization problem on graph decompositions of \( K_n \) that embed graph decompositions of \( K_v \) for \( v \leq n \). Using this formulation, optimal two-period groomings are found for small grooming ratios using techniques from the theory of graphs and designs.

Keywords: optical networks, traffic grooming, graph decomposition, combinatorial designs
1 Introduction

A massive migration to digital communication in the telecommunications infrastructure has occurred as a result of the deployment of fiber optic cables. Hastening the widespread use of fiber is the acceptance of the Synchronous Optical NETwork/Synchronous Digital Hierarchy (SONET/SDH) signal generation standard at the physical layer. A SONET system consists of switches, multiplexers, and repeaters, all connected by fiber. The topology of a SONET system can be a mesh, but is often a ring.

Wavelength division multiplexing (WDM) is an immediate and cost-effective means of increasing fiber capacity without the added investment of installing new fiber. WDM partitions the optical transmission spectrum into a number of disjoint wavelength bands, each supporting a single communication channel. This allows multiple (as many as 80 in dense WDM) wavelength channels to coexist in a single fiber.

Coupling WDM with SONET rings not only significantly increases capacity but also potentially reduces the amount of terminal equipment required. This includes add/drop multiplexers (ADMs) that enable individual wavelengths to optically bypass a vertex through a wavelength add-drop multiplexer (WADM) rather than being electronically terminated. With WADMs, the number of ADMs required in a SONET ring is equal to the number of vertices that are terminals of requests carried in the ring.

Consider the unidirectional SONET/WDM ring network with four vertices in Figure 1(a) with an ADM for every wavelength \( \lambda_i \) with \( i \in \{0, 1, 2\} \) at each of the four vertices; that is, the drop cost (number of ADMs) for this architecture is twelve. The dominant cost of an architecture is not in the cost of the optics; rather, it is in the cost of the electronics. Thus if the number of ADMs can be reduced by intelligently assigning traffic to wavelengths, this lowers the cost of the architecture. One way this can be achieved is through traffic grooming, wherein low rate signals are packed into high speed streams such that all of the traffic to and from a given vertex is carried on the minimum number of wavelengths.

To make the idea of traffic grooming more concrete suppose that each wavelength is used to support an optical carrier 48, OC-48, ring with a data rate of 2488.32 Mbps, and that the traffic requirement is for eight OC-3 circuits between each pair of vertices. With four vertices there are \( \binom{4}{2} = 6 \) vertex pairs, and the total traffic load is equal to \( 6 \times 8 = 48 \) OC-3s, or equivalently, three OC-48 rings. Consider the following two circuit assignments of traffic each carrying the same amount of total traffic. The only difference in these assignments is in how vertex pairs are assigned to wavelengths:

**Assignment 1:** \( \lambda_0 = \{0 \leftrightarrow 1, 2 \leftrightarrow 3\}, \lambda_1 = \{0 \leftrightarrow 2, 1 \leftrightarrow 3\}, \lambda_2 = \{0 \leftrightarrow 3, 1 \leftrightarrow 2\}. \)

**Assignment 2:** \( \lambda_0 = \{0 \leftrightarrow 1, 1 \leftrightarrow 2\}, \lambda_1 = \{0 \leftrightarrow 2, 2 \leftrightarrow 3\}, \lambda_2 = \{0 \leftrightarrow 3, 1 \leftrightarrow 3\}. \)

In each assignment, the grooming ratio is two since each wavelength is assigned two vertex pairs, i.e., each wavelength carries \( 2 \times 8 = 16 \) OC-3 circuits. Multiplexing the 16 OC-3s, each with a data rate of 155.52 Mbps, yields a high speed OC-48 stream.

In the first assignment, each vertex has traffic on every wavelength; \( \lambda_0 \) carries traffic between vertices 0 and 1 (0 \( \leftrightarrow \) 1) and between vertices 2 and 3 (2 \( \leftrightarrow \) 3) and similarly for
Figure 1: Drop cost is: (a) 12 without traffic grooming; (b) 9 with traffic grooming.

$\lambda_1$ and $\lambda_2$. Thus every vertex requires an ADM on every wavelength. Figure 1(a) supports such a circuit assignment of traffic. In the second assignment, now each wavelength carries traffic among three vertices. For example, $\lambda_0$ now carries traffic between vertices 0 and 1 and between vertices 1 and 2, while $\lambda_1$ and $\lambda_2$ only involve traffic among vertices 0, 2, and 3, and among vertices 0, 1, and 3, respectively. Figure 1(b) shows that this assignment of traffic reduces the drop cost from twelve to nine.

Bermond and Coudert [5] and Wan [35] formulated such grooming problems on unidirectional and bidirectional rings, respectively, in terms of graph decompositions. To see how, we continue the same example. Since the network has 4 vertices, consider the complete graph $K_4$. In general, $K_n$ has $\binom{n(n-1)}{2}$ edges, one from each vertex to every other. In the application, traffic is carried from each of $n$ source vertices to each of $n-1$ destination vertices, for a total of $n(n-1)$ connections; each requires a fixed fraction $1/C$ of the capacity of a wavelength. Traffic from $x$ to $y$ is always carried on the same wavelength as traffic from $y$ to $x$, and hence that fixed fraction of the whole ring capacity is assigned to the undirected pair $\{x \leftrightarrow y\}$. The edges of the $K_n$ therefore correspond to all possible vertex pairs, specifically $\binom{n}{2} = \frac{n(n-1)}{2}$ vertex pairs. A graph decomposition of the complete graph $K_n$ on $n$ vertices is a partition of the edges of $K_n$ into $b$ edge-disjoint subgraphs $G_i$, $i = 0, 1, \ldots, b-1$. Figure 2 shows a graph decomposition of $K_4$ into three edge-disjoint subgraphs; $G_i$ corresponds to $\lambda_i$ in Figure 1(b) for $i = 0, 1, 2$. The total number of vertices (of non-zero degree) in the graph decomposition corresponds to the drop cost of the network. That is $\sum_{i=0}^{b-1} |V(G_i)| = 9$ corresponding to the nine ADMs in the architecture in Figure 1(b).

In this paper, we consider the problem of minimizing the drop cost to support two traffic periods in SONET/WDM unidirectional rings, rather than static traffic. As in [5,35], graph decomposition and design theory are used extensively to establish bounds for such networks. Informally, each time period supports different traffic requirements. In the first time period $n$ vertices are required to support a grooming ratio of $C$, while in the second time period
Figure 2: Graph decomposition of $K_4$ into $G_0, G_1, G_2$ with $E(G_0) = \{01, 12\}$, $E(G_1) = \{02, 23\}$, and $E(G_2) = \{03, 13\}$.

A grooming ratio of $C', C' < C$, is required for $v \leq n$ vertices. This allows the two-period grooming problem to be expressed as an optimization problem on graph decompositions of $K_n$ that embed graph decompositions of $K_v$ for $v \leq n$. Using this formulation, optimal two-period groomings are found for small grooming ratios.

While grooming traffic in general mesh networks is an important emerging problem, our focus is on grooming in rings. In addition to the network architecture, the minimum drop cost depends on the traffic pattern and on the traffic demand. The traffic may have a regular pattern, such as one-to-all or all-to-all (symmetric), or an irregular pattern. The traffic demands may be uniform, i.e., all traffic has the same demand, or non-uniform. Modiano and Lin [26] provide a good survey of related work on traffic grooming in WDM networks. We highlight the main results here, emphasizing results using graph- and design-theoretic techniques.

In a network with $n$ vertices, traffic requirements are represented by the $n \times n$ array $R = [R_{ij}]$, where $R_{ij}$ is the requirement for the vertex pair $i \leftrightarrow j$. If $R_{ij} = \frac{1}{C} > 0$ for every pair of distinct vertices $i, j$, then the traffic requirements of the network are uniform symmetric with grooming ratio $C$.

In general, the traffic grooming problem is to partition the set of traffic requirements into a number of groups such that each group is carried on a single SONET/WDM ring. When the traffic requirements are uniform symmetric with grooming ratio $C$ and the ring has $n$ vertices, the grooming is denoted by $N(n, C)$. When the grooming $N(n, C)$ is optimal, i.e., minimizes the total ADM cost, then the grooming is denoted by $\mathcal{O}N(n, C)$. Whether general or optimal, the drop cost of a grooming is denoted by $\text{cost} N(n, C)$ or $\text{cost} \mathcal{O}N(n, C)$, respectively.

Modiano and Chiu [25] show that the decision problem corresponding to the traffic grooming problem is NP-complete by reducing bin packing into traffic grooming in polynomial time. However for several special classes of traffic requirements, both optimal algorithms and heuristics with good performance have been found (see, e.g., [19–22, 26, 33]). The algorithms of Modiano and Chiu [25] and of Simmons, Goldstein, and Saleh [33] appear to be near optimal for grooming in undirectional and bidirectional rings, respectively. Three different traffic scenarios are considered: uniform traffic, distance-dependent traffic where the amount of traffic between vertex pairs is inversely proportional to the distance separating them, and hub traffic where all of the traffic is going to one vertex on the ring.
In a bidirectional ring, the amount of electronics can be reduced further if routing and wavelength assignment is performed on the groomed paths since it allows end-to-end paths to share common ADMs [20]. Both grooming and wavelength assignment are often difficult to solve.

Zhang and Qiao [37] formulated the traffic grooming problem with arbitrary traffic as an integer program. Their heuristic works well for uniform traffic with large grooming ratios, and for non-uniform traffic if a good matching algorithm is utilized.

Using tools from graph and design theory the existence of an \( \mathcal{G}(n, C) \) has been completely settled for every \( C \leq 5 \) [2, 4, 5, 23]. The solution for \( C = 6 \) is complete with a small number of exceptions [3], and the solution for \( C = 7 \) is determined within an additive constant of the optimum [13]. Bermond and Coudert [5] established that an \( \mathcal{G}(n, C) \) exists if and only if there exists a graph decomposition of \( K_n \) into \( b \) edge-disjoint subgraphs \( G_i \), \( i = 0, 1, \ldots, b - 1 \), such that \( |E(G_i)| \leq C \), and \( \sum_{i=0}^{b-1} |V(G_i)| \) is minimized. Wan [35] also used tools from graph decomposition for bidirectional rings.

In each of these cases, the traffic requirement is characterized by a single (static) traffic matrix. However, traffic is often not static. Traffic demands may change slowly over time, or more rapidly as with the traffic dynamics characteristic of the Internet. In the dynamic grooming problem, the traffic requirements are given as a set of traffic matrices. In this case, the number of ADMs needed to support any traffic matrix in this set is to be minimized. As a result, existing solutions support, in a nonblocking manner, traffic that dynamically changes within the given set [6, 19, 38].

Berry and Modiano [6] apply tools from graph theory for the dynamic grooming problem. Given a set of traffic requirements \( \{R_0, \ldots, R_{K-1}\} \) they define \( R_{ij}^* = \max_{K-1}^{k=0} R_{ij}^k \); this makes every allowable traffic requirement a subset of \( R^* \). The grooming problem is then formulated as a matching problem in a bipartite graph.

In this paper we consider rings with two periods of traffic and an optimal grooming is sought in each period; this is often better than simply taking the maximum of each requirement. Generalizing the work of Bermond and Coudert [5], we show that the related dynamic grooming problem can be expressed as an optimization problem on a graph decomposition embedding another graph decomposition (see [15, 28–30]) and we produce a solution to these problems when \( C \leq 3 \).

The rest of this paper is organized as follows. Section 2 defines grooming for two-period optical networks, a form of dynamic grooming in which traffic is characterized by two periods, each with its own requirements. In addition, the notation of graph decomposition, some terminology from design theory, the correspondence between graph decompositions that embed other graph decompositions, and the problem of grooming in two-period optical networks are provided. In order to simplify the presentation here, a companion paper [16] develops all of the lower bounds needed. Sections 3, 4, and 5 then provide constructions for the matching upper bounds when \((C, C') = (2,1), (3,1), \) and \((3,2)\), respectively. Summary statements of the main theorems in each case are given in Section 6. Concluding remarks are given in Section 7.
2 Two-Period Optical Networks

First, we make the notion of a two-period optical network precise.

**Definition 2.1** Let $X = \{0, 1, \ldots, n - 1\}$ and $V \subseteq X$, with $|V| = v$. A grooming of a two-period network $N(n, v; C, C')$ with ratio $(C, C')$ is a SONET/WDM network and a partition of time into two disjoint periods $T$ and $T'$ such that:

1. For times in $T$ the traffic is symmetric and uniform with ratio $C \geq 2$ between all vertex pairs in $X$; that is, $R_{ij}^T = \frac{1}{C}$, $C \geq 2$, for every $i, j \in X$ with $i \neq j$.

2. For times in $T'$ the traffic is symmetric and uniform with ratio $C', 1 \leq C' < C$, between all vertex pairs in $V$, i.e., $R_{ij}^{T'} = \frac{1}{C'}$ for every $i, j \in V$ with $i \neq j$. For every pair $\{i, j\}$ such that either $i, j \in W = X \setminus V$ or $i \in X$ and $j \in W$, $R_{ij}^{T'} = 0$.

Intuitively, the traffic in time periods $T$ and $T'$ is specified by different requirement matrices $[R_{ij}^T]$ and $[R_{ij}^{T'}]$. In the first time period, the traffic between vertex pairs of the $n$ vertices has a grooming ratio of $C$. In the second time period only a subset of $v \leq n$ vertices are active with the traffic among them therefore able to run at a higher rate.

We model the problem of grooming in a two-period network in terms of graph decomposition. Recall that a graph decomposition of the complete graph $K_n$ on $n$ vertices is a partition of the edges of $K_n$ into $b$ edge-disjoint subgraphs $G_i$, $i = 0, 1, \ldots, b - 1$. Such a decomposition is denoted by $(X, B)$, where $X$ is the vertex set of $K_n$ and $B = \{G_0, G_1, \ldots, G_{b-1}\}$. Following terminology from design theory, $B$ is the set of blocks of the graph decomposition.

**Definition 2.2** Let $V \subseteq X$. The graph decomposition $(X, B)$ embeds the graph decomposition $(V, D)$ if there is a mapping $f : D \rightarrow B$ such that $D$ is a subgraph of $f(D)$ for every $D \in \mathcal{D}$. If $f$ is injective (i.e., one-to-one), then $(X, B)$ faithfully embeds $(V, \mathcal{D})$.

A grooming of a two-period network $N(n, v; C, C')$ coincides with a graph decomposition $(X, B)$ of $K_n$ such that $(X, B)$ is a grooming $N(n, C)$ in time period $T$, and $(X, B)$ embeds a graph decomposition of $K_n$ such that $(V, D)$ is a grooming $N(v, C')$ in time period $T'$. Furthermore, for every $B \in \mathcal{B}$ such that $f^{-1}(B) \neq \emptyset$, $\sum_{D \in f^{-1}(B)} |E(D)| \leq C'$. Simply put, an $N(n, v; C, C')$ coincides with an $N(n, C)$ that faithfully embeds an $N(v, C')$. We use the notation $\mathcal{O}(N(n, v; C, C'))$ to denote an optimal grooming $N(n, v; C, C')$ of a two-period SONET/WDM network with ratio $(C, C')$.

