On nesting of $G$-decompositions of $\lambda K_v$ where $G$ has four nonisolated vertices or less *

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Abstract

The complete multigraph $\lambda K_v$ is said to have a $G$-decomposition if it is the union of edge disjoint subgraphs of $K_v$ each of them isomorphic to a fixed graph $G$. The spectrum problem for $G$-decompositions of $\lambda K_v$ that have a nesting was first considered in the case $G = K_3$ by C.J. Colbourn and M.J. Colbourn [4] and D.R. Stinson [15]. For $\lambda = 1$ and $G = C_m$ (the cycle of length $m$) this problem was studied in many papers, see [9, 10, 11] for more details and references. For $\lambda = 1$ and $G = P_k$ (the path of length $k-1$) the analogous problem was considered in [13].

In this paper we solve the spectrum problem of nested $G$-decompositions of $\lambda K_v$ for all the graphs $G$ having four nonisolated vertices or less, leaving eight possible exceptions.

1 Introduction

Let $H = (V(H), E(H))$ be a graph. Denote by $\lambda H$ the graph $H$ in which every edge has multiplicity $\lambda$. The multigraph $\lambda H$ is said to be $G$-decomposable if it is a union of edge disjoint subgraphs of $K_v$, each of them isomorphic to a fixed graph $G$. This situation is denoted by $\lambda H \rightarrow G$; $\lambda H$ is also said to admit a $G$-decomposition $(V, B)$, where $V = V(H)$, the vertex set of $H$, and $B$ is the edge-disjoint decomposition of $\lambda H$ into copies of $G$. Usually $B$ is called the block-set of the $G$-design and any $B \in B$ is said to be a block.

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A G-decomposition of \( \lambda K_v \) is also called a \( G \)-design of order \( v \), block size \(|V(G)|\) and index \( \lambda \). A \( G \)-design \((W, A)\) is called to be a subdesign of \((V, B)\) if \( W \subseteq V \) and \( A \subseteq B \).

The set of values of \( v \) for which \( K_v \) has a \( G \)-decomposition is determined if \( G \) has four vertices or less [2].

Let \( m \)-star be the graph \( S_m = [a; a_1, a_2, \ldots, a_m] = \{a, a_1, a_2, \ldots, a_m\} \). The vertex \( a \) of degree \( m \) in \( S_m \) is called the centre of the star and the vertices \( a_i \) of degree 1 are called the terminal vertices of the star.

Definition (see [9, 13]) A nesting of a \( G \)-decomposition of \( \lambda H \) \((V, B)\) is a pair \( \{(V, S), F\} \) where \((V, S)\) is a \( \lambda H \to S_m \) and \( F : B \to S \) is a \( 1 \to 1 \) mapping such that:

\( (n_1) \) for every \( B \in B \) the centre of the \( m \)-star \( F(B) \) is not in \( V(B) \) and any terminal vertex of \( F(B) \) is in \( V(B) \);

\( (n_2) \) For every pair \( B_1, B_2 \in B \) the graphs \( B_1 \cup F(B_1) \) and \( B_2 \cup F(B_2) \) are isomorphic.

Example 1. Let \( V(K_9) = Z_9 \). For \( i \in Z_9 \) put \( B^i = (i, 1 + i, 7 + i, 2 + i) \) and \( S^i = [3 + i; 1 + i, 7 + i, 2 + i] \), reducing all sums modulo 9. Then \( (Z_9, \{B^i| i \in Z_9\}) \) is a \( 4CS \) of order 9 that has a nesting defined by \( (Z_9, \{S^i| i \in Z_9\}) \) and \( F(B^i) = S^i \).

Let \( \tilde{G} = G \cup S_m \), where the centre of \( S_m \) is not in \( V(G) \) and any terminal vertex of \( S_m \) is in \( V(G) \). It is clear that a nesting of a \( \lambda \)-decomposition of \( \lambda H \) is a \( 2\lambda H \to \tilde{G} \) \((V(H), N)\) such that:

\( (p_1) \) \((V(H), \{B_1| B \in N\})\) (where \( B_1 \) is the subgraph of \( B \) isomorphic to \( G \)) is a decomposition \( \lambda H \to G \);

\( (p_2) \) \((V(H), \{B_2| B \in N\})\) (where \( B_2 \) is the subgraph of \( B \) isomorphic to \( S_m \)) is a decomposition \( \lambda H \to S_m \).

When \( H = K_v \), we say that the nested \( G \)-decomposition of \( \lambda K_v \), \( 2\lambda K_v \to \tilde{G} \), is a \( \tilde{G} \)-design \( N(v, 2\lambda) \).

The spectrum problem for \( \tilde{C}_m \)-designs \( N(v, 2\lambda) \) was first considered in the case where \( m = 3 \) by C.J.Colbourn and M.J.Colbourn [4] and D.R. Stinson [15]. For \( \lambda = 1 \) this problem was studied by Lindner, Rodger and Stinson [10] for odd \( m \) and by Lindner and Stinson [11] for even \( m \). See also [9] for more details and references.

In the following theorem we state the known results for \( m = 3, 4 \).

Theorem 1 [4, 15, 16]. There exists a \( \tilde{C}_3 \)-designs \( N(v, 2\lambda) \) if and only if \( \lambda(v - 1) \equiv 0 \pmod{6} \), \( v \geq 4 \). For every \( v \equiv 1 \pmod{8} \) there is a \( \tilde{C}_4 \)-designs \( N(v, 2) \) except possibly if \( v \in \{57, 65, 97, 113, 185, 265\} \).

Direct constructions of \( \tilde{C}_4 \)-designs of order 65, 97 and 113 are given in [14].
In this paper, we consider the case where $G$ is a graph with four nonisolated vertices or less and we solve the spectra problem of nested $G$-decompositions of $\lambda K_v$, except possibly for eight values of $v$.

It will cause no confusion if a graph $G$ identified by its edge set $E(G)$, since no graphs have isolated vertices.

## 2 Preliminaries

In this section, we shall define some terminology and state some results which will be useful later on.

**Lemma 1** Let $(V, B)$ be a $G$-decomposition of $\lambda K_v$, $v > 1$, having a nesting \{(V, $S$), $F$\}. Then the following conditions hold (1) $v \geq 1 + |V(G)|$; (2) $\lambda v(v-1) \equiv 0 \pmod{2|E(G)|}$; (3) $\lambda(v-1) \equiv 0 \pmod{\alpha(G)}$, where $\alpha(G)$ is the greatest common divisor of the degrees of the vertices of $G$; (4) $|E(G)| = |V(S_m)| - 1 = m$; and (5) $|E(G)| \leq |V(G)|$.

**Proof.** Conditions (1), (2) and (3) are straightforward. Condition (4) follows from the equality $|B| = |S|$. To prove (5) note that $|E(G)| + 1 = m + 1 = |V(S_m)| \leq |V(G)| + 1$. \(\square\)

By Lemma 1 there is not a nested $G$-decomposition of $\lambda K_v$ for $G = K_4$ and $G = K_4 - e$ (the quadrilateral with one diagonal). Since the spectrum problem for nested $K_3$-decompositions of $\lambda K_v$ is solved (see Theorem 1), the following cases must be considered:

(i) $G = P_2 = [a_1, a_2] = \{a_1, a_2\}$, the path of length 1.
(ii) $G = P_3 = S_2 = [a_1, a_2, a_3] = \{a_2; a_1, a_3\} = \{a_1, a_2\}, \{a_2, a_3\}$, the path of length 2 or the 2-star of centre $a_2$.
(iii) $G = E_2 = [a_1, a_2; a_3, a_4] = \{a_1, a_2\}, \{a_3, a_4\}$, two edges having no common vertex.
(iv) $G = S_3 = [a; a_1, a_2, a_3] = \{a, a_1\}, \{a, a_2\}, \{a, a_3\}$, the 3-star of centre $a$.
(v) $G = P_4 = [a_1, a_2, a_3, a_4] = \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}$, the path of length 3.
(vi) $G = D = [a_1, a_2, a_3 \uparrow a_4] = \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_3, a_4\}$, the triangle with attached edge.
(vii) $G = C_4 = (a_1, a_2, a_3, a_4) = \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}$, the cycle of length 4.