We do not consider disconnected subgraphs in the decompositions formed, because our concern is with drop cost, and the drop cost of a disconnected graph is the sum of the drop costs of its components. By omitting disconnected graphs, the number of graphs in a decomposition may not coincide with the number of wavelengths needed. Throughout the remainder,

- $K_{V,W}$ denotes a complete bipartite graph in which the classes are $V$ and $W$. 


• $P_n = [a_0, a_1, \ldots, a_{n-1}]$ denotes a path, i.e., $V(P_n) = \{a_0, a_1, \ldots, a_{n-1}\}$ and $E(P_n) = \{a_0a_1, a_1a_2, \ldots, a_{n-2}a_{n-1}\}$.

• $K_n = (a_0, a_1, \ldots, a_{n-1})$ denotes a complete graph, i.e., $V(K_n) = \{a_0, a_1, \ldots, a_{n-1}\}$ and $E(K_n) = \{a_0a_1, a_0a_2, \ldots, a_0a_{n-1}, a_1a_2, a_1a_3, \ldots, a_1a_{n-1}, \ldots, a_{n-2}a_{n-1}\}$. If the notation $K_X$ is used, it represents the complete graph on the vertex set $X$.

• $K_{1,n} = [a_0; a_1, a_2, \ldots, a_n]$ is used for the star where $a_0$ is the source and $a_1, \ldots, a_n$ are sinks; i.e., $V(K_{1,n}) = \{a_0, a_1, \ldots, a_n\}$ and $E(K_{1,n}) = \{a_0a_1, a_0a_2, \ldots, a_0a_n\}$.

Figure 3 shows the notation and corresponding graphs for $P_2$, $P_3$, $K_3$, $K_{1,3}$, and $P_4$.

$$
\begin{align*}
P_2 &= [a_0, a_1] \\
P_3 &= [a_0, a_1, a_2] \\
K_3 &= (a_0, a_1, a_2) \\
K_{1,3} &= [a_0; a_1, a_2, a_3] \\
P_4 &= [a_0, a_1, a_2, a_3]
\end{align*}
$$

Figure 3: Notation and corresponding graphs for $P_2$, $P_3$, $K_3$, $K_{1,3}$, and $P_4$.

$$
\begin{align*}
B_0 &\ \\
B_1 &\ \\
B_2 &\ \\
B_3 &\ \\
B_4 &\ \\
B_5 &\ \\
B_6 &\ 
\end{align*}
$$

Figure 4: A graph decomposition of $K_7$ with optimal drop cost of 21.

We adopt the convention that vertices of $V$ are denoted by integers, while those in $W = X \setminus V$ are denoted by symbols of the form $a_i$, for $i$ an integer. Now consider the graph decomposition $(X, B)$ of $K_7$, $X = \{0, 1, 2, 3\} \cup \{a_0, a_1, a_2\}$, into seven blocks of 3-cycles $B = \{B_0, \ldots, B_6\}$ where $B_0 = (0, 1, a_0)$, $B_1 = (0, 2, a_1)$, $B_2 = (0, 3, a_2)$, $B_3 = (1, 2, a_2)$, $B_4 = (1, 3, a_1)$, $B_5 = (2, 3, a_0)$, and $B_6 = (a_0, a_1, a_2)$. Figure 4 shows this decomposition of $K_7$. This graph decomposition embeds the graph decomposition $(V, D)$ where
\( V = \{0, 1, 2, 3\} \). Now \( \mathcal{D} = \{D_0, \ldots, D_5\} \) where \( D_0 = [0, 1], D_1 = [0, 2], D_2 = [0, 3], D_3 = [1, 2], D_4 = [1, 3], \) and \( D_5 = [2, 3] \). In this embedding the drop cost is 21 and this is optimal. Thus the \( N(7, 4; 3, 1) \) in Figure 4 is an \( \mathcal{ON} \). That is, \( \text{cost } N(7, 4; 3, 1) = \text{cost } \mathcal{ON}(7, 4) = \text{cost } \mathcal{ON}(7, 3) = 21 \).

![Graph decomposition of \( K_8 \) into 16 blocks embedding \( K_6 \) with drop cost 44.]

As it turns out, an \( \mathcal{ON}(n, v; C, C') \) does not always coincide with an \( \mathcal{ON}(n, C) \). To see this, consider an \( \mathcal{ON}(8, 6; 2, 1) \), a graph decomposition \((X, \mathcal{B})\) of \( K_8 \) into 16 blocks \( \mathcal{B} = \{B_0, \ldots, B_{15}\} \) where \( B_0 = [1, 0, a_0], B_1 = [2, 0, a_1], B_2 = [2, 1, a_0], B_3 = [3, 1, a_1], B_4 = [3, 2, a_0], B_5 = [4, 2, a_1], B_6 = [4, 3, a_0], B_7 = [5, 3, a_1], B_8 = [5, 4, a_0], B_9 = [0, 4, a_1], B_{10} = [0, 5, a_0], B_{11} = [1, 5, a_1], B_{12} = [0, 3], B_{13} = [1, 4], B_{14} = [2, 5], \) and \( B_{15} = [a_0, a_1] \). Figure 5 shows this graph decomposition of \( K_8 \) embedding a graph decomposition \((V, \mathcal{D})\) where \( V = \{0, 1, \ldots, 5\} \). (As before, \( \mathcal{D} \) is unique and implied from \( \mathcal{B} \).) Indeed, in this example, \( \text{cost } \mathcal{ON}(8, 6; 2, 1) = 44 \) while \( \text{cost } \mathcal{ON}(8, 2) = 42 \). Optimality is a consequence
of the fact that any $P_3$ uses an edge of $K_{V,W}$; therefore there can be at most 12 $P_3$s which necessitate 36 ADMs, and 4 $P_2$s which necessitate 8 ADMs.

Always cost $\mathcal{O}(n, v; C, C') \geq \text{cost } \mathcal{O}(n, C)$. Of particular interest is the case when cost $\mathcal{O}(n, v; C, C') = \text{cost } \mathcal{O}(n, C)$. For every triple $(n, C, C')$ denote by $\mathbb{N}(n, C, C')$ the set of integers $v$ for which cost $\mathcal{O}(n, v; C, C') = \text{cost } \mathcal{O}(n, C)$. Evidently $1 \in \mathbb{N}(n, C, C')$ for every positive integer $n$.

3 \quad \mathcal{O}(n, v; 2, 1)

The determination of optimal two-period groomings naturally divides into three cases. Roughly speaking, when $|V|$ is “small” relative to $|X|$, the cost of the grooming on $X$ determines the overall cost; then we focus on the groomings on $X$. When $|V|$ is a “large” fraction of $|X|$, the cost of the grooming on $V$ determines the overall cost; then we concentrate on embedding groomings on $V$ into $X$ that do not increase cost. The transition from small to large involves the simultaneous construction of both groomings. We recall the basic result on grooming with grooming ratio two:

**Theorem 3.1** [5,23] For every positive integer $n$, cost $\mathcal{O}(n, 2) = \frac{3n(n-1)+\delta}{4}$, where $\delta = 0$ if $n \equiv 0, 1 \pmod{4}$, and $\delta = 2$ if $n \equiv 2, 3 \pmod{4}$.

We now treat small cases: $\mathbb{N}(3, 2, 1) = \{1, 2\}; \mathbb{N}(4, 2, 1) = \{1, 2, 3\}; \mathbb{N}(5, 2, 1) = \{1, 2, 3\};$ and cost $\mathcal{O}(5, 4; 2, 1) = 16$. For every integer $n \geq 6$ let

$$\vartheta_2(n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod{3} \\ \frac{2n+1}{3} & \text{if } n \equiv 1 \pmod{3} \\ \frac{2n-1}{3} & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

**Theorem 3.2** For every $n \geq 6$, $\mathbb{N}(n, 2, 1) = \{1, 2, \ldots, \vartheta_2(n)\}$.

**Proof.** By [16, Corollary 3.2], cost $\mathcal{O}(n, v; 2, 1) > \text{cost } \mathcal{O}(n, 2)$ when $v > \vartheta_2(n)$. So it is sufficient to prove that $\vartheta_2(n) \in \mathbb{N}(n, 2, 1)$ for every integer $n \geq 6$.

Let $V = \{0, 1, \ldots, \vartheta_2(n)-1\}$, $W = \{a_0, a_1, \ldots, a_{n-\vartheta_2(n)-1}\}$, and $X = V \cup W$. Embed the $P_2$-design on vertex set $V$ into a graph decomposition $(X, B)$ of $K_n$ that is an $\mathcal{O}(n, 2)$ (see Theorem 3.1). Let $\mathcal{D}_0 = \{[a_i, j, i+j+1] : 0 \leq j \leq 2\vartheta_2(n)-n-1, 0 \leq i \leq n-\vartheta_2(n)-1\}$; $\mathcal{D}_1 = \{[a_i, j, i+j+1] : 2\vartheta_2(n)-n \leq j \leq \vartheta_2(n)-2, 0 \leq i \leq \vartheta_2(n)-2-j\};$ and $\mathcal{D}_2 = \{[j, i+j+n-\vartheta_2(n)+1, a_{\vartheta_2(n)-n-2-i}] : 0 \leq j \leq 2\vartheta_2(n)-n-2, 0 \leq i \leq 2\vartheta_2(n)-n-2-j\}$. Further sets $\mathcal{D}_i$ are next defined that depend on the congruence class of $n$ modulo 12.

When $n \equiv 1 \pmod{3}$, the $P_3$s in $\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ cover the edges of the graphs $K_V$ and $K_{V,W}$. Let $(W, \mathcal{D}_3)$ be an $\mathcal{O}(n-\vartheta_2(n), 2)$ on $K_W$, and let $B = \bigcup_{a=0}^{3} \mathcal{D}_a$.

When $n \equiv 0 \pmod{3}$, the $P_3$s in $\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ cover the edges of $K_V$ and $K_{V,W} \setminus \mathcal{E}$, where $\mathcal{E} = \{a_{n-\vartheta_2(n)-1}, 2\vartheta_2(n)-n+j\} : 0 \leq j \leq n-\vartheta_2(n)-1\}$. If $n \equiv 0$ or 6 (mod 12) then $|\mathcal{E}|$ is even and hence there is a set $\mathcal{D}_3$ of $P_3$s covering the edges in $\mathcal{E}$. Let $(W, \mathcal{D}_4)$ be an $\mathcal{O}(n-\vartheta_2(n), 2)$ on $K_W$, and let $B = \bigcup_{a=0}^{4} \mathcal{D}_a$. If instead $n \equiv 3$ or 9 (mod 12) then $|\mathcal{E}|$ is
odd and hence there is a set $D_3$ of $P_3$s covering the edges in $E \setminus \{\{a_n - \varphi_2(n - 1), 2\varphi_2(n) - n\}\}$. Let $(W, D_4)$ be an $\mathcal{ON}(n - \varphi_2(n), 2)$ on $K_W$. If $n \equiv 3 \pmod{12}$ then $(W, D_4)$ is a $P_3$-design and $B = \bigcup_{a=0}^{4} D_a \cup \{\{a_n - \varphi_2(n - 1), 2\varphi_2(n) - n\}\}$. If $n \equiv 9 \pmod{12}$ then $(W, D_4)$ is $\varphi_2(n)/(\varphi_2(n) - 1) - 2$ copies of $P_3$ and $P_2$: without loss of generality, the $P_2$ is $e = [a_n - \varphi_2(n - 1), a_n - \varphi_2(n - 2)]$. Then $B = (\bigcup_{a=0}^{4} D_a) \setminus \{e\} \cup \{2\varphi_2(n) - n, a_n - \varphi_2(n - 1), a_n - \varphi_2(n - 2)\}$.

Let $D_3 = \{[a_n - \varphi_2(n - 2), 2\varphi_2(n) - n + 1 + j, a_n - \varphi_2(n - 1)] | 0 \leq j \leq n - \varphi_2(n) - 2\}$ when $n \equiv 2 \pmod{3}$. The $P_3$s in $\bigcup_{a=0}^{3} D_a$ cover the edges of $K_V$ and $K_W \setminus \{\{a_n - \varphi_2(n - 1), 2\varphi_2(n) - n\}\}$. If $n \equiv 2$ or 11 (mod 12), let $(W, D_4)$ be an $\mathcal{ON}(n - \varphi_2(n), 2)$, and let $B = (\bigcup_{a=0}^{4} D_a) \cup \{[a_n - \varphi_2(n - 1), 2\varphi_2(n) - n]\}$. If instead $n \equiv 5$ or 8 (mod 12), let $(W, D_4)$ be an $\mathcal{ON}(n - \varphi_2(n), 2)$ for which $e = [a_n - \varphi_2(n - 1), a_n - \varphi_2(n - 2)]$ is the unique $P_2$ in the decomposition $D_4$, and let $B = (\bigcup_{a=0}^{4} D_a) \setminus \{e\} \cup \{[a_n - \varphi_2(n - 2), a_n - \varphi_2(n - 1), 2\varphi_2(n) - n]\}$.

Figure 5 shows a solution when $v = 6$ and $n = 8$ that is generalized in the following:

**Theorem 3.3** Let $n$ and $v$ be two integers such that $n \geq 6$ and $\varphi_2(n) < v < n$. Then cost $\mathcal{ON}(n, v; 2, 1) \leq$ cost $\mathcal{ON}(n - v, 2) + v(n - 1)$.

**Proof.** Let $(V, E)$ be a $P_2$-design with $V = \{0, 1, \ldots, v - 1\}$ and let $(W, F)$ be an $\mathcal{ON}(n - v, 2)$ with $W = \{a_0, a_1, \ldots, a_{n-v-1}\}$. Let $D_0 = \{[a_i, j, i + j] : 0 \leq j \leq 2v - n - 1, 0 \leq i \leq n - v - 1\}$; $D_1 = \{[i, j + i + j - 2, n - v - 1-i] : 0 \leq j \leq n - v - 1, 0 \leq i \leq n - v - 1 - j\}$; and $D_2 = \{[i, j, i + j + 1] : 2v - n - 1 \leq j \leq v - 2, 0 \leq i \leq v - 2 - j\}$ if $\varphi_2(n) < v < n - 2$, or $D_2 = \emptyset$ if $v = n - 1$. Let $E_1$ be the set of edges of $K_V$ covered by the blocks in $\bigcup_{a=0}^{2} D_a$. Then cost $(V \cup W, (\bigcup_{a=0}^{2} D_a) \cup F \cup (E \setminus E_1)) =$ cost $\mathcal{ON}(n - v, 2) + v(n - 1)$.

## 4 $\mathcal{ON}(n, v; 3, 1)$

We recall the complete characterization for optimal groomings with a grooming ratio of three:

**Theorem 4.1** [2] Let $n$ be a positive integer.

- When $n$ is odd, cost $\mathcal{ON}(n, 3) = \binom{n}{2} + \delta$, where $\delta = 0$ if $n \equiv 1, 3 \pmod{6}$ and $\delta = 2$ if $n \equiv 5 \pmod{6}$.
- When $n$ is even, cost $\mathcal{ON}(n, 3) = \binom{n}{2} + \lceil \frac{n}{4} \rceil + \delta$, where $\delta = 1$ if $n \equiv 8 \pmod{12}$ and $\delta = 0$ otherwise.