Let $(V, B)$ be a $\tilde{G}$-design $N(v, 2\lambda)$. If $|E(G)| < |V(G)|$ then every block $B \in B$ contains exactly $|V(G)| - |E(G)|$ vertices missing on the vertex set of $F(B)$. So, to satisfy $(n_2)$ of Definition, it is necessary to decide the position of these vertices into the block $B$.

**Example 2. (2.1)** $(Z_9, \{Q^i = [i, 3 + i, 7 + i \uparrow 8 + i]\} i \in Z_9)$ is a $D$-decomposition of $K_9$ that has a nesting defined by $(Z_9, \{S^i = [2 + i; i, 3 + i, 7 + i, 8 + i]\} i \in Z_9)$, reducing all the sums modulo $9$, and $F(Q^i) = S^i$. 

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(2.2) \( Z_5, \{Q_1^i = [i,1+i,3+i,2+i], Q_2^i = [3+i,4+i,2+i,i][i \in Z_5]\) is a \( P_4 \)-decomposition of 3\( K_5 \) that has a nesting defined by
\( (Z_5, \{S_1^i = [4+i;i,1+i,3+i],[S_2^i = [1+i;3+i,4+i,2+i,i][i \in Z_5]\}) \) and \( F(Q_1^i) = S_1^i, \rho = 1,2. \)

(2.3) \( (Z_7, \{Q^i = [5+i;1+i,6+i,i][i \in Z_7]\}) \) is a \( P_4 \)-decomposition of \( K_7 \) that has a nesting defined by \( (Z_7, \{S^i = [4+i;5+i,6+i,i][i \in Z_7]\}) \) and \( F(Q^i) = S^i. \)

(2.4) Let \( Q_1^i = [(i,1);(1+i,0),(2+i,0),(i,1)], Q_2^i = [(i,0);(1+i,1),(2+i,1),(3+i,1)], Q_3^i = [(i,1);(1+i,0),(1+i,1),(3+i,1)], Q_4^i = [(i,1);(1+i,2),(2+i,2),(3+i,2)], Q_5^i = [(i,2);(1+i,2),(3+i,2),(0,0)], Q_6^i = [(i,0);(1+i,2),(2+i,2),(3+i,2)], Q_7^i = [(i,2);(1+i,0),(2+i,1),(1+i,1)], S_1^i = [(3+i,0),(1+i,0),(2+i,0),(i,1)], S_2^i = [(2+i,0),(1+i,1),(2+i,1),(3+i,1)], S_3^i = [(4+i,0),(1+i,0),(1+i,1),(3+i,1)], S_4^i = [(1+i,1),(i,2),(1+i,2),(2+i,2)], S_5^i = [(4+i,2),(1+i,2),(3+i,2),(i,0)], S_6^i = [(1+i,0),(1+i,2),(2+i,2),(3+i,2)] \) and \( S_7^i = [(4+i,2),(1+i,0),(2+i,1),(1+i,1)] \), where \( i \) is in \( Z_5 \) and the sum is \((\text{mod} \ 5). \) Then \( (Z_5 \times Z_3, \{Q_1^i \in Z_5, \rho = 1,2,\ldots, 7\}) \) is a \( S_3 \)-decomposition of \( K_{15} \) that has a nesting defined by \( (Z_5 \times Z_3, \{S_1^i \in Z_5, \rho = 1,2,\ldots, 7\}) \) and \( F(Q_1^i) = S_1^i. \)

(2.5) Let \( Q_1^i = [i;\infty,1+i,3+i], Q_2^i = [i;\infty,1+i,2+i], Q_3^i = [i;\infty,3+i,2+i], Q_4^i = [i;1+i,2+i,4+i], S_1^i = [2+i;\infty,1+i,1+i], S_2^i = [3+i;\infty,1+i,1+i], S_3^i = [4+i;\infty,3+i,1+i] \) and \( S_4^i = [4+i;3+i,1+i,2+i], \) where \( i \) is in \( Z_7 \) and the sum is \( \text{(mod 7)}. \) Then \( (Z_7 \cup \{\infty\}, \{Q_1^i \in Z_7, \rho = 1,2,3,4\}) \) is a \( S_3 \)-decomposition of \( 3K_8 \) that has a nesting defined by \( (Z_7 \cup \{\infty\}, \{S_1^i \in Z_7, \rho = 1,2,3,4\}) \) and \( F(Q_1^i) = S_1^i. \)

(2.6) \( (Z_7 \cup \{\infty\}, \{Q_1^i = [\infty,2+i;1+i,4+i], Q_2^i = [i,1+i;5+i,3+i][i \in Z_7]\}) \) is an \( E_2 \)-decomposition of \( K_9 \) that has a nesting defined by \( (Z_7 \cup \{\infty\}, \{S_1^i = [i;\infty,1+i], S_2^i = [2+i;1+i,5+i][i \in Z_7]\}) \) and \( F(Q_1^i) = S_1^i, \rho = 1,2. \)

(2.7) \( (Z_5, \{Q_1^i = [i,1+i;2+i,4+i], Q_2^i = [2+i,4+i;1+i][i \in Z_5]\}) \) is an \( E_2 \)-decomposition of \( 2K_5 \) that has a nesting defined by \( (Z_5, \{S_1^i = [3+i,1+i,1+i], S_2^i = [3+i;2+i,4+i][i \in Z_5]\}) \) and \( F(Q_1^i) = S_1^i, \rho = 1,2. \)

To denote the graph \( \hat{G} \) we will use the following notation (the symbol \( \hat{x} \) means that \( x \) is a vertex of \( G \) missing on the vertex set of \( S_m \)):

(b1) For \( G = P_2 = [a_1, a_2] \) it is \( \hat{G} = \hat{P}_2 = < a_1, \hat{a}_2; a > = [a_1, a_2] \cup [a; a_1] \) (see fig. 1).
(b_2) For \( G = E_2 = [a_1, a_2; a_3, a_4] \) it is either
\[
\begin{align*}
\hat{G} \neq E_2 &=< a_1, \bar{a}_2; a_3, a_4; a > = [a_1, a_2; a_3, a_4] \cup [a; a_1, a_3] \\
\hat{G} \neq E_2 &=< a_1, a_2; \bar{a}_3, a_4; a > = [a_1, a_2; a_3, a_4] \cup [a; a_1, a_2] \quad \text{(see fig. 2)}.
\end{align*}
\]

\[
\begin{align*}
\hat{G} \neq E_2 &=< a_1, a_2; a_3, \bar{a}_4; a > = [a_1, a_2; a_3, a_4] \cup [a; a_1, a_2] \quad \text{(see fig. 2)}.
\end{align*}
\]

(b_3) For \( G = S_3 = [a; a_1, a_2, a_3] \) it is either
\[
\begin{align*}
\hat{G} \neq S_3 &=< a; a_1, a_2, a_3; c > = [a; a_1, a_2, a_3] \cup [c; a_1, a_2, a_3] \\
\hat{G} \neq S_3 &=< a; a_1, a_2, \bar{a}_3; c > = [a; a_1, a_2, a_3] \cup [c; a, a_1, a_2] \quad \text{(see fig. 3)}.
\end{align*}
\]

(b_4) For \( G = P_k = [a_1, a_2, \ldots, a_k] \) \((k = 3, 4)\), it is either
\[
\begin{align*}
\hat{G} \neq P_k &=< a_1, a_2, \ldots, \bar{a}_k; a > = [a_1, a_2, \ldots, a_k] \cup [a; a_1, a_2, \ldots, a_{k-1}] \\
\hat{G} \neq P_k &=< a_1, \bar{a}_2, \ldots, a_k; a > = [a_1, a_2, \ldots, a_k] \cup [a; a_1, a_3, \ldots, a_k] \quad \text{(see fig. 4)}.
\end{align*}
\]

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(b_5) For \( G = D = [a_1, a_2, a_3 \bowtie a_4] \), it is
\[
\hat{G} = \hat{D} = \langle a_1, a_2, a_3 \bowtie a_4; a \rangle = [a_1, a_2, a_3 \bowtie a_4] \cup [a; a_1, a_2, a_3, a_4] \] (see fig. 5).