Most graphs in an optimal decomposition are $K_3$s (triples, or triangles). We review some relevant terminology from design theory. A decomposition of a subgraph $G$ of $K_n$ into triples is a packing by triples, or a partial triple system (PTS); when it has the largest possible number of triples for any packing on $n$ vertices, it is a maximum packing by triples, and is denoted by MPT($n$). For any packing on $n$ vertices, the leave is the graph that is the complement in $K_n$ of the edges appearing in triples of the packing; in other words, the leave contains all of the remaining edges. When the leave is empty, the MPT is a Steiner triple.
system (STS). A group-divisible design with block size 3 and order n (3-GDD for short) is a partial triple system whose leave consists of a number of disjoint complete subgraphs. When the leave of a partial triple system of order n consists of \( u_i \) copies of \( K_{g_i} \) for \( 0 < i \leq t - 1 \) and \( n = \sum_{i=0}^{t-1} u_i g_i \), the type of the 3-GDD is written in exponential notation as \( g_0^{u_0} \cdots g_{t-1}^{u_{t-1}} \).

We recall two results on 3-GDDs:

**Theorem 4.2** [14] Let \( g \), \( t \), and \( u \) be nonnegative integers. There exists a 3-GDD of the type \( g^t u \) if and only if these conditions are all satisfied:

1. if \( g > 0 \), then \( t \geq 3 \) or \( t = 2 \) and \( u = g \), or \( t = 1 \) and \( u = 0 \) or \( t = 0 \);
2. \( u \leq g(t-1) \) or \( gt = 0 \);
3. \( g(t-1) + u \equiv 0 \) (mod 2) or \( gt = 0 \);
4. \( gt \equiv 0 \) (mod 2) or \( u = 0 \);
5. \( \frac{1}{2} g^2 t(t-1) + gtu \equiv 0 \) (mod 3).

**Theorem 4.3** [11] Let \( u \), \( r \), and \( t \) be positive integers. Then there exists a 3-GDD of type \( u^r 1^t \) with \( t \geq 1 \) and \( r \geq 1 \) if and only if these conditions are all satisfied:

1. \( u \equiv 1 \) (mod 2);
2. \( r + t \equiv 1 \) (mod 2);
3. if \( r = 1 \), \( t \geq u + 1 \);
4. if \( r = 2 \), \( t \geq u \);
5. \( \binom{1}{2} +rut + \binom{r}{2} u^2 \equiv 0 \) (mod 3).

If the set \( B \) of blocks of the graph decomposition consists of \( b \) copies of the same graph \( G \), then \( (X, B) \) is a \( G \)-design of order \( n \). When the two decompositions are a \( G \)-design and an \( H \)-design, Definition 2.2 coincides with the embedding definition in [28,29]; see also [12,15,30].

We also employ certain graph-theoretic concepts. A proper edge-colouring of a graph \( G = (V,E) \) is an assignment of colours to edges so that no two edges that share a vertex receive the same colour. The edges of each colour form a matching. When every vertex of the graph is contained in an edge of the matching, it is a 1-factor; when every colour class is a 1-factor, the edge-colouring is a 1-factorization of \( G \). When \( V \) has an odd number of vertices, the graph cannot have a 1-factorization; indeed it cannot contain even one 1-factor. Therefore a near 1-factor is defined to be a matching in which all but one of the vertices has an incident edge in the matching; a near 1-factorization is a partition of the edges of the graph into near 1-factors.
Lemma 4.4

Let $v$ be a positive odd integer. It is well known that

$$F_i = \left\{ \{i - j, i + j\} : 1 \leq j \leq \frac{v - 1}{2} \right\}, \ i = 0, 1, \ldots, v - 1$$

(arithmetic modulo $v$) is a near 1-factorization of $K_v$ such that $i$ is the missing vertex of $F_i$. See [18] for related definitions and results on designs and graphs.

We begin by treating the cases when $v$ is “small” with respect to $n$, then cases when $v$ is more than half of $n$, and finally some exceptional cases when $v$ is nearly half of $n$. For every integer $n \geq 6$ let

$$\vartheta_3(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{2} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

4.1 $\mathcal{O}(n, v; 3, 1)$: Small values of $v$

In this section we handle the cases $v \leq \vartheta_3(n)$. Because $\mathcal{N}(3, 3, 1) = \{1, 2\}; \mathcal{N}(4, 3, 1) = \{1, 2, 3\}; \mathcal{N}(5, 3, 1) = \{1, 2, 3\};$ and cost $\mathcal{O}(5, 4, 3; 1) = 14$, we assume that $n \geq 6$.

Lemma 4.4 For every integer $n \geq 6$, $n \not\equiv 2 \pmod{4}$, $\mathcal{N}(n, 3, 1) = \{1, 2, \ldots, \vartheta_3(n)\}$.

Proof. By [16, Corollary 4.2], cost $\mathcal{O}(n, v; 3, 1) > \mathcal{O}(n, 3, 1)$ when $v > \vartheta_3(n)$. So it is sufficient to prove that $\vartheta_3(n) \in \mathcal{N}(n, 3, 1)$. For $n \equiv 1, 3 \pmod{6}$ and $n \equiv 11 \pmod{12}$ the result is well known [18]. The case when $n = 7$ and $v = 4$ is shown in Figure 4.

Suppose that $n \equiv 5 \pmod{12}$. Let $(X, B)$ be an MPT$(n)$ with leave $\{x_0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_0\}$. A maximal independent set in $(X, B)$ is a set of vertices containing at most two vertices of each triple, to which no further vertex can be adjoined without violating this condition. An oval in $(X, B)$ is a maximal independent set $\theta \subset X$ with the property that for every $x \in \theta$ there is only one block $B \in B$ such that $B \cap \theta = \{x\}$. In the case when $n \equiv 5 \pmod{12}$, Milici and Quattrocchi [24] proved the existence of a maximum packing $(X, B)$ with an oval of size $\vartheta_3(n) = \frac{n-1}{2}$ such that $x_0, x_1 \in \theta$. Then $(X, B)$ is an $\mathcal{O}(n, \vartheta_3(n); 3, 1)$.

Let $n \equiv 0, 4 \pmod{12}$. $V = \{0, 1, \ldots, \frac{n}{2} - 1\}$, $W = \{a_0, a_1, \ldots, a_{\frac{n}{2} - 1}\}$, and $X = V \cup W$. Let $(W, D_0)$ be an MPT($\frac{n}{2}$) with leave $\{\{a_0, a_{\frac{n}{2}}\}, \{a_1, a_{\frac{n}{2} + 1}\}, \ldots, \{a_{\frac{n}{2} - 1}, a_{\frac{n}{2} + \frac{n}{2} - 1}\}\}. Let F_0, F_1, \ldots, F_{\frac{n}{2} - 2}$ be a 1-factorization of $K_{\frac{n}{2}}$ such that there is a 1-factor $G$ containing at most one edge from each of the $(F_i)$ (this is easily constructed). Without loss of generality, $|E(G) \cap E(F_i)| = 1$ for every $i = 0, 1, \ldots, \frac{n}{2} - 1$ and $|E(G) \cap E(F_i)| = 0$ for every $i = \frac{n}{4}, \frac{n}{4} + 1, \ldots, \frac{n}{2} - 2$. Let $D_1$ consist of the triangles $(a_i, \alpha, \beta)$ for every $i = 0, 1, \ldots, \frac{n}{2} - 2$ and $\{a_i, \alpha, \beta\} \in F_i \setminus G$; the triangles $(a_{\frac{n}{2} - 1}, \alpha, \beta)$ for every $\{\alpha, \beta\} \in G$; and the stars $[a_i; a_{\frac{n}{2} + i}, \alpha, \beta]$ where $\{\alpha, \beta\} \in F_i \cap G$, $i = 0, 1, \ldots, \frac{n}{4} - 1$. Then $(X, D_0 \cup D_1)$ is an $\mathcal{O}(n, \vartheta_3(n); 3, 1)$.

Let $n \equiv 8 \pmod{12}$, $n \geq 20$. Choose the MPT($\frac{n}{2}$), $(W, D_0)$, with leave $\{\{a_0, a_{\frac{n}{2}}\}, \{a_1, a_{\frac{n}{2} + 1}\}, \ldots, \{a_{\frac{n}{4} - 3}, a_{\frac{n}{4} + \frac{n}{4} - 3}\}\} \cup \{\{a_{\frac{n}{2} - 1}, a_{\frac{n}{2} - 2}, a_{\frac{n}{2} - 1}, a_{\frac{n}{2} - 2}\}\},$ and proceed as in the case $n \equiv 0, 4 \pmod{12}$. Let $D_1$ consist of the triangles $(a_i, \alpha, \beta)$ for every $i = 0, 1, \ldots, \frac{n}{2} - 2$ and $\{a_i, \alpha, \beta\} \in F_i \setminus G$; the triangles $(a_{\frac{n}{2} - 1}, \alpha, \beta)$ for every $\{\alpha, \beta\} \in G$; the stars $[a_i; a_{\frac{n}{2} + i}, \alpha, \beta]$ where
\{\alpha, \beta\} \in F_i \cap G, i = 0, 1, \ldots, \frac{n}{2} - 3; the stars \{a_{\frac{n}{2} - 3 + i}; a_{\frac{n}{2} - 1}, \alpha, \beta\} where \{\alpha, \beta\} \in F_{\frac{n}{2} - i} \cap G and i \in \{1, 2\}; and the \(P_2\) \{a_{\frac{n}{2} - 1}, a_{\frac{n}{2} - 2}\}. Then \((X, D_0 \cup D_1)\) is an \(\mathcal{O}(n, \vartheta_3(n); 3, 1)\).

Let \(n = 8\). Let \(F_0, F_1, F_2\) be a 1-factorization of \(K_V\). The blocks of an \(\mathcal{O}(n, \vartheta_3(n); 3, 1)\) are: \([a_3; a_0, a_1, a_2], [a_3; 0, 1, 2], (a_0, a_1, a_2), [a_3, 3]\), and \((a, \alpha, \beta)\) for every \{\alpha, \beta\} \in F_i and \(i \in \{0, 1, 2\}\).

To treat cases when \(n \equiv 2 \pmod{4}\), an auxiliary object is employed. Let \((\gamma_0, \gamma_1, \ldots, \gamma_{t-1})\) be a sequence of integers from \(\{0, 1, \ldots, d-1\}\). For \(i = 0, 1, \ldots, t-1\), define \(\delta_i = 2\gamma_{i+1} - \gamma_i\) (arithmetic is modulo \(d\) and \(i+1\) is computed modulo \(t\)). Then \((\gamma_0, \gamma_1, \ldots, \gamma_{t-1})\) is a \(t\)-laced sequence of order \(d\) if all entries in the collection \{\(\gamma_i, \delta_i\) : \(i = 0, 1, \ldots, t-1\)\} are distinct.

**Lemma 4.5** Let \(s \geq 3\) be an integer. Suppose that there is an \(s\)-laced sequence of order \(2s + 1\) when \(s \equiv 0 \pmod{3}\), or an \((s-1)\)-laced sequence of order \(2s + 1\) when \(s \equiv 1, 2 \pmod{3}\). Then \(\mathcal{N}(4s + 2, 3, 1) = \{1, 2, \ldots, \vartheta_3(4s + 2)\}\).

**Proof.** Form the solution on \(\{0, \ldots, 2s\} \times \{0, 1\}\), using the short form \(i\) for the vertex \((i, 0)\) and \(i\) for the vertex \((i, 1)\). Let \((\gamma_0, \gamma_1, \ldots, \gamma_{t-1})\) be the laced sequence of order \(2s + 1\) with \(t = s\) when \(s \equiv 0 \pmod{3}\) and \(t = s - 1\) otherwise. For each \(0 \leq i \leq 2s\), define the set of pairs \(F_i = \{-j + i, j + i\} : j = 1, \ldots, s\) so that \(i\) is the only vertex from \(\{0, \ldots, 2s\}\) not appearing in an edge of \(F_i\). Remove the pair \(\delta_i, \gamma_i\) from \(F_{\frac{s}{2} + 1}\) for \(0 \leq i < t\).

For \(0 \leq i < 2s\), form the triangle \((\overline{i}, a, b)\) for each \(\{a, b\} \in F_i\). Then for \(0 \leq j < t\), add the triangle \((\gamma_j, \overline{\gamma_j}, \overline{\gamma_{j+1}})\) and the \(K_{1,3} [\delta_j; \gamma_j, \overline{\gamma_j}, \overline{\gamma_{j+1}}]\) to the decomposition.

Now let \(U = \{0, \ldots, 2s\} \setminus \{\delta_0, \delta_1, \ldots, \delta_{t-1}, \gamma_0, \gamma_1, \ldots, \gamma_{t-1}\}\). When \(s \equiv 0 \pmod{3}\), \(U\) contains a single vertex \(u\). Add the \(P_2 [u, \overline{\gamma}]\) to the decomposition. When instead \(s \equiv 1, 2 \pmod{3}\), \(U\) contains three vertices: \(u, v, \) and \(w\). In this case, add the \(K_{1,3} [u; v, \overline{u}, \overline{w}]\) and the \(P_4 [v, \overline{u}, \overline{w}, w]\) to the decomposition.

All pairs now appear in subgraphs of the decomposition except some pairs of the form \(\{\overline{i}, j\}\); among these, those already appearing form precisely a cycle of length \(s\) when \(s \equiv 0 \pmod{3}\), or the disjoint union of a cycle of length \(s - 1\) and a cycle of length \(3\) (on \(\overline{u}, \overline{v}, \overline{w}\)) when \(s \equiv 1, 2 \pmod{3}\). By the quadratic leaves theorem in [17], the remaining edges can be partitioned into \(K_3s\).

**Lemma 4.6** For \(n \leq 34\) and \(n \equiv 2 \pmod{4}\), \(\mathcal{N}(n, 3, 1) = \{1, 2, \ldots, \vartheta_3(n)\}\).

**Proof.** For \(\mathcal{O}(6, 3; 3, 1)\), take \(V = \{0, 1, 2\}\), \(X = V \cup \{a_0, a_1, a_2\}\), and \(B = \{(a_0, a_1, 1), (a_0, a_2, 0), (a_2, 1, 2), [2; a_0, a_1, 0], [1, 0, a_1, a_2]\}\). For \(\mathcal{O}(10, 5; 3, 1)\), take \(V = \{0, 1, 2, 3, 4\}\), \(X = V \cup \{a_0, a_1, a_2, a_3, a_4\}\), and \(B = \{(0, 1, a_0), (0, 3, a_1), (a_0, a_1, a_2), (0, 4, a_3), (4, a_2, a_4), (0, a_2, a_4), (3, a_0, a_3), (1, 4, a_1), (a_1, a_3, a_4), (1, 2, a_3), (2, 3, a_4), (2, 4, a_2), [2; 0, a_0, a_1], [1; 3, a_2, a_4], [4, 3, a_2, a_3]\}\).

For the remaining cases, apply Lemma 4.5 using the laced sequences in Table 1.

**Lemma 4.7** For every \(n \equiv 2 \pmod{4}\), \(\mathcal{N}(n, 3, 1) = \{1, 2, \ldots, \vartheta_3(n)\}\).

**Proof.** Write \(n = 12m + 4x + 2\) with \(x \in \{0, 1, 2\}\). When \(m \in \{0, 1, 2\}\), apply Lemma 4.6 to form the decompositions. Now assume that \(m \geq 3\).
Let \( \rho < n \) and \( \vartheta \). If \( v > \rho \) then add two more vertices \( \infty \) and \( \overline{\infty} \). For every group \( G \) of size six in the GDD, on the \( 12 \) vertices together with the vertices \( \infty \) and \( \overline{\infty} \), place an \( \mathcal{G}(14,7;3,1) \) with the \( P_2 \) on \( \{\infty, \overline{\infty}\} \) and the embedded decomposition on the non-overlined vertices; remove the \( P_2 \) from each. Now on the \( 4x \) vertices from the group of size \( 2x \), together with \( \infty \) and \( \overline{\infty} \), place an \( \mathcal{G}(4x+2,2x+1;3,1) \) from Lemma 4.6.

	\( 4.2 \quad \mathcal{G}(n,v;3,1): \) Large values of \( v \)

Now we construct an \( \mathcal{G}(n,v;3,1) \) for every \( v \) such that \( \vartheta_3(n) < v < n \).

**Theorem 4.8** Let \( n \geq 6 \) be an integer. Let \( v \) be an even integer with \( \vartheta_3(n) < v < n \). Then

\[
\text{cost } \mathcal{G}(n,v;3,1) \leq \text{cost } \mathcal{G}(n - v,3) + 2\left(\frac{v}{2}\right) + \frac{1}{2}(n-v).
\]

**Proof.** Let \( V = \{0,1,\ldots,v-1\} \), \( W = \{a_0,a_1,\ldots,a_{n-v-1}\} \) and \( X = V \cup W \). Because \( v > \vartheta_3(n) \geq \frac{n-\rho}{3} \), \( v-1 \geq n-v \). Let \( F_0, F_1, \ldots, F_{n-2} \) be a 1-factorization of \( K_V \). Let \( B \) contain the triangles \( (a_i,x,y) \) for every \( \{x,y\} \in F_i, i = 0,1,\ldots,n-v-1 \); the \( P_2s \) \([x,y]\) for \( \{x,y\} \in F_i, i = n-v, n-v+1,\ldots,v-2 \); and the blocks of an \( \mathcal{G}(n-v,3) \) on \( W \).

**Theorem 4.9** Let \( n \geq 6 \) be an integer. Let \( v \) be an odd integer with \( \vartheta_3(n) < v < n \), and \( \vartheta_3(n)+1 \neq v \) when \( n \equiv 1 \pmod{4} \). Then

\[
\text{cost } \mathcal{G}(n,v;3,1) \leq 2\left(\frac{v}{2}\right) + \left(\frac{n-v}{2}\right) + \frac{1}{2}(v+1)(n-v).
\]

**Proof.** Let \( V = \{0,1,\ldots,v-1\} \), \( W = \{a_0,a_1,\ldots,a_{n-v-1}\} \), and \( X = V \cup W \). Let \( \{F_i : 0 \leq i \leq v-1\} \) be the near 1-factorization of \( K_V \) given in (1).

First consider the case when \( n \) is even. Because \( v > \vartheta_3(n) = \frac{n}{2} \), \( v > n-v \). Then, since \( v \) and \( n-v \) are odd, \( v \geq n-v+2 \). Set \( y_\rho = v - \rho - 2 \) for \( \rho = 0,\ldots,\min(n-v-1,\frac{v-3}{2}) \). If \( n-v > \frac{v-1}{2} \) further set \( y_\rho = v - \rho - 4 \) for \( \rho = \frac{v-1}{2}, \frac{v+1}{2},\ldots,n-v-1 \). Now \( \{\rho, y_\rho\} : 0 \leq \rho < n-v \) contains distinct edges of \( F_{n-2} \cup P_{v-1} \).

If \( n-v \equiv 1,3 \pmod{6} \) then let \( B \) contain the triangles \( (a_i,x,y) \) for every \( \{x,y\} \in F_i, i = 0,1,\ldots,n-v-1 \); the \( P_2s \) \([a_\rho,\rho, y_\rho]\) for \( \rho = 0,1,\ldots,n-v-1 \); the \( P_2s \) \([x,y]\) for every \( \{x,y\} \in \bigcup_{i=n-v}^{n-1} F_i \setminus \{\rho, y_\rho\} : \rho = 0,1,\ldots,n-v-1 \} \); and the blocks of an STS\((n-v)\).
If \( n - v \equiv 5 \pmod{6} \), then let \( \mathcal{B} \) contain the triangles \((a_i, x, y)\) for every \( \{x, y\} \in F_i, i = 0, 1, \ldots, n-v-1\); the blocks of an MPT\((n-v)\) having leave \(\{a_0, a_1\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_0\}\); the paths \([y_0, 0, a_0, a_1], [y_1, 1, a_1, a_2], [y_2, 2, a_2, a_3], [y_3, 3, a_3, a_0]\); the paths \(\{a_\rho, \rho, y_\rho\} : \rho = 4, 5, \ldots, n-v-1\)\}; and the \(P_2s\) \([x, y]\) for every \(\{x, y\} \in \bigcup_{j=n-v}^{n-1} F_j \setminus \{\{\rho, y_\rho\} : \rho = 0, 1, \ldots, n-v-1\}\).

Then \((X, \mathcal{B})\) is an \(\mathcal{O}N(n, v; 3, 1)\), completing the case when \(n\) is even.

Now consider the case when \(n\) is odd. Because \(v > \vartheta_3(n) + 1 = \frac{n+1}{2}, v \geq n-v+3\). Let \(D_1\) consist of the triangles \((a_i, x, y)\) for every \(\{x, y\} \in F_i, i = 0, 1, \ldots, n-v-1\). Let \(D_2\) consist of the triangles of an MPT\((n-v)\) having leave \(\{a_j, a_j+\frac{n-v}{2}\} : j = 0, 1, \ldots, \frac{n-v}{2} - 1\) if \(n-v \equiv 0, 2 \pmod{6}\), or leave \(\{a_{n-v-1}, a_0\}, \{a_{n-v-1}, a_1\}, \{a_{n-v-1}, a_2\}\) \(\cup \{a_j, a_j+\frac{n-v}{2}\} : j = 3, 4, \ldots, \frac{n-v}{2}\) if \(n-v \equiv 4 \pmod{6}\). The edge of the leave containing vertex \(a_i\) is called \(\ell_i\); when \(n-v \equiv 0, 2 \pmod{6}\) they are \(\ell_0, \ldots, \ell_{(n-v)/2}\), and when \(n-v \equiv 4 \pmod{6}\) they are \(\ell_0, \ldots, \ell_{(n-v)/2}\). Let \(y_j\) be the vertex of \(\ell_j \setminus \{a_j\}\); \(D_3\) consists of the paths \([v-2-j, x, y_j]\) for \(j \in \{0, 1, \ldots, \frac{n-v}{2}\}\) when \(n-v \equiv 0, 2 \pmod{6}\), and for \(j \in \{0, 1, \ldots, \frac{n-v}{2}\}\) when \(n-v \equiv 4 \pmod{6}\). The first edge of the path is in \(F_{n-v}\). To define \(D_4\), let \(N = \{\frac{2-v}{2}, \frac{n-v}{2}+1, \ldots, n-v-1\}\) if \(n-v \equiv 0, 2 \pmod{6}\), or \(N = \{\frac{n-v}{2} + 1, \frac{n-v}{2}+2, \ldots, n-v-1\}\) if \(n-v \equiv 4 \pmod{6}\). As \(v-2 > n-v\), all the edges of \(F_{v-2}\) have not been used in \(D_1\). For every \(\{x, y\} \in F_{v-2}\) include \([a_x, x, y, a_y]\) in \(D_4\) if \(x, y \in N\), or include \([a_x, x, y]\) if \(x \in N\) and \(y \not\in N\). Let \(D_5\) consist of the \(P_2s\) \([x, y]\) for every \(\{x, y\} \in \bigcup_{j=n-v}^{n-1} F_j \setminus \text{not used in } D_3\) and \(D_4\). Then \((X, D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5)\) is an \(\mathcal{O}N(n, v; 3, 1)\), completing the case when \(n\) is odd. \qed

For the special case when \(n \equiv 1 \pmod{4}\) and \(v = \vartheta_3(n) + 1 = \frac{n+1}{2}\), a remarkable theorem is used:

**Theorem 4.10** (Plantholt, [27]) Let \(G = (V, E)\) be a graph with an odd number \(2n+1\) of vertices, having at most \(2n^2\) edges and having a vertex of degree \(2n\). Then \(G\) has “chromatic index” at most \(2n\)—that is, the edges in \(E\) can be partitioned into \(2n\) classes, each containing at most \(n\) edges, so that edges within a class never share a vertex.

**Theorem 4.11** For \(n = 4s+1\) with \(s \geq 2\), and \(v = 2s+1\), cost \(\mathcal{O}N(n, v; 3, 1) \leq 8s^2 + 3s = \binom{s}{2} + \frac{1}{2}(v+1)(n-v)\).

**Proof.** Let \(s = 2\). Let \(V = \{0, 1, \ldots, 4\}\), \(W = \{a_0, a_1, a_2, a_3\}\), and \(\mathcal{B} = \{(a_0, 1, 4), (a_0, 2, 3), (a_1, 1, 2), (a_1, 3, 4), (a_2, 2, 1), (a_2, 0, 4), (a_3, 2, 4), (a_3, 1, 0), (a_0, a_1, 0), (a_1, a_2, a_3), [0, 2, a_2, a_0], [0, 3, a_3, a_0]\}\). Then \((V \cup W, \mathcal{B})\) is the required \(\mathcal{O}N(9, 4; 3, 1)\) whose cost is 38.

Now let \(s \geq 3\). Let \(V = \{0, \ldots, v-1\}\) and \(W = \{a_i, b_i : i = 0, \ldots, s-1\}\). Use Theorem 4.10 to form a near 1-factorization of \(K_{2s+1} - C_s\) on \(V\), where \(C_s\) is the s-cycle \((0, 1, \ldots, s-1)\). Let the near 1-factors be \(F_0, \ldots, F_{s-1}\) and \(G_0, \ldots, G_{s-1}\), where \(F_i\) and \(G_i\) are the near 1-factors that have \(i\) as missing vertex for \(i \in \{0, \ldots, s-1\}\). For \(0 \leq i < s\), form triangles by placing \(a_i\) together with each pair of \(F_i\), and \(b_i\) together with each pair of \(G_i\).

When \(s \equiv 0, 1 \pmod{3}\), place the triples of an MPT\((2s)\) on the \(\{a_i\}\) and \(\{b_i\}\) so that \(\{a_i, b_i\} \equiv 0, \ldots, s-1\}\) forms the leave. Next form triangles \(\{(i, a_i, b_i) : i = 0, \ldots, s-1\}\) finally include the \(P_2s\) \([i, i+1 \pmod{s} : i = 0, \ldots, s-1]\). This completes the construction.
When $s \equiv 2 \pmod{3}$, place the triples of an MPT($2s$) on the $\{a_i\}$ and $\{b_i\}$ so that $\{(a_i, b_i) : i = 2, \ldots, s-1\}$ is in the leave, and the $K_{1,3} [a_1; a_0, b_0, b_1]$ is in the leave. Form triangles $\{(i, a_i, b_i) : i = 1, \ldots, s-1\}$. Include $P_2s \{[i, i+1] : i = 1, \ldots, s-2\}$. The edges not yet appearing are: $\{0, 1\}$, $\{0, s-1\}$, $\{a_0, a_1\}$, $\{b_0, a_1\}$, $\{0, a_0\}$, and $\{0, b_0\}$, so form two $P_3s \{1, 0, a_0, a_1\}$ and $\{s-1, 0, b_0, a_1\}$.

5 \ \mathcal{O}(n, v, 3, 2)

Every $N(n, v; 3, 1)$ satisfies the traffic demand of an $N(n, v; 3, 2)$, and hence $\text{cost } \mathcal{O}(n, v; 3, 2) = \text{cost } \mathcal{O}(n, 3)$ when $v \leq \vartheta_3(n)$. Moreover, $\mathcal{N}(n, 3, 1) \subseteq \mathcal{N}(n, 3, 2)$. Since $\text{cost } \mathcal{O}(n, 5, 4; 3, 2) = 12$, $\mathcal{N}(5, 3, 2) = \{1, 2, 3, 4\}$. In this section we settle the existence of an $\mathcal{O}(n, v; 3, 2)$ for every integer $v$ such that $\vartheta_3(n) < v < n$ and $n \geq 6$.

A simple decomposition theorem is used throughout:

**Theorem 5.1** [8, 36] A graph $G$ can be decomposed into paths on two edges ($P_3$s) if and only if every connected component of $G$ has an even number of edges. Moreover, a graph $G$ can be decomposed into one edge, together with paths on two edges ($P_3$s), if and only if all but one connected component of $G$ has an even number of edges, and one has an odd number.

5.1 Large Values of $v$

In this section we treat all but two exceptional cases, handled in Sections 5.2 and 5.4.

**Theorem 5.2** Let $v$ be an even integer. Let $n$ be an integer such that $n \geq 6$ and $n \leq 2v-3$. Then $\text{cost } \mathcal{O}(n, v; 3, 2) \leq \left[\frac{3}{2}v^2 + \text{cost } \mathcal{O}(n, v, 3) + \frac{3}{4}v(n-v)\right]$.

**Proof.** Let $V = \{0, 1, \ldots, v-1\}$, $W = \{a_0, a_1, \ldots, a_{n-v-1}\}$, and $X = V \cup W$. Place an $\mathcal{O}(n-v, 3)$ on $W$ (noting that $n \leq 2v-3$). Then choose $n-v$ edge-disjoint 1-factors on $V$, and associate each with a vertex of $W$ to form $K_3$s. Partition the remaining edges of $V$ into $P_3$s (and one $P_2$ when $v \equiv 2 \pmod{4}$ and $n-v$ is odd) using Theorem 5.1.

**Theorem 5.3** Let $v$ be an odd integer and $n \geq 6$ an integer so that $\vartheta_3(n) < v$ and $n \neq 2v-2$. Then $\text{cost } \mathcal{O}(n, v; 3, 2) \leq \left[\frac{3}{2}v^2 + \left(\frac{n-v}{2}\right) + \frac{3}{4}v(n-v) + \frac{n-v}{4} + \delta\right]$, where $\delta = \frac{1}{2}$ if $n-v \equiv 4 \pmod{6}$, $\delta = 1$ if $n-v \equiv 5 \pmod{6}$, and $\delta = 0$ otherwise.