(b_6) For \( G = C_4 = (a_1, a_2, a_3, a_4) \), it is
\[
\hat{G} = \hat{C}_4 = \langle a_1, a_2, a_3, a_4; a \rangle = (a_1, a_2, a_3, a_4) \cup [a; a_1, a_2, a_3, a_4] \] (see fig. 6).
The Authors studied in [13] the spectrum problem for $\overline{P}_k$-designs $N(v, 2)$. In the following theorem we state the known results for $k = 2, 3, 4$.

**Theorem 2** ([13]). For every $v \geq 3$, there is a $\overline{P}_2$-design $N(v, 2)$. For every $v \equiv 0$ or $1 \pmod{4}$, $v \geq 4$, there is a $\overline{P}_3$-design $N(v, 2)$. For every $v \equiv 0$ or $1 \pmod{3}$, $v \geq 5$ there is a $\overline{P}_4$-design $N(v, 2)$ except possibly if $v \in \{16, 39, 52, 70\}$.

**Remark 1.** We admit repeated blocks. So it will be sufficient for each $G$ and each $v$ to solve the spectrum problem only for the smallest positive $\lambda$ such that a $2\lambda K_v \rightarrow \hat{G}$ can exist.

Generally, two well-known methods are used in construction: the difference method (see e.g. [6]) and the composition method (see e.g. [21], [2] and [3]).

Usually, using the difference method, we will give only the base blocks of the decomposition as illustrated in the following examples.

**Example 3. (3.1)** The base block of the $\overline{D}$-design $N(9, 2)$ given in Example (2.1) is $<0, 3, 7 \equiv 8; 2 > \pmod{9}$.

(3.2) The base blocks of the $\overline{P}_4$-design $N(5, 6)$ given in Example (2.2) are $<0, 1, 3, \hat{2}; 4 >$ and $<3, 4, 2, \hat{0}; 1 > \pmod{5}$.

(3.3) The base block of the $\overline{P}_4$-design $N(7, 2)$ given in Example (2.3) is $<5, \hat{1}, 6, 0; 4 > \pmod{7}$.

(3.4) Put $V(K_{15}) = Z_5 \times Z_3$. The base blocks of the $\overline{S}_3$-design $N(15, 2)$ given in Example (2.4) are:

- $<(0, 0); (1, 0), (2, 0), (0, 1); (3, 0) >$, $<(0, 0); (1, 1), (2, 1), (3, 1); (2, 0) >$,
- $<(0, 1); (1, 0), (1, 1), (3, 1); (4, 1) >$, $<(0, 1); (0, 2), (1, 2), (2, 2); (1, 1) >$,
- $<(0, 2); (1, 2), (3, 2), (0, 0); (4, 2) >$, $<(0, 0); (1, 2), (2, 2), (3, 2); (1, 0) >$ and
- $<(0, 2); (1, 0), (2, 1), (1, 1); (4, 2) > \pmod{(5, -)}$. 

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(3.5) Put $V(K_8) = Z_7 \cup \{\infty\}$. The base blocks of the $\hat{S}_3$-design $N(8,6)$ given in Example (2.5) are:

- $< 0; \infty, 1, 3; 2 >$
- $< 0; \infty, 1, 3, \hat{2}; 3 >$
- $< 0; \infty, 3, \hat{2}; 4 >$

and $< 0; 1, 2, \hat{4}; 3 > \pmod{7}$.

(3.6) Put $V(K_8) = Z_7 \cup \{\infty\}$. The base blocks of the $\hat{E}_2$-design $N(8,2)$ given in Example (2.6) are:

- $< \infty, \hat{2}, 1, 4; 0 >$ and $< 0, 1; 5, 3; 2 > \pmod{7}$.

(3.7) The base blocks of the $\hat{E}_2$-design $N(5,4)$ given in Example (2.7) are:

- $< 0, 1; \hat{2}, 4; 3 >$ and $< 2, 4; \hat{0}, \hat{1}; 3 > \pmod{5}$.

Let $Y$ be a finite set of points, $C$ a family of distinct subsets of $Y$ called groups which partition $Y$, and $A$ a collection of subsets of $Y$ called blocks. Let $v$ and $\lambda$ be positive integers and $K$ and $M$ sets of positive integers. The triple $(Y, C, A)$ is called a group divisible design (GDD) $GD[K; \lambda, M; v]$ if:

- $(c_1)$ $|Y| = v$;
- $(c_2)$ $\{|C| | C \in C\} \subseteq M$;
- $(c_3)$ $\{|B| | B \in A\} \subseteq K$;
- $(c_4)$ $|C \cap B| \leq 1$ for every $C \in C$ and every $B \in A$;
- $(c_5)$ every pairset $\{x, y\} \subseteq Y$ such that $x$ and $y$ belong to distinct groups is contained in exactly $\lambda$ blocks of $A$.

If $C$ contains $t_i$ groups of size $m_{i}$, for $i = 1, 2, \ldots, s$, we call $m_1t_1m_2t_2\ldots m_st_s$ the group type of the GDD. When $K = \{k\}$ we will write $GD[k; \lambda, M; v]$ instead of $GD[\{k\}; \lambda, M; v]$.

A $GD[K; \lambda, \{1\}; v]$ with group type $1^v$ is called a pairwise balanced design, denoted by $(Y, A)$ or $(v, K, \lambda)$-PBD. A $(v, k, \lambda)$-PBD is simply a $K_k$-design.

A $GD[k, 1, \{m\}; km]$ is called a transversal design, denoted by $TD[k, m]$.

Let $2\lambda K_{n_1, n_2, \ldots, n_h}$ be the complete multipartite multigraph on vertices $\cup_{i=1}^{h}X_i$, $|X_i| = n_i$, with exactly $2\lambda$ edges joining each pair of vertices from different sets $X_i, X_j, i \neq j$. The composition method is based on the following lemmas.

**Lemma 2** Suppose there exist a $\hat{G}$-design $N(w + n_i, 2\lambda)$ containing a subdesign $N(w, 2\lambda)$ (it could be $w = 0, 1$), $i = 1, 2, \ldots, h$, and a $2\lambda K_{n_1, n_2, \ldots, n_h} \rightarrow \hat{G}$. Then there exists a $\hat{G}$-design $N(w + n_1 + n_2 + \ldots + n_h, 2\lambda)$.

**Lemma 3** [2, 3]. If $2K_{n,n,n} \rightarrow \hat{G}$ then $2K_{pn, pn, pn} \rightarrow \hat{G}$ for every positive integer $p$.

**Lemma 4** [3]. If $2K_{n,n,n} \rightarrow \hat{G}$ then $2K_{pn, pn, pn} \rightarrow \hat{G}$ for every positive integer $p \neq 2, 6$.

**Lemma 5** [3]. If $2K_{n,n,n} \rightarrow \hat{G}$ and $2K_{n,n,n} \rightarrow \hat{G}$, then $2K_{pn, pn, pn} \rightarrow \hat{G}$ for $p \neq 2, 6, 0 \leq q \leq p$.

**Lemma 6** [3, 18]. If $2K_{n,n,n,n,n} \rightarrow \hat{G}$ then $2K_{pn, pn, pn, pn, pn} \rightarrow \hat{G}$ for every positive integer $p \neq 2, 3, 6, 10$. 8
Lemma 7 Let \((Y, A)\) be a \((v, K, \lambda_1)\)-PBD. If there exists a \(2\lambda_2 K_n \rightarrow \tilde{G}\) for every \(n \in K\), then there exists a \(2\lambda_1 \lambda_2 K_v \rightarrow \tilde{G}\).

For a very complete survey about the existence of \((v, K, 1)\)-PBD it is possible to see [1]. In the next lemma we report only the results we need in our proofs.

Lemma 8 [1]. Let \(K, V, A,\) and \(B\) be the sets defined in Table 1. Then for every \(v \in V\) and \(v \notin A \cup B\) there exists \((v, K, 1)\)-PBD. Note that the values in \(A\) are genuine exceptions whereas the values in \(B\) are possible exceptions.