**Proof.** Let $V = \{0, 1, \ldots, v-1\}$, $W = \{a_i : i \in \{0, 1, \ldots, \left\lceil \frac{n-v}{2} \right\rceil - 1\}\} \cup \{b_i : i \in \{0, 1, \ldots, \left\lfloor \frac{n-v}{2} \right\rfloor - 1\}\}$, and $X = V \cup W$. Write $v = 2s + 1$. The constructions in each case for $n-v \pmod{6}$ are similar.

Except when $n-v \equiv 4 \pmod{6}$, use Theorem 4.10 to form a near 1-factorization of $K_{2s+1} - C_s$ on $V$, where $C_s$ is the $s$-cycle $(0, 1, \ldots, s-1)$. Let the near 1-factors be $F_0, \ldots, F_{s-1}$ and $G_0, \ldots, G_{s-1}$, where $F_i$ and $G_i$ are the near 1-factors that have $i$ as missing vertex for $i \in \{0, \ldots, s-1\}$. Form triangles by placing $a_i$ together with the pairs of $F_i$ for $i \in \{0, 1, \ldots, \left\lceil \frac{n-v}{2} \right\rceil - 1\}$, and $b_i$ together with the pairs of $G_i$ for $i \in \{0, 1, \ldots, \left\lfloor \frac{n-v}{2} \right\rfloor - 1\}$.
When \( n - v \equiv 1, 3 \pmod{6} \), place an STS\((n - v)\) on the vertices of \( W \).

When \( n - v \equiv 5 \pmod{6} \), place the triples of an MPT\((n - v)\) on the \( \{a_i\} \) and \( \{b_i\} \) so that \( \{\{a_0, b_0\}, \{b_0, a_1\}, \{a_1, b_1\}, \{b_1, a_0\}\} \) forms the leave. Use the edges of the leave and \( \{a_0, 0\}, \{b_0, 0\} \) to form the \( P_4 \) \( \{a_0, b_1, a_1, b_0\} \) and the triangle \( \{a_0, b_0, 0\} \).

When \( n - v \equiv 0, 2 \pmod{6} \), initially suppose that \( s > 2 \). Place the triples of an MPT\((n - v)\) on the \( \{a_i\} \) and \( \{b_i\} \) so that \( \{\{a_i, b_i\} : i = 0, \ldots, \frac{n-v}{2} - 1\} \) forms the leave. Next form triangles \( \{(i, a_i, b_i) : i = 0, \ldots, \frac{n-v}{2} - 1\} \).

Let \( n - v \equiv 4 \pmod{6} \) and \( n - v \geq 10 \). Apply Theorem 4.10 to \( K_{2s+1} \) minus all edges in the path \([0, 1, \ldots, s]\) to form a near 1-factorization \( \{F_i : 0 \leq i < s\} \cup \{G_i : 0 \leq i < s\} \) on \( V \). Exactly two near 1-factors miss \( i \) for \( 1 \leq i < s \); exactly one near 1-factor misses \( i \) for \( i \in \{0, s\} \); and no vertex not in \( \{i : 0 \leq i \leq s\} \) is missed by any near 1-factor. Without loss of generality, the near 1-factor containing the edge \( \{0, s\} \) is \( F_0 \); it misses one of the vertices \( q \) on the path other than \( 0 \) and \( s \). Let \( F_1 \) be the other near 1-factor missing \( q \). Let \( G_0 \) be the only near 1-factor missing \( 0 \) and let \( G_1 \) be the only near 1-factor missing \( s \). Define a bijection \( \mu : \{1, \ldots, s - 1\} \mapsto \{1, \ldots, s - 1\} \) for which \( \mu(1) = q \). Ensure that the near 1-factors \( \{F_i, G_i\} \) miss vertex \( \mu(i) \) for \( i \in \{2, \ldots, s - 1\} \). Now add \( n - v \) vertices \( a_i \) and \( b_i \) for \( i = 0, 1, \ldots, \frac{n-v}{2} - 1 \). Form the triangles: \( \{a_0, a_1, q\}, \{a_0, b_0, 0\}, \{a_0, b_1, s\} \) together with the pairs of \( F_0 \setminus \{[0, s]\}, b_0 \) together with the pairs of \( G_0, a_i \) together with the pairs of \( F_i \) and \( b_i \) together with the pairs of \( G_i \) for \( i \in \{1, 2, \ldots, \frac{n-v}{2} - 1\} \). Place the triples of an MPT\((n - v)\) on the \( \{a_i\} \) and \( \{b_i\} \) so that \( \{\{a_0, a_1\}, \{a_0, b_0\}, \{a_0, b_1\}\} \cup \{\{a_i, b_i\} : i \in \{2, \ldots, \frac{n-v}{2} - 1\}\} \) forms the leave. Now form the triangles \( \{\mu(i), a_i, b_i) : i \in \{2, \ldots, \frac{n-v}{2} - 1\}\} \).

The edges that remain include: (1) the cycle \( (0, 1, \ldots, s - 1) \) when \( n - v \not\equiv 4 \pmod{6} \) or the cycle \( (0, 1, \ldots, s) \) together with edges \( \{0, q\} \) and \( \{q, s\} \) when \( n - v \equiv 4 \pmod{6} \); (2) the edges of \( 2v - n - 1 \) near 1-factors when \( n < 2v - 1 \); (3) \( n - v \) edges with one end in \( V \) and one in \( W \) when \( n - v \) is odd. Under the stated conditions, the graph induced on \( V \) by the edges in (1) and (2) is connected, so apply Theorem 5.1 to partition it into \( P_3 \)s plus possibly one \( P_2 \). When \( n - v \) is odd, \( n - v \leq v - 4 \), but each vertex of \( V \) appears in at most two of the edges in (3); hence the edges of (3) can be adjoined to \( n - v \) different \( P_3 \)s, in each case forming either a \( P_4 \) or a \( K_{1,3} \).

A similar technique treats the case when \( (n, v) = (9, 5) \).

\[\Box\]

5.2 \( \mathcal{ON}(2v - 2, v; 3, 2) \) when \( v \) is odd

**Theorem 5.4** Let \( v \geq 5 \) be odd. Then cost \( \mathcal{ON}(2v - 2, v; 3, 2) \) \( \leq \binom{2v-2}{2} + \left\lceil \frac{5v-7}{8} \right\rceil + \delta \), where \( \delta = 1 \) if \( v \equiv 11, 17 \pmod{24} \) and \( \delta = 0 \) otherwise.

**Proof.** Let \( V = \{0, \ldots, v - 1\} \) and \( W = \{a_0, \ldots, a_{v-3}\} \). On \( V \setminus \{0\} \) place a 1-factorization \( F_0, \ldots, F_{v-3} \), and form triangles by associating \( a_i \) with each pair of \( F_i \) for \( 0 \leq i < v - 2 \). Next add the stars \( [0; 2i + 1, 2i + 2, a_i] \) for \( 0 \leq i < \frac{v-1}{2} \). These exhaust all edges on \( V \) and all edges with one end in \( V \) and one in \( W \) except the edges \( E = \{0, a_i : \frac{v-1}{2} \leq i < v - 2\} \). All edges on \( W \) remain.

We treat cases for \( v \) modulo 24, and later subdivide into cases modulo 48. When \( v \equiv 5, 11, 17 \pmod{24} \), include the \( P_2 \) \([0, a_{(v-1)/2}] \) in the decomposition and remove it from
| $v \pmod{8}$ | $v = 48s+$ | $v - 2 = 48s+$ | $(v - 2)/2 \pmod{3}$ | Drop Cost | Type | Note | $\Delta_1$ | $|\mathcal{E}|$ |
|----------------|------------|----------------|-----------------------|----------|---------|-------|-----------|--------|
| 1              | 49         | 47             | 1                     | 6$s + 6$ | $D^1 F^{2s+1}$ | *     | 12$s + 11$ | 24$s + 23$ |
| 9              | 7          | 0              | 6$s + 1$             | $A^{3s} G^1$ | *     | 12$s + 1$ | 24$s + 3$  |
| 17             | 15         | 0              | 6$s$                 | $A^{3s}$  |        | 12$s$ | 24$s$     |
| 25             | 23         | 1              | 6$s + 3$             | $A^{3s+1} G^1$ | *     | 12$s + 5$ | 24$s + 11$ |
| 33             | 31         | 0              | 6$s + 4$             | $F^{2s+1} G^1$ | *     | 12$s + 7$ | 24$s + 15$ |
| 41             | 39         | 0              | 6$s + 3$             | $F^{2s+1}$ |        | 12$s + 6$ | 24$s + 12$ |
| 3              | 3          | 1              | 6$s$                 | $A^{3s}$  |        | 12$s$ | 24$s$     |
| 11             | 9          | 0              | 6$s$                 | $A^{3s}$  |        | 12$s$ | 24$s$     |
| 19             | 17         | 1              | 6$s + 2$             | $A^{3s+1}$ |        | 12$s + 4$ | 24$s + 8$  |
| 27             | 25         | 0              | 6$s + 3$             | $F^{2s+1}$ |        | 12$s + 6$ | 24$s + 12$ |
| 35             | 33         | 0              | 6$s + 3$             | $F^{2s+1}$ |        | 12$s + 6$ | 24$s + 12$ |
| 43             | 41         | 1              | 6$s + 5$             | $A^1 F^{2s+1}$ |        | 12$s + 10$ | 24$s + 20$ |
| 5              | 5          | 3              | 6$s$                 | $A^{3s}$  |        | 12$s$ | 24$s$     |
| 13             | 11         | 1              | 6$s + 2$             | $A^{3s} C^1$ | *     | 12$s + 2$ | 24$s + 5$  |
| 21             | 19         | 0              | 6$s + 3$             | $F^{2s} I^1$ | *     | 12$s + 4$ | 24$s + 9$  |
| 29             | 27         | 0              | 6$s + 3$             | $F^{2s+1}$ |        | 12$s + 6$ | 24$s + 12$ |
| 37             | 35         | 1              | 6$s + 5$             | $A^1 F^{2s} I^1$ | *     | 12$s + 8$ | 24$s + 17$ |
| 45             | 43         | 0              | 6$s + 6$             | $A^{3s+2} C^1$ | *     | 12$s + 10$ | 24$s + 21$ |
| 7              | 7          | 5              | 6$s + 1$             | $A^{3s} B^1$ |        | 12$s + 1$ | 24$s + 2$  |
| 15             | 13         | 0              | 6$s + 2$             | $F^{2s} J^1$ |        | 12$s + 3$ | 24$s + 6$  |
| 23             | 21         | 0              | 6$s + 3$             | $F^{2s} H^1$ |        | 12$s + 5$ | 24$s + 10$ |
| 31             | 29         | 1              | 6$s + 4$             | $B^1 F^{2s+1}$ |        | 12$s + 7$ | 24$s + 14$ |
| 39             | 37         | 0              | 6$s + 5$             | $A^{3s+2} B^1$ |        | 12$s + 9$ | 24$s + 18$ |
| 47             | 45         | 0              | 6$s + 6$             | $F^{2s+1} H^1$ |        | 12$s + 11$ | 24$s + 22$ |

Table 2: Constructions for $\mathcal{O}_N(2v - 2, v; 3, 2)$ when $v$ is odd. 

$\mathcal{E}$. When $v \equiv 11, 17 \pmod{24}$, also include the star $[0; a_{(v+1)/2}, a_{(v+3)/2}, a_{(v+5)/2}]$ and remove the three corresponding edges from $\mathcal{E}$. When $v \equiv 17 \pmod{24}$, also include the star $[0; a_{(v+7)/2}, a_{(v+9)/2}, a_{(v+11)/2}]$ and remove the three corresponding edges from $\mathcal{E}$. Now $\mathcal{E}$ has $\frac{v-17}{2}$ edges when $v \equiv 17 \pmod{24}$, $\frac{v-11}{2}$ edges when $v \equiv 11 \pmod{24}$, $\frac{v-5}{2}$ edges when $v \equiv 5 \pmod{24}$, and $\frac{v-3}{2}$ edges otherwise.

Next partition all edges on $W$ and the edges of $\mathcal{E}$. In Table 2, all cases for $v$ modulo 48 are listed. For each, the drop cost required to meet the bound is given, along with the congruence class of the number of edges on $W$ and the number of edges remaining in $\mathcal{E}$. The remaining three columns, ‘Type’, ‘Note’, and ‘$\Delta_1$’ give an abbreviated form of a general construction.

In the type specified, nine different capital letters are used to indicate ingredients. For each occurrence of each letter in the type, use Table 3 as follows. Edges are written in
the form $xy$ to represent edge $\{x, y\}$, and triangle $(x, y, z)$ is written as $xyz$. First examine the size $s$ given for the ingredient. Reserve $s$ distinct vertices in $W$ (that have not been used by previous ingredients) so that all vertices appearing in the edges of the factor are endvertices of edges of $\mathcal{E}$; if the ingredient is C, D, G, or I, exactly one other vertex $q$ is also an endvertex of an edge of $\mathcal{E}$; and no other vertex is an endvertex of an edge of $\mathcal{E}$. Having identified vertices in this way with vertices of $W$, place all triples of the ingredient on the corresponding vertices of $W$, place all graphs in the specified leave in the decomposition, and place all triangles of the form $(0, x, y)$ for each edge $xy$ in the specified factor. (The total number of such triangles is given in the column ‘$\Delta_1$’.) When the ingredient is C, D, G, or I (marked with the note $\star$ in Table 2), add edge $\{0, q\}$ to the $P_3$ of the leave; this does not change the drop cost.

This exhausts all edges of $\mathcal{E}$, and in addition employs certain edges on $W$. Indeed when there are $m$ ingredients with sizes $s_1, s_2, \ldots, s_m$, to partition the remaining edges into triangles, a 3-GDD of type $s_1^1s_2^1\cdots s_m^1$ $1v-2-\sum_{i=1}^ms_i$ is required. It remains to establish that such a 3-GDD exists. When $v \equiv 3, 5, 7, 11, 13, 17, 19, 45 \pmod{48}$ and $v \notin \{7, 19, 39\}$, choose all sizes for the ingredients to equal 11 and then apply Theorem 4.3 to form the 3-GDD. When $v \equiv 9, 25 \pmod{48}$ and $v \neq 25$, proceed in the same manner ignoring the group of type $G$; then simple counting shows that there is a block of the 3-GDD meeting none of the groups of size 11, and its deletion then forms the group of size 3. When $v \equiv 15, 21, 23, 27, 29, 35, 41, 47 \pmod{48}$ and $v \notin \{21, 23, 27\}$, choose all sizes for the ingredients to equal 13 and then apply Theorem 4.3 to form the 3-GDD. The case when $v \equiv 33 \pmod{48}$ is again obtained by deleting a triple that meets no group of size 13.

In the remaining cases, choose ingredients $A$, $B$, and $D$ to have size 11, and ingredients $F$ and $I$ to have size 15. Then write $v = 48s + x$ for $x \in \{31, 37, 43, 49\}$ and suppose that $v \notin \{31, 37, 43, 49, 79\}$. Use Theorem 4.2 to form a 3-GDD of type $15^{2s+2}(18s + x - 30)^1$. On the large group, place the blocks of a 3-GDD of type $1^{18s+x-41}11^1$, also from Theorem 4.2. On one group of size 15, place the blocks of an STS(15). The result is a 3-GDD of type $15^{2s+1}11^11^{18s+x-26}$, which completes the construction.