Table 1. Let \(m = \text{min}(K)\). In the sets \(A\) and \(B\), nonnegative integers less than \(m\) are omitted, since \(0\) and \(1\) are always present and the remaining integers are always absent.

(1.1) \(K = \{4, 5, 6, 7\}, V = \{v \in N | v \geq 4\}, A = \{8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}, B = \emptyset\).

(1.2) \(K = \{5, 6, 7, 8, 9\}, V = \{v \in N | v \geq 5\}, A = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34\}, B = \emptyset\).

(1.3) \(K = \{5, 7, 8, 9\}, V = \{v \in N | v \geq 5\}, A = \{6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 38, 39\}, B = \{42, 43, 44, 46, 51, 52, 60, 94, 95, 96, 98, 99, 100, 102, 104, 106, 107, 108, 110, 111, 116, 138, 139, 140, 142, 143, 146, 150, 154, 156, 158, 162, 163, 166, 167, 170, 172, 173, 174, 206, 228, 243\}.

(1.4) \(K = \{5, 8, 9\}, V = \{v \in N | v \equiv 0 \mod 4\}, A = \{12, 13, 16, 17, 20, 24, 28, 29, 32, 33, 44\}, B = \{52, 60, 68, 84, 92, 96, 100, 104, 108, 112, 113, 116, 124, 132, 140, 156, 172, 173, 192, 204, 212, 228, 244, 252, 268, 272, 300, 308, 312\}.

(1.5) \(K = \{5, 9, 13\}, V = \{v \in N | v \equiv 1 \mod 4\}, A = \{17, 29, 33\}, B = \emptyset\).

Lemma 9 (truncation of groups of a transversal design [8]). Let \(k\) be an integer, \(k \geq 2\). Let \(K = \{k, k + 1, \ldots, k + s\}\). Suppose that there exists a \(TD[k + s, m]\). Let \(g_1, g_2, \ldots, g_s\) be integers satisfying \(0 \leq g_i \leq m\), \(i = 1, 2, \ldots, s\). Then there exists a GDD of type \(m^k g_1 g_2 \ldots g_s\) with block sizes in \(K\).

Lemma 10 Suppose there exists a \(GD[t, 1, M; v]\), a \(2\lambda K_{n_1, n_2, \ldots, n_t} \rightarrow \tilde{G}\) (with \(n_1 = n_2 = \ldots = n_t = n\)) and for any \(m \in M\) a \(\tilde{G}\)-design \(N(mn + w, 2\lambda)\) containing a subdesign \(N(w, 2\lambda)\) (it could be \(w = 0, 1\)). Then there exists a \(\tilde{G}\)-design \(N(nv + w, 2\lambda)\).

Example 4. Let \(G = D\). Let \(X_i = \{i, 5 + i\} \in Z_5\), and \(V(K_{2,2,2,2,2}) = \cup_{i=0}^{1} X_i\). For a \(2K_{2,2,2,2,2} \rightarrow \tilde{G}\) take the base block \(<0, 4, 3 \equiv 5; 1 >\) (mod 10).

Put \(w = 1\) in Lemma 10. Since there exists a \(GD[5, 1, \{4\}; 20]\) [8] and a \(N(9, 2)\) (take \(<0, 3, 7 \equiv 8; 2 >\) as base block), then Lemma 10 implies the existence of a \(N(41, 2)\).

Lemma 11 Suppose there exists a \((v, t, \lambda_1)\)-PBD, a \(2\lambda_2 K_{n_1, n_2, \ldots, n_t} \rightarrow \tilde{G}\) (with \(n_1 = n_2 = \ldots = n_t = n\)) and a \(\tilde{G}\)-design \(N((t - 1)n + w, 2\lambda_2)\) containing a subdesign \(N(w, 2\lambda_2)\) (it could be \(w = 0, 1\)). Then there exists a \(\tilde{G}\)-design \(N(nv + w, 2\lambda_1 \lambda_2)\).
Lemma 12 Suppose there exists: a $\tilde{G}$-design (with $G \in \{P_3, P_4, C_4\}$) $N(v, 2\lambda)$; a $\tilde{G}$-design $N(w, 2\lambda)$ containing a subdesign $N(q, 2\lambda)$ (it could be $q = 0, 1$). Then there is a $\tilde{G}$-design $N(v(w - q) + q, 2\lambda)$.

Proof. We prove the lemma only for $G = P_4$. Similarly it is possible to prove the remaining cases. Let $(Z_w, B)$ be a $\tilde{P}_4$-design $N(v, 2\lambda)$. Let $(Z_{w q}, \cdot)$ be a quasigroup of order $w - q$. Let $T = \{ \infty_1, \infty_2, \ldots, \infty_q \}$ if $q > 0$ and $T = \emptyset$ if $q = 0$. Then define a $\tilde{P}_4$-design of order $v(w - q) + q$, $((Z_v \times Z_{w q}) \cup T, D)$ as follows:

$\tilde{d}_1$ For every $< a, b, c, d; x > \in B$ put in $D$ the following blocks

$< (a, i), (b, j), (c, i) \cdot (d, j); (x, i \cdot j) >$ for every $i, j \in Z_{w q}$.

$\tilde{d}_2$ For every $a \in Z_v$, put in $D$ the blocks of a $N(w, 2\lambda)$ on the point set $\{a\} \times Z_{w q}) \cup T$ containing a subdesign $N(q, 2\lambda)$ on the point set $T$.

Lemma 13 Suppose there exist: a $\tilde{D}$-design $N(v, 2\lambda)$; a $\tilde{D}$-design $N(w, 2\lambda)$ containing a subdesign $N(q, 2\lambda)$ (or $q = 0, 1$); two orthogonal quasigroups of order $w - q$. Then there is a $\tilde{D}$-design $N(v(w - q) + q, 2\lambda)$.

Proof. Let $(Z_w, B)$ be a $\tilde{D}$-design $N(v, 2\lambda)$. Let $(Z_{w q}, \cdot)$ and $(Z_{w q}, \circ)$ be two orthogonal quasigroups of order $w - q$ (it is well-known [20] that these quasigroups exist for every $w - q \neq 2, 6$). Let $T = \{ \infty_1, \infty_2, \ldots, \infty_q \}$ if $q > 0$ and $T = \emptyset$ if $q = 0$. Then define a $\tilde{D}$-design of order $v(w - q) + q$, $((Z_v \times Z_{w q}) \cup T, D)$ as follows:

$\tilde{d}_1$ For every $< a, b, c, d; x > \in B$ put in $D$ the blocks

$< (a, i), (b, j), (c, i) \cdot (d, i); (x, i \circ j) >$, $i, j \in Z_{w q}$.

$\tilde{d}_2$ For every $a \in Z_v$, put in $D$ the blocks of a $N(w, 2\lambda)$ on the point set $\{a\} \times Z_{w q}) \cup T$ containing a subdesign $N(q, 2\lambda)$ on the point set $T$.

We complete this section by collecting some results for small values of $v$ given in [14].

Theorem 3 ([14]) The following nested $G$-design are determined:

$\tilde{P}_3$-design $N(16, 2)$;
$\tilde{P}_3$-design $N(v, 4)$ for $v = 6, 7, 10, 11, 14, 15, 18, 19, 23$;
$\tilde{P}_3$-design $N(v, 6)$ for $v = 5, 8, 11, 14, 17, 20, 23, 32$;
$\tilde{P}_3$-design $N(v, 2)$ for $v = 9, 10, 15, 22, 34$;
$\tilde{P}_3$-design $N(v, 6)$ for $v = 5, 8, 11, 14, 16, 17, 20, 23, 32$;
$\tilde{S}_3$-design $N(v, 2)$ for $v = 9, 10, 22$;
$\tilde{S}_3$-design $N(v, 2)$ for $v = 9, 10, 15, 22$;
$\tilde{S}_3$-design $N(v, 6)$ for $v = 5, 8, 11, 14, 17, 20, 23, 29, 32$;
$C_4$-design $N(v, 2)$ for $v = 65, 97, 113$;
$C_4$-design $N(v, 4)$ for $v = 5, 13, 21, 29$;
$C_4$-design $N(v, 16)$ for $v = 12, 24, 2^n, n \geq 3$;
$\tilde{D}$-design $N(v, 2)$ for $v = 8, 16, 17, 24, 25, 32, 33, 56$;
$\tilde{D}$-design $N(v, 4)$ for $v = 5, 12, 13, 28, 29, 52, 84$. 
The following nested decompositions are determined:

\[ 2K_{13,13,13} \rightarrow \bar{P}_4; \]
\[ 2K_{13,13,13,13} \rightarrow \bar{P}_4; \]
\[ 2K_{v,v,v} \rightarrow \bar{P}_4, \text{ for } v = 3, 7, 13; \]
\[ 2K_{2,2,2,2} \rightarrow \bar{P}_4; \]
\[ 2K_{v,v,v} \rightarrow \tilde{S}_3 \text{ and } 2K_{v,v,v} \rightarrow \tilde{S}_3, \text{ for } v = 3, 5, 7, 13; \]
\[ 2K_{v,v,v,v} \rightarrow \tilde{S}_3 \text{ and } 2K_{v,v,v,v} \rightarrow \tilde{S}_3, \text{ for } v = 3, 10. \]

3 Nesting of G-designs for \( G = P_2, P_3, E_2, P_4 \) and \( S_3 \)

In this section we deal with the problem of constructing a nested G-design of order \( v \) for all the graphs \( G \) having four or less nonisolated vertices and at most three edges.

The spectrum of \( \bar{P}_2 \)-designs \( N(v, 2\lambda) \) is an immediate consequence of Lemma 1, Theorem 2 and Remark 1.

**Theorem 4** The necessary and sufficient condition for the existence of a \( \bar{P}_2 \)-design \( N(v, 2\lambda) \) is that \( v \geq 3 \).

**Theorem 5** The necessary and sufficient condition for the existence of a \( \bar{P}_3 \)-design \( N(v, 2\lambda) \) is that:

1. \( v \equiv 0 \text{ or } 1 \pmod{4}, v \geq 4, \text{ if } \lambda \equiv 1 \pmod{2}. \)
2. \( v \geq 4, \text{ if } \lambda \equiv 0 \pmod{2}. \)

**Proof.** The necessity follows from Lemma 1. Theorem 2 and Remark 1 get the sufficiency for odd \( \lambda \). By composition method, Theorems 2 and 3, Lemmas 7 and 8 (Table (1.1)) we obtain the sufficiency for \( \lambda = 2 \). Remark 1 completes the proof.

**Theorem 6** The necessary and sufficient condition for the existence of a \( \bar{P}_3 \)-design \( N(v, 2\lambda) \) is that:

1. \( v \equiv 0 \text{ or } 1 \pmod{4}, v \geq 4, \text{ if } \lambda \equiv 1 \pmod{2}. \)
2. \( v \geq 4, \text{ if } \lambda \equiv 0 \pmod{2}. \)

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for \( \lambda = 1 \) if \( v \equiv 0 \text{ or } 1 \pmod{4} \) and for \( \lambda = 2 \) if \( v \equiv 2 \text{ or } 3 \pmod{4} \).

Let \( v = 4h, h \geq 1 \), and let \( V(K_v) = Z_{v-1} \cup \{\infty\} \). The base blocks \( \pmod{4h-1} \) of a \( \bar{P}_3 \)-design \( N(4h, 2) \), are: \( < \infty, 0, 1; 2 > \), and for \( h \geq 2, < 2\rho + 2, 0, 2\rho + 3; 4\rho + 5 > \rho = 0, 1, \ldots, h - 2. \)

Let \( v = 1 + 4h, h \geq 1 \), and let \( V(K_v) = Z_v \). The base blocks \( \pmod{1 + 4h} \) of a \( \bar{P}_3 \)-design \( N(4h + 1, 2) \), are: \( < 2\rho + 1, 0, 2\rho + 2; 4\rho + 3 > \rho = 0, 1, \ldots, h - 1. \)

Let \( v = 4h+2, h \geq 1 \), and let \( V(K_v) = Z_{v-1} \cup \{\infty\} \). The base blocks \( \pmod{1 + 4h} \) of a \( \bar{P}_3 \)-design \( N(4h + 2, 4) \), are: \( < \infty, 0, 1; 2 >, < \infty, 0, 2; 4 >, < 1, 0, 2; 3 >, \) and for \( h \geq 2 \), two copies of \( < 2\rho + 3, 0, 2\rho + 4; 4\rho + 7 >, \rho = 0, 1, \ldots, h - 2. \)
Let \( v = 4h + 3 \), \( h \geq 1 \), and let \( V(K_v) = Z_v \). The base blocks \((\text{mod } 3 + 4h)\) of a \( \widehat{P}_4\)-design \( N(4h + 3, 4) \), are: \( < 1, 0, 2; 3 > \), \( < 1, 0, 2; 3 > \), \( < 1, 0, 2; 5 > \), and for \( h \geq 2 \), two copies of \( < 2\rho + 4, \bar{0}, 2\rho + 5; 4\rho + 9 > \), \( \rho = 0, 1, \ldots, h - 2 \). \( \Box \)

**Theorem 7** The necessary and sufficient condition for the existence of a \( \widehat{E}_2\)-design \( N(v, 2\lambda) \) is that:

1. \( v \equiv 0 \) or \( 1 \) \((\text{mod } 4)\), \( v \geq 5\), if \( \lambda \equiv 1 \) \((\text{mod } 2)\).
2. \( v \geq 5\), if \( \lambda \equiv 0 \) \((\text{mod } 2)\).

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for \( \lambda = 1 \) if \( v \equiv 0 \) or \( 1 \) \((\text{mod } 4)\) and for \( \lambda = 2 \) if \( v \equiv 2 \) or \( 3 \) \((\text{mod } 4)\).

Let \( v = 4h, h \geq 2 \). Let \( V(K_v) = Z_{v-1} \cup \{\infty\} \). The base blocks \((\text{mod } 4h - 1)\) are: \( < \infty, \frac{2}{3}; 1, \frac{2}{3}; 0 > , < 0, 2\rho - 1; 4\rho + 1, 2\rho + 1 >, \rho = 1, 2, \ldots, h - 1 >. \)

Let \( v = 1 + 4h, h \geq 1 \). Let \( V(K_v) = Z_v \). The base blocks \((\text{mod } 1 + 4h)\) are:

\[
< 0, 2\rho; 4\rho - 4, 2\rho - 2; 2\rho - 1 >, \rho = 1, 2, \ldots, h.
\]

Let \( v = 2 + 4h, h \geq 1 \). Take \( V(K_v) = Z_{v-1} \cup \{\infty\} \) and base blocks \((\text{mod } 1 + 4h)\):

\[
< \infty, \frac{2}{3}; 1, \frac{2}{3}; 0 > , < 0, 2\rho - 1; 4\rho + 1, 2\rho + 1 >, \rho = 1, 2, \ldots, h - 1 >. \]

Let \( v = 3 + 4h, h \geq 1 \). Take \( V(K_v) = Z_v \) and base blocks \((\text{mod } 3 + 4h)\):

\[
< 0, 4h + 2 + 2, 3; 1 >, \text{two copies of } < 0, 2\rho - 1; 4h + 1, 2\rho - 1 >, \rho = 1, 2, \ldots, h - 1 >. \]

\( \Box \)

**Theorem 8** The necessary condition for the existence of a \( \widehat{E}_2\)-design \( N(v, 2\lambda) \) is that:

1. \( v \equiv 0 \) or \( 1 \) \((\text{mod } 4)\), \( v \geq 5\), if \( \lambda \equiv 1 \) \((\text{mod } 2)\).
2. \( v \geq 5\), if \( \lambda \equiv 0 \) \((\text{mod } 2)\).

This necessary condition is also sufficient except possibly for \( v = 5 \) if \( \lambda \equiv 1 \) \((\text{mod } 2)\).

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for \( \lambda = 1 \) if \( v \equiv 0 \) or \( 1 \) \((\text{mod } 4)\) and for \( \lambda = 2 \) if \( v \equiv 2 \) or \( 3 \) \((\text{mod } 4)\).