It remains only to treat the small cases. For $v = 7$, use ingredient $B$ of size 5. For $v = 21$, use a 3-GDD of type $9^11^{10}$ and ingredient $I$ of size 9. For $v = 25$, form a 3-GDD of type $11^11^{12}$, and place ingredient $A$ of size 11 on the group of size 11. Then delete a triple that does not involve any endvertex of an edge of $\mathcal{E}$ to form the required ‘group’ of size 3, and place ingredient $B$ on it. For $v = 31$, use a 3-GDD of type $11^11^{17}1^{11}$ from [10]; place ingredient $A$ on the group of size 11, and partition the group of size 7 using triples bcf, acg, beg, dfg; factor ab cd ef; and graphs $[a;d,e,f]$ and $[b;d,e,c]$. For $v = 37$, use a 3-GDD of type $11^19^11^{15}$ from [10]. For $v = 43$, use a 3-GDD of type $13^11^{11}1^{17}$ from [10]. For $v = 49$, use a 3-GDD of type $13^11^{11}1^{23}$ from [10]. For $v = 79$, use Theorem 4.2 to form a 3-GDD of type $15^417^1$, and fill the large group with a 3-GDD of type $1^{125}1$, using size 5 for the ingredient of type $B$. For $v \in \{19, 23, 27\}$, form ingredient $A, H, F$ on 17, 21, and 25 vertices, respectively (specific solutions are omitted here).
<table>
<thead>
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<th>Factor</th>
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<td>A</td>
<td>11</td>
<td>ab cd ef gh</td>
<td>[a;c,e,g], [b;d,f,h]</td>
<td>adj afi ahk bcj beg bik ceh cfk cgi dek dfi dfg dhi eij fhi gjk</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>ab</td>
<td>[a,c,d,b]</td>
<td>ade bce</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>ab</td>
<td>[a,c,d,b]</td>
<td>adh aek afj agi bcf beg bhi bjk ceh cfk cgh cij dej dfi djk efj fik ghj</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>ab cd</td>
<td>[b;c,d,e], [a,f,e]</td>
<td>ack ade agi ahj bfg bhi bjk cef cgf dfj dfg djk ejg ehi eik egk fik gjk</td>
</tr>
<tr>
<td>D</td>
<td>11</td>
<td>ab cd ef gh ij</td>
<td>[a,k,c], [b,f,h,j], [d;e,g,i]</td>
<td>aci adh aej afg bce bdk bgi cfi cij dfj efj gfh ghj gik hij hjk</td>
</tr>
<tr>
<td>F</td>
<td>13</td>
<td>ab cd ef gh ij kl</td>
<td>[a,f,g,h], [b;c,i,j], [d;e,k,l]</td>
<td>acg adh aej aik alm bdm beh bfl bgk cef cgh chf chk clj dfi dfg egl fem gir hik hij hkl jkl</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>ab cd ef gh ij kl</td>
<td>[a,f,g,h], [b;c,i,j], [d;e,k,l]</td>
<td>ack adm ael agi alm ajo bdo bfr bgl bhm cef cfo cgl chl cij dfi dgn dhj egj ehh emo fjf fnm gko hio ikn ilm jkm lno</td>
</tr>
<tr>
<td>G</td>
<td>3</td>
<td>ab</td>
<td>[a,c,b]</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>13</td>
<td>ab cd ef gh ij</td>
<td>[b;f,i,j], [d;e,g,h], [a,c]</td>
<td>aci adh aej afg bce bdk bgi cfi cij dfj efj gfh ghj gik hij hjk</td>
</tr>
<tr>
<td>I</td>
<td>9</td>
<td>ab cd ef gh</td>
<td>[c;a,e,g], [f;d,h,i], [b,g,i]</td>
<td>adi aeh afg bcf bdl bei chi deg</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>ab cd ef gh</td>
<td>[b;c,h,i], [d;e,f,g], [a,j,i]</td>
<td>ack adi aeh afl agm bdl bem bfl bgk cef cgl chf dhj dkm egj ekl flk fim gik hil jlm jmn</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>ab cd ef gh</td>
<td>[b;c,h,i], [d;e,f,g], [a,j,i]</td>
<td>ace ado afg ahl ail akm bdm bej bfl bgk bmo cfi cgj cin cko clm dlm dik djl egm ehi ekn elo fim fjk fno gio glm hjo hkl jmn</td>
</tr>
<tr>
<td>J</td>
<td>13</td>
<td>ab cd ef</td>
<td>[a,g,h,c], [d;b,e,f]</td>
<td>acm adi aeh afk ajl bcf bei bgj bhk bhm cef cgl cil dgm dhj dkl egl ephem fgi fhl fjf hlm ilm jkm lno</td>
</tr>
</tbody>
</table>

Table 3: Ingredients for GDD types.
5.3 Semiframes

We detour to introduce new definitions and new results that are needed in Section 5.4.

Consider the complete multipartite graph $\Gamma$ with $x + 1$ parts (groups), of which $x$ have size $g$, one has size $u$, and $g$ and $u$ are both even. When $p \leq g$ and $p \leq u$, a $p$-semiframe of type $g^xu^1$ is a partition of the edge set of $\Gamma$ into $px + 2(g - p)x + 2(u - p)$ classes; the first $px$ are 1-factors of $\Gamma$ (or parallel classes). For each of the $x$ groups, there are $g - p$ classes that contain disjoint edges that span all vertices not in the selected group (frame parallel classes). The last $u - p$ classes are frame parallel classes for the group of size $u$. Now $px(xg + u) + x(g - p)[(x - 1)g + u] + (u - p)xg = x(x - 1)g^2 + 2xgu$.

When no parallel classes are selected, $p = 0$ and a $p$-semiframe is a frame. When $p = g = u$, a $p$-semiframe is a 1-factorization of the complete multipartite graph having $x + 1$ parts of size $p$. Both special cases have been well studied when all groups have the same size. Rees settled the existence of semiframes when $g$ is even, with $g = u$ (the uniform case):

Lemma 5.5 [31] For even $g$,

1. A 0-semiframe of type $g^{x+1}$ exists for all $x \geq 2$.
2. A $g$-semiframe of type $g^{x+1}$ exists for all $x \geq 1$.
3. A $p$-semiframe of type $g^{x+1}$ with $0 < p < g$ exists provided that $x \geq 2$.

Lemma 5.6 There is a 1-factorization of $K_{2x+2y}$ that contains a 1-factorization of $K_{2y}$ if and only if $x = 0$ or $x \geq y$. Hence a $2x$-semiframe of type $2^x(2y)^1$ exists.

Proof. For existence of 1-factorizations with sub-1-factorizations, see [34]. To obtain the semiframe, delete the edges of the sub-1-factorization, truncating in the process $2y - 2$ of the 1-factors to form frame parallel classes. The remaining $2x$ 1-factors have no edges removed in this way.

Theorem 5.7 If a pairwise balanced design exists on $x + 1$ vertices, for which (1) there is an vertex contained only in blocks of size at least four, and for every size $k$ of a block that contains this vertex there is a 2-semiframe of type $(2g)^{k-1}(2y)^1$ and (2) every other block has size at least three, then there exists a 2-semiframe of type $(2g)^x(2y)^1$.

Proof. Let $(V, B)$ be a pairwise balanced design, and let $V = \{v_0, v_1, \ldots, v_x\}$. We treat $v_0$ as a distinguished vertex. Suppose that all blocks of $B$ have size at least three and that those containing $v_0$ have size at least four. Form the semiframe on $(\{v_0\} \times \{1, \ldots, 2y\}) \cup (\{v_1, \ldots, v_x\} \times \{1, \ldots, 2g\})$. When $B \in B$ does not contain $v_0$, form a frame on $B \times \{1, \ldots, 2g\}$ whose groups are $\{\{w\} \times \{1, \ldots, 2g\} : w \in B\}$ (this is uniform and exists by Lemma 5.5). When $B \in B$ does contain $v_0$, form a 2-semiframe of type $(2g)^{|B|^{-1}}(2y)^1$ on $((B \setminus \{v_0\}) \times \{1, \ldots, 2g\}) \cup (\{v_0\} \times \{1, \ldots, 2y\})$, aligning groups in the natural manner.

First form the $2x$ parallel classes. For $v_i$ for $i = 1, \ldots, x$ in turn, let $B$ be the unique block containing $v_0$ as well as $v_i$. Let $B_1, \ldots, B_r$ be the remaining blocks containing $v_i$. 

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Taking one parallel class from the semiframe arising from $B$, and one frame parallel class from each frame arising from $B_1, \ldots, B_r$ produces a parallel class of the semiframe under construction. Repeat this to form a second parallel class. Now carrying this out for each of the $x$ undistinguished vertices produces $2x$ parallel classes. In the process, we exhaust all parallel classes of the ingredient semiframes, two of the $2g$ frame parallel classes at each undistinguished vertex for each ingredient frame, and none of the frame parallel classes for the group of size $2y$ arising from the distinguished vertex. Now simply union the $2y - 2$ frame parallel classes for the group of size $2y$ in each ingredient semiframe to form the $2y - 2$ frame parallel classes of the frame under construction. Finally for each undistinguished vertex, merge the $2(g - 1)$ remaining frame parallel classes.

**Corollary 5.8** If a 2-semiframe of type $(2g)^x(2y)^1$ exists for each $x \in \{3, 4, 5, 6, 7, 8, 10, 11, 13, 14\}$, then a 2-semiframe of type $(2g)^x(2y)^1$ exists for all $x \geq 3$.

**Proof.** When $x \geq 12$ and $x \not\in \{13, 14, 17, 18, 22\}$, there exists a pairwise balanced design on $x + 1$ vertices in which every block has size 4, 5, 6, or 9 [1]. There exist 3-GDDs of types $3^3, 3^45^1, 3^6$, and $4^61^1$ [10]. Adding a new vertex and extending each group to include the new vertex, yields pairwise balanced designs in which one vertex is only in blocks of size at least four on 10, 18, 19, and 23 vertices.

**Theorem 5.9** Suppose that there is a group divisible design of type $u^{x+1}$ in which every block has size at least four, having $\ell$ parallel classes of blocks $P_1, \ldots, P_\ell$. Let $v_1, \ldots, v_u$ be the vertices of the first group. Further suppose that if $v_i$ belongs to a block of size $k$ in $P_j$, there exists a $p_j$-semiframe of type $g^{k-1}(y_i)^1$. Then there exists a $(\sum_{j=1}^\ell p_j)$-semiframe of type $(gu)^x(\sum_{i=1}^u y_i)^1$.

**Proof.** Give weight $y_i$ to $v_i$, and weight $g$ to each vertex in the remaining groups. For $i = 1, \ldots, \ell$, for parallel class $P_i$, place $p_i$-semiframes, and form unions of the parallel classes to obtain parallel classes of the semiframe under construction. On all remaining blocks, place frames. Each vertex is associated with frame parallel classes in the frames and semiframes, which are merged to form the required frame parallel classes.

**Theorem 5.10** A 2-semiframe of type $8^x(2y)^1$ exists when $1 \leq y \leq 6$ and $x \geq 3$.

**Proof.** The case when $y = 4$ is treated by the third statement of Lemma 5.5. For the remaining cases, we first form a set of “small” semiframes. Using a hill-climbing algorithm adapted to find partial triple systems [10], we have produced 2-semiframes of type $8^x(2y)^1$ for all $3 \leq x \leq 15$ and $1 \leq y \leq 6$. We do not provide the solutions here in the interest of conserving space; they can be obtained from the authors. Apply Corollary 5.8.
5.4 $\mathcal{O}\mathcal{N}(2v - 2, v; 3, 2)$ when $v$ is even

We now treat the final case, which has been left for last as it is the most technical. Figure 6 depicts a graph, $G_{8,2}$, that can be partitioned into two $K_{1,3^8}$, a $P_3$, and a triangle. It accounts for the correct number of inside edges (6) for the number of cross edges consumed (4), and has eight odd degree vertices in $V$ (on the left) and two in $W$ (on the right). Indeed both are odd in the subgraph induced on $W$ as well. It would suffice (at least when the numerical conditions work) to form an appropriate partition into triangles whose leaves is $\frac{v}{8}$ copies of the graph at right, along with $\frac{3v}{16} - \frac{1}{2}$ stars entirely on $W$. This is driven by the detailed lower bound analysis in [16]. We follow this strategy throughout.

![Figure 6: $G_{8,2}$.](image)

**Lemma 5.11** There is a partial triple system on $\{0, \ldots, 7\} \cup \{a_0, \ldots, a_5\}$ missing exactly the edges of $G_{8,2}$ (with vertices on the left labelled with $0, \ldots, 7$) together with all edges $\{\{a_i, a_j\} : 0 \leq i < j \leq 5\}$.

**Proof.** Form the following partition of the edges of the complement of $G_{8,2}$:

<table>
<thead>
<tr>
<th>$i$</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0,5}$, ${2,6}$, ${4,7}$</td>
</tr>
<tr>
<td>1</td>
<td>${0,4}$, ${2,5}$, ${1,7}$</td>
</tr>
<tr>
<td>2</td>
<td>${1,5}$, ${3,7}$, ${4,6}$, ${0,2}$</td>
</tr>
<tr>
<td>3</td>
<td>${0,3}$, ${1,6}$, ${2,4}$, ${5,7}$</td>
</tr>
<tr>
<td>4</td>
<td>${1,2}$, ${3,6}$, ${4,5}$, ${0,7}$</td>
</tr>
<tr>
<td>5</td>
<td>${0,6}$, ${1,4}$, ${2,7}$, ${3,5}$</td>
</tr>
</tbody>
</table>

Form triangles by adding vertex $a_i$ to each of its associated edges, for $0 \leq i \leq 5$.

Table 4 provides information concerning the construction to be used for each congruence class for $v$ modulo 48. The last column, ‘Total Cost’, indicates the number of non-triangle graphs required in the entire decomposition in order to meet the bound. We treat the easiest case first.

**Lemma 5.12** When $v \equiv 0 \pmod{8}$ and $v \neq 16$, cost $\mathcal{O}\mathcal{N}(2v - 2, v; 3, 2) \leq \left(\frac{2v-2}{2}\right) + \left\lceil\frac{9v}{16} - \frac{1}{2}\right\rceil + \delta$, where $\delta = 1$ if $v \equiv 8, 40 \pmod{48}$ and $\delta = 0$ otherwise.
\[ V \cdot G \] denotes a copy of \( G \) entirely on \( V \), and symmetrically \( W \cdot G \) denotes one entirely on \( W \).