Let \( v = 4h, h \geq 2 \). Take \( V(K_v) = Z_{v-1} \cup \{\infty\} \) and the following base blocks \((\text{mod } 4h - 1)\):

\[
< \infty, 2, 1, \frac{2}{3}; 3 > , < 0, 2h - 3; 2, \frac{2}{3}; 2h - 1 >, \text{and for } h \geq 3, < 0, 2\rho - 1; 1, 2\rho + 1 >, \rho = 1, 2, \ldots, h - 2 >. \]

Let \( v = 1 + 4h, h \geq 2 \). Take \( V(K_v) = Z_v \) and base blocks \((\text{mod } 1 + 4h)\):

\[
\text{If } h \neq 1 \text{ (mod 3) and } h \neq 3 \text{ (mod 5), then the base blocks (mod } 1 + 4h \text{) are:}
\]

\[
< 4\rho - 1, 6\rho - 2; 0, 2\rho; 5\rho - h - 2 >, \rho = 1, 2, \ldots, h. \]

\[
\text{If } h = 1 + 3\alpha, \alpha \geq 1, \text{ then the base blocks (mod } 1 + 4h \text{) are:}
\]

\[
< 4\rho - 1, 6\rho - 2; 0, 2\rho; 5\rho - h - 2 >, \text{ for } \rho \in \{1, 2, \ldots, h\} - \{1 + \alpha\}, \text{ and }
\]

\[
< 3 + 4\alpha, 4 + 6\alpha; 0, 2 + 2\alpha; 5 + 8\alpha >.
\]

\[
\text{If } h = 3 + 5\alpha, \alpha \geq 1, \text{ then the base blocks (mod } 1 + 4h \text{) are:}
\]

\[
< 4\rho - 1, 6\rho - 2; 0, 2\rho; 5\rho - h - 2 >, \text{ for } \rho \in \{1, 2, \ldots, h\} - \{1 + \alpha\}, \text{ and }
\]

\[
< 3 + 4\alpha, 4 + 6\alpha; 0, 2 + 2\alpha; 7 + 10\alpha >.
\]
Let \( v = 2 + 4h, h \geq 2 \). Take \( V(K_v) = Z_{v-1} \cup \{\infty\} \) and base blocks \( \pmod{1+4h} \):

\[
< \infty, 0; 1 \pmod{2}; 2h >, < \infty, 0; \bar{1} \pmod{2}; 2h-1 >, < 0; 2; 1 \pmod{3}; 1+2h >, \quad \text{and 2 copies of} \quad < 0, 2; 1, 3; 1+2p; 1+p > \quad \text{for odd} \quad \rho \in \{1, 2, \ldots, h-1\}, < 1, 3+2p; 0, 1+2p; \rho + 2h + 2 >, \quad \text{for even} \quad \rho \in \{1, 2, \ldots, h-1\}.
\]

Let \( v = 3 + 4h, h \geq 1 \). Take \( V(K_v) = Z_v \) and base blocks \( \pmod{3+4h} \):

\[
< 1, 3+4h; 2, 3; 2+2h >, \quad \text{and 2 copies of} \quad < 1, 2+2p; 0, 2p; 2+p > \quad \text{for odd} \quad \rho \in \{1, 2, \ldots, h\}, < 0, 2p; 1, 2+2p; \rho + 2h + 2 >, \quad \text{for even} \quad \rho \in \{1, 2, \ldots, h\}.
\]

To complete our proof note that a \( \overline{E}_2 \)-design \( N(5, 4) \) is given by Example (3.7) and a \( \overline{E}_2 \)-design \( N(6, 4) \) is the following: \( V(K_6) = Z_5 \cup \{\infty\} \) and the base blocks \( \pmod{5} \) are \( < \infty, 0; \bar{1} \pmod{2}; 3; 0 >, < \infty, 0; \bar{1} \pmod{2}; 4 >, < 0, 2; \bar{1} \pmod{3}; 3; 4 >. \) \( \square \)

**Remark 2.** It is easy to verify that there is not a \( \overline{E}_2 \)-design \( N(5, 2\lambda) \) for \( \lambda = 1, 3 \).

But we are unable to prove the nonexistence of these designs for every odd \( \lambda \).

**Theorem 9** The necessary and sufficient condition for the existence of a \( \overline{P}_4 \)-design \( N(v, 2\lambda) \) is that:

1. \( v \equiv 0 \) or 1 \( \pmod{3} \), \( v \geq 6 \), if \( \lambda \equiv 1 \) or 2 \( \pmod{3} \).
2. \( v \geq 5 \), if \( \lambda \equiv 0 \) \( \pmod{3} \).

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for \( \lambda = 1 \) if \( v \equiv 0 \) or 1 \( \pmod{3} \) and for \( \lambda = 3 \) if \( v \equiv 2 \) \( \pmod{3} \).

Let \( v \equiv 0 \) or 1 \( \pmod{3} \), then the sufficiency follows from Theorem 2 except possibly for \( v = 16, 39, 52, 70 \). By Theorem 3 there is a \( \overline{P}_4 \)-design \( N(16, 2) \). A \( \overline{P}_4 \)-design \( N(v, 2) \) for \( v = 39, 52 \) is given by Lemma 2 where we put \( w = 0 \), \( n_i = 13 \) and \( h = 3, 4 \) respectively (a \( \overline{P}_4 \)-design \( N(13, 2) \) there is by Theorem 2 and the decompositions \( 2K_{13,13,13} \rightarrow P_4 \) and \( 2K_{13,13,13,13} \rightarrow P_4 \) by Theorem 3). Lemma 12 with \( v = 10, w = 7 \) and \( q = 0 \) implies the existence of a \( \overline{P}_4 \)-design \( N(70, 2) \).

Let \( v \equiv 2 \) \( \pmod{3} \). The existence of a \( \overline{P}_4 \)-design \( N(29, 6) \) follows from Lemma 7 since there is a decomposition \( 3K_{29} \rightarrow K_7 \) ([12]) and a \( \overline{P}_4 \)-design \( N(7, 2) \) (Theorem 2). The remaining cases follow from Theorem 3, the composition method, Lemma 7 and Lemma 8 (Table (1,2)). \( \square \)

**Theorem 10** The necessary condition for the existence of a \( \overline{P}_4 \)-design \( N(v, 2\lambda) \) is that:

1. \( v \equiv 0 \) or 1 \( \pmod{3} \), \( v \geq 5 \), if \( \lambda \equiv 1 \) or 2 \( \pmod{3} \).
2. \( v \geq 5 \), if \( \lambda \equiv 0 \) \( \pmod{3} \).

This necessary condition is also sufficient except possibly for \( v = 52 \) if \( \lambda \equiv 1 \) or 2 \( \pmod{3} \).

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for \( \lambda = 1 \) if \( v \equiv 0 \) or 1 \( \pmod{3} \) and for \( \lambda = 3 \) if \( v \equiv 2 \) \( \pmod{3} \).

Suppose at first \( v \equiv 0 \) or 1 \( \pmod{3} \). Let \( v = 6h, h \geq 1 \). Take \( V(K_v) = Z_{v-1} \cup \{\infty\} \) and base blocks \( \pmod{6h-1} \):

\[
< 1, 6h-2, 0; \infty; 3h-1 >, \quad \text{and if}
\]

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\[ h \geq 2, < 6h - 2 - 3\rho, 1, 6h - 1 - 3\rho, 0; 6h - 2 - \frac{3(\rho - 1)}{2} >, \text{for odd } \rho \in \{1, 2, \ldots, h - 1\}, \]
\[ < 6h - 2 - 3\rho, 1, 6h - 1 - 3\rho, 0; 3h - 3 \rho >, \text{for even } \rho \in \{1, 2, \ldots, h - 1\}. \]

Let \( v = 1 + 6h, h \geq 1. \) Take \( V(K_v) = Z_v \) and base blocks \( \pmod{6h + 1}: \)
\[ < 6h - 1 - 3\rho, 1, 6h - 3\rho, 0; 4h - \rho >, \rho = 0, 1, \ldots, h - 1. \]

Let \( v \equiv 3 \) or \( 4 \pmod{6}. \) A \( \overrightarrow{P}_4 \)-design \( N(21, 2) \) is obtained by Lemma 2 with \( w = 0, n_i = 7 \) and \( h = 3 \) (a \( 2K_7 \to \overrightarrow{P}_4 \) is given in Example (3.3) and a \( 2K_{7,7,7} \to \overrightarrow{P}_4 \) there is by Theorem 3). A \( \overrightarrow{P}_4 \)-design \( N(33, 2) \) can be constructed by Lemma 10 with \( t = 4, v = 16, n = 2, w = 1 \) and \( m = 4 \) (it is well-known that there is a \( GD[4, 1, \{4\}; 16] \), and a \( 2K_{22,2,2} \to \overrightarrow{P}_4 \) there is by Theorem 3). A \( \overrightarrow{P}_4 \)-design \( N(39, 2) \) is obtained by Lemma 2 with \( w = 0, n_i = 13 \) and \( h = 3 \) (a \( 2K_{13,13,13} \to \overrightarrow{P}_4 \) there is by Theorem 3). The existence of a \( \overrightarrow{P}_4 \)-design \( N(40, 2) \) follows from Lemma 4 with \( n = 2 \) and \( p = 5 \) and Lemma 2 with \( w = 0, n_i = 10 \) and \( h = 4. \)

The sufficiency for the remaining values of \( v \geq 6, v \equiv 3 \) or \( 4 \pmod{6}, v \neq 52, \)
follow from Theorem 3, Lemma 10 (with \( w = 0 \) if \( v \equiv 3 \pmod{6} \) and \( w = 1 \) if \( v \equiv 4 \pmod{6} \)), and the existence of a \( 2K_{33,3} \to \overrightarrow{P}_4, \) and the following GDDs ([5]):
\[ GD[3, 1, \{3, 5\}; 23], GD[3, 1, \{3\}; 3 + 6\alpha] \] for any \( \alpha \geq 1, \)
\[ GD[3, 11, \{3\}; 11 + 6\alpha] \] for any \( \alpha \geq 1. \)

Now suppose \( v \equiv 2 \pmod{3}, v \geq 3. \) For \( v = 29 \) the proof follows from Lemma 7 and the existence of a decomposition \( 3K_{29} \to K_7 ([12]) \) and the \( \overrightarrow{P}_4 \)-designs \( N(7, 2) \) above constructed.

The composition method, Lemma 7, Lemma 8 (Table (1.2)) and Theorem 3 complete the proof. \( \square \)

**Theorem 11** The necessary and sufficient condition for the existence of a \( \overrightarrow{S}_3 \)-design \( N(v, 2\lambda) \) is that:

(1) \( v \equiv 0 \) or \( 1 \pmod{3}, v \geq 6, \text{if } \lambda \equiv 1 \) or \( 2 \pmod{3}. \)

(2) \( v \geq 5, \text{if } \lambda \equiv 0 \pmod{3}. \)

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for \( \lambda = 1 \) if \( v \equiv 0 \) or \( 1 \pmod{3} \) and for \( \lambda = 3 \) if \( v \equiv 2 \pmod{3}. \)

Suppose at first \( v \equiv 0 \) or \( 1 \pmod{6}. \) If \( v = 6h, h \geq 1, \) then take \( V(K_v) = Z_v \cup \{\infty\} \) and base blocks, \( \pmod{6h - 1}: < 0; \infty, 1, 2, 3 > \) and, if \( h \geq 2, < 0; 3\rho + 3, 3\rho + 4, 3\rho + 5; 6\rho + 8 >, \rho = 0, 1, \ldots, h - 2. \)

If \( v = 1 + 6h, h \geq 1, \) then take \( V(K_v) = Z_v \) and base blocks, \( \pmod{1 + 6h}, < 0; 3\rho + 1, 3\rho + 2, 3\rho + 3; 6\rho + 4 >, \rho = 0, 1, \ldots, h - 1. \)

Let now \( v \equiv 3 \) or \( 4 \pmod{6}. \) The sufficiency for \( v = 9, 10, 22 \) is given in Theorem 3. For \( v = 15 \) see the Example (3.4). It is easy to apply Lemmas 2, 3, 4 and Theorem 3 to prove the sufficiency for \( v = 16, 21, 27, 28, 39, 40, 63. \) As an example we prove the case \( v = 16 \) leaving the remaining ones to the reader: Lemma 3 (with \( n = 5, p = 3 \) and the \( 2K_{5,5,5} \to \overrightarrow{S}_3 \) given in Theorem 3) gets a \( 2K_{15,15,15} \to \overrightarrow{S}_3. \) Then Lemma 2 with \( w = 1, h = 3 \) and \( n_i = 15 \) completes the proof.

Lemmas 3 and 5 imply a \( 2K_{3p,3p,3p} \to \overrightarrow{S}_3 \) (for any \( p \)) and a \( 2K_{3p,3p,3p,q} \to \overrightarrow{S}_3 \) (for \( p \neq 2, 6, 0 \leq q \leq p \)). Therefore by Lemma 2 we obtain that if for \( w = 0,1 \)
there is a $2K_{3p+w} \rightarrow \overline{S}_3$ (or a $2K_{3p+w} \rightarrow \overline{S}_3$ and a $2K_{3q+w} \rightarrow \overline{S}_3$) then there exists a $2K_{3p+w} \rightarrow \overline{S}_3$ (or a $2K_{3p+3q+w} \rightarrow \overline{S}_3$, $p \neq 2, 6$, respectively). By induction, starting with the $\overline{S}_3$-designs $N(v, 2)$ for $v = 16, 21, 27, 28, 39, 40, 63$, we complete the proof for $v \equiv 3$ or $4 \pmod{6}$. As an example we prove the sufficiency for $v = 51, 52$. Let $p = 5$ and $q = 2$, then there is a $2K_{51,15,15,6} \rightarrow \overline{S}_3$. A $2K_6 \rightarrow \overline{S}_3$ is given above and a $2K_{15} \rightarrow \overline{S}_3$ there is by Theorem 3. Then we obtain a $2K_{51} \rightarrow \overline{S}_3$ (for $w = 0$) and a $2K_{52} \rightarrow \overline{S}_3$ (for $w = 1$).

At last we prove the sufficiency for $v \equiv 2 \pmod{3}$.

Let $v = 5 + 6h$, $h \geq 0$. Take $V(K_v) = Z_v$ and base blocks (mod $5 + 6h$):

- $< 0; 1, 2 + 3h, 3 + 6h; 4 + 6h >$, $< 0; 1 + 3h, 2 + 3h, 4 + 6h; 3 + 6h >$, and, for $h \geq 1$,
- $< 0; 3\rho + 1, 3\rho + 2, 3\rho + 3; 6\rho + 4 >$, $< 0; 3\rho + 2, 3\rho + 3, 3\rho + 4; 6\rho + 6 >$
- $< 0; 3\rho + 3, 3\rho + 4, 3\rho + 5; 6\rho + 8 >$, $\rho = 0, 1, \ldots, h - 1$.

Let $v = 2 + 6h$, $h \geq 1$. Take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks (mod $1 + 6h$):

- $< 0; \infty, 1, 2; 3 >$, $< 0; \infty, 1, 3; 4 >$, $< 0; \infty, 2, 3; 5 >$, $< 0; 1, 2, 3; 4 >$ and, for $h \geq 2$,
- three copies of $< 0; 3\rho + 4, 3\rho + 5, 3\rho + 6; 6\rho + 10 >$, $\rho = 0, 1, \ldots, h - 2$.

\[ \square \]

**Theorem 12** The necessary and sufficient condition for the existence of a $\overline{S}_3$-design $N(v, 2\lambda)$ is that:

1. $v \equiv 0$ or $1 \pmod{3}$, $v \geq 6$, if $\lambda \equiv 1$ or $2 \pmod{3}$.
2. $v \geq 5$, if $\lambda \equiv 0 \pmod{3}$.

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1 \pmod{3}$ and for $\lambda = 3$ if $v \equiv 2 \pmod{3}$.