Table 4: Ingredients for Theorem 5.16.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( v \) & Master & Filler & \( G_{8,2} \) & \( K_{1,3} \) & Other & Total Cost \\
\hline
48x & 4^{9x-3}(12x + 10)^{1} & 2^{6x-10^{1}} & 6x & 9x - 3 & \( W \cdot K_{1,9} \) & 27x \\
48x + 2 & 4^{9x}(12x)^{1} & 2^{6x} & 6x & 9x & \( V \cdot K_{2} \) & 27x + 1 \\
48x + 4 & 4^{9x}(12x + 2)^{1} & 2^{6x+1} & 6x & 9x & \( G_{4,2} \) & 27x + 2 \\
48x + 6 & 4^{9x+1}(12x)^{1} & 2^{6x} & 6x & 9x + 1 & \( G_{6,0} \) & 27x + 3 \\
48x + 8 & 4^{9x}(12x + 6)^{1} & 2^{6x+3} & 6x + 1 & 9x + 2 & \( 2 \times W \cdot K_{2} \) & 27x + 5 \\
48x + 10 & 4^{9x}(12x + 8)^{1} & 2^{6x+8} & 6x + 1 & 9x + 1 & \( V \cdot K_{2}; W \cdot K_{1,7} \) & 27x + 6 \\
48x + 12 & 4^{9x+1}(12x + 6)^{1} & 2^{6x+3} & 6x + 1 & 9x + 1 & \( G_{4,2}; W \cdot K_{2} \) & 27x + 7 \\
48x + 14 & 4^{9x+3}(12x)^{1} & 2^{6x} & 6x + 1 & 9x + 2 & \( G_{6,0}; \star \) & 27x + 8 \\
48x + 16 & 4^{9x+3}(12x + 2)^{1} & 2^{6x+1} & 6x + 2 & 9x + 2 & \( \star \) & 27x + 9 \\
48x + 18 & 4^{9x+1}(12x + 12)^{1} & 2^{6x+28} & 6x + 2 & 9x + 3 & \( V \cdot K_{2}; W \cdot K_{1,7} \) & 27x + 10 \\
48x + 20 & 4^{9x+3}(12x + 6)^{1} & 2^{6x+3} & 6x + 2 & 9x + 3 & \( G_{4,2} \) & 27x + 11 \\
48x + 22 & 4^{9x+3}(12x + 8)^{1} & 2^{6x+4} & 6x + 2 & 9x + 3 & \( G_{6,0}; 2 \times W \cdot K_{2} \) & 27x + 13 \\
48x + 24 & 4^{9x+4}(12x + 6)^{1} & 2^{6x+3} & 6x + 3 & 9x + 4 & \( \star \) & 27x + 13 \\
48x + 26 & 4^{9x+3}(12x + 12)^{1} & 2^{6x+28} & 6x + 3 & 9x + 3 & \( V \cdot K_{2}; W \cdot K_{1,7} \) & 27x + 15 \\
48x + 28 & 4^{9x+3}(12x + 14)^{1} & 2^{6x+38} & 6x + 3 & 9x + 3 & \( G_{4,2}; W \cdot K_{1,7} \) & 27x + 16 \\
48x + 30 & 4^{9x+4}(12x + 12)^{1} & 2^{6x+28} & 6x + 3 & 9x + 4 & \( \star \) & 27x + 17 \\
48x + 32 & 4^{9x+6}(12x + 6)^{1} & 2^{6x+3} & 6x + 4 & 9x + 5 & \( V \cdot K_{2} \) & 27x + 18 \\
48x + 34 & 4^{9x+6}(12x + 8)^{1} & 2^{6x+4} & 6x + 4 & 9x + 6 & \( G_{12,0} \) & 27x + 19 \\
48x + 36 & 4^{9x+7}(12x + 6)^{1} & 2^{6x+3} & 6x + 3 & 9x + 7 & \( G_{6,0}; 2 \times W \cdot K_{2} \) & 27x + 20 \\
48x + 38 & 4^{9x+6}(12x + 12)^{1} & 2^{6x+6} & 6x + 4 & 9x + 7 & \( 2 \times W \cdot K_{2} \) & 27x + 22 \\
48x + 40 & 4^{9x+6}(12x + 14)^{1} & 2^{6x+7} & 6x + 5 & 9x + 6 & \( V \cdot K_{2}; W \cdot K_{2} \) & 27x + 23 \\
48x + 42 & 4^{9x+7}(12x + 12)^{1} & 2^{6x+6} & 6x + 5 & 9x + 7 & \( G_{4,2}; W \cdot K_{1,7} \) & 27x + 24 \\
48x + 44 & 4^{9x+6}(12x + 18)^{1} & 2^{6x+8} & 6x + 5 & 9x + 6 & \( G_{6,0}; \star \) & 27x + 25 \\
48x + 46 & 4^{9x+9}(12x + 8)^{1} & 2^{6x+4} & 6x + 5 & 9x + 8 & \( \star \) & 27x + 26 \\
\hline
\end{tabular}
\end{table}

**Proof.** If \( v = 8 \), apply Lemma 5.11. Then form a 3-GDD of type \( 2^{3} \) on \( W \), aligning one group of the GDD with the edge of the \( G_{8,2} \).

Otherwise write \( v = 48r + 8z + 8 \) with \( 0 \leq z \leq 5 \). Let the vertex set be \( V \cup W \) with \( V = \{ \ell_{ij} : 0 \leq i \leq 6r + z, 0 \leq j < 8 \} \) and \( W = \{ p_{ij} : 0 \leq i < 6r + z, 0 \leq j < 2 \} \cup \{ f_{ij} : 0 \leq i \leq 6r + z, 0 \leq j < 6 \} \). Form a 2-semiframe of type \( 8^{6r+z+1} \) on \( V \) (from Theorem 5.10), aligning groups on \( \{ \ell_{ij} : 0 \leq j < 8 \} : 0 \leq i \leq 6r + z \}. \) Let \( \{ \mathcal{P}_{ij} : 0 \leq i < 6r + z, 0 \leq j < 2 \} \) be the 2\((6r+z)\) parallel classes of the semiframe, and for \( 0 \leq i \leq 6r + z \), let \( \{ \mathcal{F}_{ij} : 0 \leq j < 6 \} \) be the six frame parallel classes for group \( \{ \ell_{ij} : 0 \leq j < 8 \} \).

Now for \( 0 \leq i < 6r + z \) and \( j \in \{ 0, 1 \} \), add vertex \( p_{ij} \) to each edge of \( \mathcal{P}_{ij} \) to form a triangle. For \( 0 \leq i \leq 6r + z \) and \( 0 \leq j < 6 \), add vertex \( f_{ij} \) to each edge of \( \mathcal{F}_{ij} \) to form a triangle. For \( 0 \leq i \leq 6r + z \), place on \( \{ \ell_{ij} : 0 \leq j < 8 \} \cup \{ f_{ij} : 0 \leq j < 6 \} \) the partial triple system from Lemma 5.11, aligning vertices \( 0, \ldots, 7 \) with vertices in \( V \). At this point
all edges inside $V$ and between $V$ and $W$ are used except for $6r + z + 1$ copies of $G_{8,2}$. Noting that each $G_{8,2}$ employs one edge inside $W$, we require that these $6r + z + 1$ edges not be covered by triangles to be added that lie entirely on $W$.

To handle the edges on $W$, we require a partial triple system on the $v - 2$ vertices in $W$. Follow the prescription in Table 4 as follows. Now write $v = 48x + y$ with $y \in \{0, 8, 16, 24, 32, 40\}$. Consider the case when $y = 24$. Then $6x + 3$ disjoint edges are needed for the $G_{8,2}$s, to which $9x + 4$ disjoint copies of $K_{1,3}$ are added. A partial triple system on $48x + 22$ vertices whose leave consists of $6x + 3$ edges and $9x + 4$ $K_{1,3}$s is then needed. To make this, form the GDD indicated in the column labelled ‘Master’—in this case, a 3-GDD of type $4^{9x+4}(12x + 6)^1$. Then fill the large group with the 3-GDD indicated in the column labelled ‘Filler’, in this case filling with a 3-GDD of type $2^{6x+3}$ to form a 3-GDD of type $4^{9x+426x+3}$. Place a triangle on each group of size 4 to obtain the required partial triple system. The GDDs employed all exist by Theorem 4.2. It remains to determine the cost. Each $G_{8,2}$ decomposes into three graphs each of cost 1, and each $K_{1,3}$ costs 1; hence the cost is $3(6x + 3) + 9x + 4 = 27x + 13$, meeting the bound.

The remaining cases are variants of this. When $v \equiv 8, 40 \pmod{48}$, the GDD constructed produces two extra edges that are not used in copies of $G_{8,2}$ (in order to make the congruence modulo 3 in the partial triple system correct). This is indicated by $2 \times W$-$K_2$ in the ‘Other’ column. When $v \equiv 16, 32 \pmod{48}$, the GDD does not make enough isolated edges to form all of the copies of $G_{8,2}$; in each case, one more is needed. Using one edge of a $K_{1,3}$ in the final $G_{8,2}$ then leaves a path in place of the star. This is indicated by ‘*’ in Table 4. Finally when $v \equiv 0 \pmod{48}$, the partial triple system is formed to have one $K_{1,9}$ in its leave obtained by placing the blocks of a Steiner triple system of order 9 on the 10-vertex group, indicated by $W$-$K_{1,9}$ (as well as $9x - 3$ isolated $K_{1,3}$s and $6x$ isolated edges). The $K_{1,9}$ can be decomposed into three $K_{1,3}$s, so that the cost is $3(6x) + 9x - 3 + 3 = 27x$, as required. Indeed in each case the cost is realized by the prescription given.

Lemma 5.13 When $v \equiv 2 \pmod{8}$ and $v \not\in \{10, 18\}$, $\cost \mathcal{O}(2v - 2, v; 3, 2) \leq \left(\frac{2v-2}{2}\right) + \left[\frac{9v}{16} - \frac{1}{2}\right]$.

Proof. If $v = 2$, the decomposition contains a single $K_2$. Otherwise write $v = 48r + 8z + 2$ with $0 \leq z \leq 5$. Let the vertex set be $V \cup W$ with $V = \{\ell_{ij} : 0 \leq i < 6r + z, 0 \leq j < 8\} \cup \{\sigma_j : 0 \leq j < 2\}$ and $W = \{p_{ij} : 0 \leq i < 6r + z, 0 \leq j < 2\} \cup \{f_{ij} : 0 \leq i < 6r + z, 0 \leq j < 6\}$. Form a 2-semiframe of type $8^{6r+2}2^1$ on $V$ (from Theorem 5.10), aligning groups on \{\{\ell_{ij} : 0 \leq j < 8\} : 0 \leq i < 6r + z\} and \{\sigma_j : 0 \leq j < 2\}. Let \{\mathcal{P}_{ij} : 0 \leq i < 6r + z, 0 \leq j < 2\} be the $2(6r + z)$ parallel classes of the semiframe, and for $0 \leq i < 6r + z$, let \{\mathcal{F}_{ij} : 0 \leq j < 6\} be the six frame parallel classes for group \{\ell_{ij} : 0 \leq j < 8\}. Group \{\sigma_j : 0 \leq j < 2\} has no frame parallel classes.

As in Lemma 5.12, for $0 \leq i < 6r + z$ and $j \in \{0, 1\}$, add vertex $p_{ij}$ to each edge of $\mathcal{P}_{ij}$ to form a triangle. For $0 \leq i < 6r + z$ and $0 \leq j < 6$, add vertex $f_{ij}$ to each edge of $\mathcal{F}_{ij}$ to form a triangle. For $0 \leq i < 6r + z$, place on \{\ell_{ij} : 0 \leq j < 8\} \cup \{f_{ij} : 0 \leq j < 6\} the partial triple system from Lemma 5.11, aligning vertices $0, \ldots, 7$ with vertices in $V$. Leave the extra group empty. All edges inside $V$ and between $V$ and $W$ are used except for $6r + z$ copies of $G_{8,2}$ and one isolated copy of $K_2$. 25
To handle the edges on $W$, follow the prescription in Table 4 as in Lemma 5.12. Specifically, when $v \equiv 2, 34 \pmod{48}$, disjoint stars and edges are formed on $W$ and all edges are consumed in copies of $G_{8,2}$. When $v \equiv 42 \pmod{48}$, one isolated edge on $W$ does not appear in a $G_{8,2}$. When $v \equiv 18 \pmod{48}$, the decomposition on $W$ contains, in addition to isolated stars and edges, one $K_{1,7}$ that in turn decomposes into two $K_{1,3}$s and a $P_2$. The most complicated case occurs when $v \equiv 10, 26 \pmod{48}$. A decomposition into isolated stars and edges, and one $K_{1,7}$, is formed. All isolated edges appear in $G_{8,2}$s and one edge of the $K_{1,7}$ also appears in a $G_{8,2}$ (indicated by ‘†’ in the table). The $K_{1,6}$ that remains partitions into two copies of $K_{1,3}$.

**Lemma 5.14** When $v \equiv 4 \pmod{8}$ and $v \not\in \{12, 20\}$, cost $\Theta N(2v - 2, v; 3, 2) \leq \binom{2v-2}{2} + \left\lceil \frac{9v}{16} - \frac{1}{2} \right\rceil$.

**Proof.** Figure 7 depicts a graph, $G_{4,2}$, with four edges inside $V$ and four between $V$ and $W$. It consumes the correct number of cross edges for the number of inside edges. It decomposes into one triangle, one path on three edges, and one star. It can be completed using two triangles to $K_6$. Indeed this solution meets the bound for $v = 4$.

![Figure 7: $G_{4,2}$](image)

Otherwise when $v \not\equiv 36 \pmod{48}$ write $v = 48r + 8z + 4$ with $0 \leq z \leq 5$. Let the vertex set be $V \cup W$ with $V = \{\ell_{ij} : 0 \leq i < 6r + z, 0 \leq j < 8\} \cup \{\sigma_j : 0 \leq j < 4\}$ and $W = \{p_{ij} : 0 \leq i < 6r + z, 0 \leq j < 2\} \cup \{f_{ij} : 0 \leq i \leq 6r + z, 0 \leq j < 6\} \cup \{h_j : 0 \leq j < 2\}$. Form a 2-semiframe of type $8^{6r+z}4^1$ on $V$ (from Theorem 5.10), aligning groups on $\{\ell_{ij} : 0 \leq j < 8\}$ and $\{\sigma_j : 0 \leq j < 4\}$. Let $\{P_{ij} : 0 \leq i < 6r + z, 0 \leq j < 2\}$ be the 2$(6r+z)$ parallel classes of the semiframe. For $0 \leq i \leq 6r + z$, let $\{F_{ij} : 0 \leq j < 6\}$ be the six frame parallel classes for group $\{\ell_{ij} : 0 \leq j < 8\}$. Group $\{\sigma_j : 0 \leq j < 4\}$ has two frame parallel classes $H_0$ and $H_1$.

Proceed as in Lemma 5.12, but in addition, first form triangles by adding $h_j$ to each edge of $H_j$ for $j \in \{0, 1\}$, and secondly place a copy of the solution for $v = 4$ (leaving $G_{4,2}$) on $\{\sigma_j : 0 \leq j < 4\} \cup \{h_j : 0 \leq j < 2\}$.

To handle the edges on $W$, follow the prescription in Table 4 as in Lemma 5.12. The only variation here is that $G_{4,2}$ consumes an edge on $W$, in addition to those consumed by each $G_{8,2}$.

It remains to treat the case when $v \equiv 36 \pmod{48}$. Use the graph $G_{12,0}$ consisting of four stars each having center and two leaves in $V$, obtained by identifying the leaves in $W$ from each star. Then write $v = 48r + 8z + 12$, and form a 2-semiframe of type $8^{6r+z}12^1$ on $V$ (from Theorem 5.10). The only issue of substance is to decompose the complement of $G_{12,0}$
into nine parallel classes (of six edges each) and one partial parallel class of four edges. To do this form a 1-factorization of type $3^4$ to form nine parallel classes; then place one edge on each group to form the partial parallel class. The rest of the proof parallels the earlier cases.

**Lemma 5.15** When $v \equiv 6 \pmod{8}$ and $v \not\in \{14, 22\}$, cost $\mathcal{O}N(2v - 2, v; 3, 2) \leq \binom{2v-2}{2} + \left\lceil \frac{9v}{16} - \frac{1}{2} \right\rceil + \delta$, where $\delta = 1$ if $v \equiv 22, 38 \pmod{48}$ and $\delta = 0$ otherwise.

**Proof.** Use the graph $G_{6,0}$ consisting of two stars each having center and two leaves in $V$, obtained by identifying the leaves in $W$ from each star. The complement of $G_{6,0}$ partitions into three 1-factors and one partial parallel class on two edges. Completing these to triangles using four vertices in $W$, and placing on the four vertices a single triangle, yields a solution for $v = 6$.

For $v = 48r + 8z + 6 \geq 30$, use a 2-semiframe of type $8^{6r+z}1^1$ on $V$ (from Theorem 5.10) along with the prescription in Table 4. In the cases when $v \equiv 22, 38 \pmod{48}$, two isolated edges on $W$ remain unused in copies of $G_{8,2}$ while for $v \equiv 14, 46 \pmod{48}$ one edge of a star on $W$ is consumed in a $G_{8,2}$. Finally for $v \equiv 30 \pmod{48}$ two stars and an edge on $W$ form a $K_{1,7}$. The details parallel Lemma 5.12.

**Theorem 5.16** When $v \equiv 0 \pmod{2}$, cost $\mathcal{O}N(2v - 2, v; 3, 2) \leq \binom{2v-2}{2} + \left\lceil \frac{9v}{16} - \frac{1}{2} \right\rceil + \delta$, where $\delta = 1$ if $v \equiv 8, 22, 38, 40 \pmod{48}$ and $\delta = 0$ otherwise.

**Proof.** We first settle the cases with $10 \leq v \leq 22$. In each case, form a 1-factorization $\mathcal{F}$ of $K_v$ on $\{0, \ldots, v - 1\}$ in which $\{\{2i, 2i + 1\} : 0 \leq i < v/2\}$ is a 1-factor. In $\mathcal{F}$ some of the edges in two, four, or six further 1-factors are also specified, as in Table 5. Edges in different rows appear in different 1-factors, those in the same row are in the same 1-factor. Such an $\mathcal{F}$ is easily constructed and we omit the full description here.

<table>
<thead>
<tr>
<th>$v$</th>
<th>${3,4} {5,6} \cdots$</th>
<th>${1,3} \cdots$</th>
<th>${11,12} {13,14} \cdots$</th>
<th>${9,11} \cdots$</th>
<th>${3,4} {5,6} \cdots$</th>
<th>${1,3} \cdots$</th>
<th>${11,12} {13,14} \cdots$</th>
<th>${9,11} \cdots$</th>
<th>${3,4} {5,6} \cdots$</th>
<th>${1,3} \cdots$</th>
<th>${11,12} {13,14} \cdots$</th>
<th>${9,11} {17,18} {19,20} \cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = 10$</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v = 16$</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$v = 22$</td>
<td>22</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Table 5: Factorizations for the small cases.
Let $L_{12}$, $L_{60}$, and $L_{82}$ be the set of edges induced on $V$ by a copy of $G_{12}$, $G_{60}$, or $G_{82}$, respectively. Delete the 1-factor $\{\{2i, 2i+1\} : 0 \leq i < v/2\}$ from $F$; replace edges $\{3, 4\}$ and $\{5, 6\}$ by edge $\{4, 5\}$; and delete edge $\{1, 3\}$. For $v = 10$, the resulting matchings cover all edges except for components $L_{82}$ and $K_2$. For $v = 12$, delete edges $\{9, 10\}$ and $\{9, 11\}$ from the 1-factors in which they appear, so that the resulting matchings leave the components $L_{82}$ and $L_{42}$. Proceeding similarly, we obtain matchings that leave components $L_{82}$ and $L_{60}$.

For $v = 14$, instead remove $\{9, 10\}$ and $\{11, 12\}$ from the factor in which they are contained, and add edge $\{10, 11\}$, to form matchings that leave components $L_{82}$ and $L_{60}$. Proceeding similarly, we obtain matchings that leave $L_{82} \cup L_{82}$ for $v = 16$, $L_{82} \cup L_{82} \cup K_2$ for $v = 18$, $L_{82} \cup L_{82} \cup L_{42}$ for $v = 20$, and $L_{82} \cup L_{82} \cup L_{60}$ for $v = 22$.

Now we have formed $v - 2$ matchings $M_0, \ldots, M_{v-3}$ on the vertex set $\{0, \ldots, v - 1\}$. Add $v - 2$ new vertices $\{a_0, \ldots, a_{v-3}\}$, and for $0 \leq i \leq v - 3$ form triangles by adding $a_i$ to each edge of $M_i$. By renaming the new symbols, the completion of the copies of $G_{82}$ and $G_{42}$, when present, can be taken to use edges $\{\{a_0, a_1\}\}$ for $v \in \{10, 14\}$, $\{\{a_0, a_1\}, \{a_2, a_3\}\}$ when $v \in \{12, 16, 18, 22\}$, and $\{\{a_0, a_1\}, \{a_2, a_3\}, \{a_4, a_5\}\}$ when $v = 20$. Call these treated pairs on $W$. To complete the decompositions, we must decompose the set of all edges on $W$ except the treated pairs. For $v = 10, 12, 14, 16, 18, 20, 22$, form a partial triple system on $v - 2$ vertices with leave $K_{1, 7}; K_{1, 3} \cup 3K_2; 3K_{1, 3} \cup K_2; K_{1, 7} \cup 2K_2 \cup K_1; 3K_{1, 3} \cup 3K_3$; and $3K_{1, 3} \cup 4K_2$, respectively. When possible, include the treated pairs as $K_2$ in the leaves; if this cannot be done, place the remaining treated pair in a $K_{1, 7}$, and if this cannot be done, in a $K_{1, 3}$. The verification is straightforward.

For $v \leq 8$ and for $v \geq 24$, apply Lemmas 5.12, 5.13, 5.14, and 5.15.

The technique used for the smaller cases could be used generally if one could always complete a “small” number of 1-factors to a 1-factorization. Unfortunately the best result known in this regard completes only $.177v$ or fewer 1-factors to a 1-factorization [9]; a generalization of our approach would necessitate the completion of approximately $.25v$ 1-factors. Such a strong completion result may be true, but it appears to be far from a solution. Hence we preferred to improve current knowledge about semiframes.

## 6 The Main Theorems

The lower bound in [16, Lemma 3.1] and [16, Corollary 3.2], and upper bounds in Theorem 3.2 when $v \leq \vartheta_2(n)$ and in Theorem 3.3 when $v > \vartheta_2(n)$, establish:

**Theorem 6.1** Let $n$ and $v$ be two positive integers such that $n \geq 6$. Let $\vartheta_2(n) = \frac{2n}{3}$ if $n \equiv 0 \pmod{3}$; $\frac{2n+1}{3}$ if $n \equiv 1 \pmod{3}$; and $\frac{2n-1}{3}$ if $n \equiv 2 \pmod{3}$. Then

$$\text{cost } \mathcal{N}(n, v; 2, 1) = \begin{cases} \text{cost } \mathcal{N}(n, 2) & \text{if } v \leq \vartheta_2(n) \\ \text{cost } \mathcal{N}(n - v, 2) + v(n - 1) & \text{if } \vartheta_2(n) < v < n \end{cases}$$

Moreover $\text{cost } \mathcal{N}(3, v; 2, 1) = \text{cost } \mathcal{N}(3, 2) = 3$ for $v \in \{1, 2\}$; $\text{cost } \mathcal{N}(4, v; 2, 1) = \text{cost } \mathcal{N}(4, 2) = 9$ for $v \in \{1, 2, 3\}$; $\text{cost } \mathcal{N}(5, v; 2, 1) = \text{cost } \mathcal{N}(5, 2) = 15$ for $v \in \{1, 2, 3\}$; and $\text{cost } \mathcal{N}(5, 4; 2, 1) = 16$. 

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The lower bound in [16, Theorem 4.3], upper bounds in Lemmas 4.4 and 4.7 when $v \leq \vartheta_3(n)$, and in Theorems 4.8, 4.9, and 4.11 when $v > \vartheta_3(n)$ establish:

**Theorem 6.2** Let $n$ and $v$ be two positive integers such that $n \geq 6$. Let $\vartheta_3(n) = \frac{n-1}{2}$ if $n \equiv 1 \pmod{4}$; $\frac{n}{2}$ if $n \equiv 0 \pmod{2}$; and $\frac{n+1}{2}$ if $n \equiv 3 \pmod{4}$. When $v \leq \vartheta_3(n)$, 

$$\text{cost } \mathcal{O}(n; v; 3,1) = \text{cost } \mathcal{O}(n,3).$$

When $v > \vartheta_3(n)$,

1. When $v$ is even, $\text{cost } \mathcal{O}(n; v; 3,1) = 2^{(v)} + \frac{1}{2}v(n - v)$.

2. When $v$ is odd, $\text{cost } \mathcal{O}(n; v; 3,1) = 2^{(v)} + \frac{1}{2}(n - v) + \frac{1}{2}(v + 1)(n - v)$.

Moreover $\text{cost } \mathcal{O}(3; v; 3,1) = \text{cost } \mathcal{O}(3,3) = 3$ for $v \in \{1,2\}$; $\text{cost } \mathcal{O}(4; v; 3,1) = \text{cost } \mathcal{O}(4,3) = 7$ for $v \in \{1,2,3\}$; $\text{cost } \mathcal{O}(5; v; 3,1) = \text{cost } \mathcal{O}(5,3) = 12$ for $v \in \{1,2,3\}$; and $\text{cost } \mathcal{O}(5; 4; 3,1) = 14$.

The lower bound in [16, Theorem 4.9], and upper bounds in Theorem 6.2 when $v \leq \vartheta_3(n)$ (since $\text{cost } \mathcal{O}(n; v; 3,2) \leq \text{cost } \mathcal{O}(n,3)$), in Theorems 5.2 and 5.3 when $v > \vartheta_3(n)$ and $2v - 2 \neq n$, and in Theorems 5.4 and 5.16 when $2v - 2 = n$ establish:

**Theorem 6.3** Let $n$ and $v$ be two integers such that $n \geq 6$. When $v \leq \vartheta_3(n)$, $\text{cost } \mathcal{O}(n; v; 3,2) = \text{cost } \mathcal{O}(n,3)$. When $v > \vartheta_3(n)$,

1. When $v$ is even,

   (a) If $n = 2v - 2$ then $\text{cost } \mathcal{O}(2v - 2; v; 3,2) = \left(2^{(v-2)} + \frac{v}{16}v - \frac{1}{2}\right) + \delta$, where $\delta = 1$ if $v \equiv 8, 22, 38, 40 \pmod{48}$ and $\delta = 0$ otherwise.

   (b) If $n \leq 2v - 3$ then $\text{cost } \mathcal{O}(n; v; 3,2) = \left[\frac{3}{2}v\right] + \text{cost } \mathcal{O}(n - v, 3) + \frac{3}{4}v(n - v)$.

2. When $v$ is odd,

   (a) If $n = 2v - 2$ then $\text{cost } \mathcal{O}(2v - 2; v; 3,2) = \left(2^{(v-2)} + \left\lceil \frac{v}{8}\right\rceil\right) + \delta$, where $\delta = 1$ if $v \equiv 11, 17 \pmod{24}$ and $\delta = 0$ otherwise.

   (b) If $n = 2v - 1$ or $n \leq 2v - 3$ then $\text{cost } \mathcal{O}(n; v; 3,2) = \left[\frac{3}{2}v\right] + \left\lceil \frac{v}{2}\right\rceil + \frac{3}{4}v(n - v) + \frac{n-v}{4} + \delta$, where $\delta = 1$ if $n - v \equiv 4 \pmod{6}$; $\delta = 1$ if $n - v \equiv 5 \pmod{6}$; and $\delta = 0$ otherwise.

Moreover $\text{cost } \mathcal{O}(3; v; 3,2) = \text{cost } \mathcal{O}(3,3) = 3$ for $v \in \{1,2\}$; $\text{cost } \mathcal{O}(4; v; 3,2) = \text{cost } \mathcal{O}(4,3) = 7$ for $v \in \{1,2,3\}$; and $\text{cost } \mathcal{O}(5; v; 3,2) = \text{cost } \mathcal{O}(5,3) = 12$ for $v \in \{1,2,3,4\}$.

Indeed the determination of $\mathcal{N}(n, 3, 2)$ is completed as well:

**Corollary 6.4** $\mathcal{N}(3, 3, 2) = \{1, 2\}$, $\mathcal{N}(4, 3, 2) = \{1, 2, 3\}$, $\mathcal{N}(n, 3, 2) = \{1, 2, 3, 4\}$ for $n = 5, 6, 7, 9$, $\mathcal{N}(8, 3, 2) = \{1, 2, 3, 4, 5\}$, $\mathcal{N}(10, 3, 2) = \{1, 2, 3, 4, 5, 6\}$.

For every integer $n \geq 11$, $\mathcal{N}(n, 3, 2) = \{1, 2, \ldots, \vartheta_3(n)\}$.

**Proof.** For every $n$ and $v$ with $n \geq 11$ and $\vartheta_3 < v < n$, $\text{cost } \mathcal{O}(n; v; 3,2) > \text{cost } \mathcal{O}(n; 3)$. Theorem 6.3 completes the proof. $lacksquare$
Techniques from graph decompositions prove to be quite useful in determining the optimal costs of two-period groomings. Although the proofs are technically involved, standard design-theoretic tools such as partial triple systems and group divisible designs and graph-theoretic results on path-factorizations play a substantial role in simplifying the proofs. Nevertheless the characterizations are by no means standard applications of tools from design and graph theory in many cases; indeed, powerful results such as Plantholt’s theorem, new results on leaves of partial triple systems, and new tools such as semiframes, have all been needed.

In each case explored, there is a relatively sharp transition as \( v \) grows for fixed \( n \), from cases in which the grooming on \( n \) vertices determines the overall cost to ones in which the grooming on \( v < n \) vertices forces an increase. In the end, what is most striking is that two-period groomings can be cast in a natural manner as graph decompositions embedding smaller decompositions.

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References