Suppose at first $v \equiv 0$ or $1 \pmod{6}$. Let $v = 6h$, $h \geq 1$. Take $V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks (mod $6h - 1$):

- $< 0; \infty, 2, 1; 3h >$ and, if $h \geq 2$,
- $< 0; 3\rho, 3\rho + 1, 3\rho + 2; \frac{3\rho - 1}{2} >$ for odd $\rho \in \{1, 2, \ldots, h - 1\}$,
- $< 0; 3\rho, 3\rho + 1, 3\rho + 2; 3h - 1 + \frac{3\rho}{2} >$ for even $\rho \in \{1, 2, \ldots, h - 1\}$.

Let $v = 1 + 6h$, $h \geq 1$. Take $V(K_v) = Z_v$ and base blocks (mod $1 + 6h$):

- $< 0; 2, 3, 1; 3h + 1 >$ and, if $h \geq 2$,
- $< 0; 3\rho + 1, 3\rho + 2, 3\rho + 3; \frac{3\rho + 1}{2} >$ for odd $\rho \in \{1, 2, \ldots, h - 1\}$,
- $< 0; 3\rho + 2, 3\rho + 3, 3\rho + 1; 3h + 1 + \frac{3\rho}{2} >$ for even $\rho \in \{1, 2, \ldots, h - 1\}$.

Let now $v \equiv 3$ or $4 \pmod{6}$. The sufficiency for $v = 9, 10, 15, 22$ follows from Theorem 3. It is easy to apply Lemmas 2, 3, 4 and Theorem 3 to prove the sufficiency for $v = 16, 21, 27, 28, 39, 40, 63$. As an example we prove the case $v = 16$ leaving the remaining ones to the reader: Lemma 3 (with $n = 5$, $p = 3$ and the $2K_{5,5,5} \rightarrow \overline{S}_3$ given in Theorem 3) gets a $2K_{15,15,15} \rightarrow \overline{S}_3$. Then Lemma 2 with $w = 1$, $h = 3$ and $n_1 = 15$ completes the proof.

To complete the proof of the sufficiency when $v \equiv 3$ or $4 \pmod{6}$ proceed as in the above Theorem 11.

By Lemma 8 (Table (1.2)) and Theorem 3 it follows the proof for $\lambda = 3$ and $v \equiv 2 \pmod{3}$, $v \geq 5$. \[ \square \]
4 Nesting of $G$-designs for $G = C_4$ and $D$

In this section we deal with the problem of constructing a nested $G$-design of order $v$ for all the graphs $G$ having four nonisolated vertex and four edges.

**Theorem 13** The necessary condition for the existence of a $2\lambda K_v \rightarrow \overline{C}_4$ is that:

1. $v \equiv 1 \pmod{8}$, $v \geq 9$, if $\lambda \equiv 1 \pmod{2}$;
2. $v \equiv 1 \pmod{4}$, $v \geq 5$, if $\lambda \equiv 2 \pmod{4}$;
3. $v \equiv 1 \pmod{2}$, $v \geq 5$, if $\lambda \equiv 4 \pmod{8}$;
4. $v \geq 5$, if $\lambda \equiv 0 \pmod{8}$.

**Proof.** There is not a nested $C_4$-design $(V, \mathcal{B})$ of order $v \equiv 0 \pmod{4}$ and index $\lambda \equiv 2 \pmod{4}$.

Suppose a such nested $C_4$-design existed. Let $(V, \mathcal{S})$ be the associated $S_4$-design. Put $\lambda = 4\rho + 2$ and $v = 4h$. Then the number of 4-cycles of $\mathcal{B}$ meeting the same vertex $a \in V$ is given by $(2\rho + 1)(4h - 1)$. Clearly $a$ appears as terminal vertex of $(2\rho + 1)(4h - 1)$ 4-stars of $\mathcal{S}$. Let $x$ be the number of 4-stars of $\mathcal{S}$ containing $a$ as a centre. Therefore it is $4x + (2\rho + 1)(4h - 1) = (4\rho + 2)(4h - 1)$. This equality is impossible.

Similarly it is possible to prove that there is not a nested $C_4$-design of order $v \equiv 0 \pmod{2}$ and index $\lambda \equiv 4 \pmod{8}$.

Lemma 1 completes the proof. $\square$

**Theorem 14** The necessary condition for the existence of a $\overline{C}_4$-design $N(v, 2\lambda)$ given in the above Theorem 13 is also sufficient except possibly if $v = 57, 185, 265$ and $\lambda \equiv 1 \pmod{2}$.

**Proof.** By Remark 1 it is enough to prove the sufficiency for the smallest values of $\lambda$.

Case (1). Let $v \equiv 1 \pmod{8}$, $v \geq 9$ and $\lambda = 1$. The proof follows from Theorem 1 and 3.

Case (2). Let $v \equiv 1 \pmod{4}$, $v \geq 5$ and $\lambda = 2$. By Case (1), Lemma 8 (Table (1.5)) and Theorem 3 we obtain the proof.

Case (3). Let $v \equiv 1 \pmod{2}$ and $\lambda = 4$. A $\overline{C}_4$-design $N(v, 8)$ is given by $V(K_v) = Z_v$ and base blocks $<x, 2x, v - x, v - 2x; 0>, x = 1, 2, \ldots, \frac{v-1}{2}$, $(\pmod{v})$.

Case (4). Let $v \geq 5$ and $\lambda = 8$.

We start by proving the sufficiency for $v = 6, 10, 14, 18, 22, 26, 30, 34$. Suppose it is possible to construct by the difference method a $\overline{C}_4$-design $N(w, 2\lambda)$ $(V, \mathcal{B})$ with $\lambda \in \{1, 2\}$. Let $\mathcal{B}^*$ be the set of base blocks of $\mathcal{B}$. Suppose $\infty \not\in V$ and $<x, y, z, t; u> \in \mathcal{B}^*$. If $\lambda = 1$ then the base blocks of a $\overline{C}_4$-design $N(w + 1, 8)$ are:

$<\infty, x, y, z; u>, <\infty, y, z, t; u>, <\infty, z, t, x; u>, <\infty, t, x, y; u>$,

$<x, y, z, t; \infty>, 9 - 4\lambda$ copies of $<x, y, z, t; u>$ and $12 - 4\lambda$ copies of each $b \in \mathcal{B}^*$, $b \neq <x, y, z, t; u>$.

Therefore it is sufficient to construct, by difference method, a $\overline{C}_4$-design $N(w, 2\lambda)$ with $w \in \{5, 9, 13, 17, 21, 25, 29, 33\}$ and either $\lambda = 1$ or $\lambda = 2$.

For $w \in \{5, 13, 29\}$ the existence of a such $\overline{C}_4$-design $N(w, 4)$ is proved in Case (2).
Base blocks of a $\tilde{C}_4$-design $N(w, 2)$ for $w \in \{9, 17, 25, 33\}$ are found in [16] and for $w = 21$ in Theorem 3.

The sufficiency for $v = 2^a, n \geq 3, v = 12$ and $v = 24$ is given by direct construction in Theorem 3.

Cases $v = 20, 28$ follow from Lemma 7, the existence of the decompositions ([12]) $4K_{20} \rightarrow K_5$, $2K_{28} \rightarrow K_7$ and that of the $\tilde{C}_4$-designs $N(5, 4)$ and $N(7, 8)$.

By above results, Lemma 7 and Lemma 8 (Table (1.2)) we obtain the proof. \(\square\)

**Theorem 15** The necessary condition for the existence of a $\tilde{D}$-design $N(v, 2\lambda)$ is that:

1. $v \equiv 0$ or $1$ (mod 8), $v \geq 8$, if $\lambda \equiv 1$ (mod 2).
2. $v \equiv 0$ or $1$ (mod 4), $v \geq 5$, if $\lambda \equiv 2$ (mod 4).
3. $v \geq 5$ if $\lambda \equiv 0$ (mod 4).

This necessary condition is also sufficient except possibly for $v = 124, 212$ if $\lambda \equiv 2$ (mod 4) and $v = 6$ if $\lambda \equiv 0$ (mod 4).

**Proof.** The necessity follows from Lemma 1. By Remark 1 it is enough to prove the sufficiency for $\lambda = 1$ if $v \equiv 0$ or $1$ (mod 8), for $\lambda = 2$ if $v \equiv 4$ or $5$ (mod 8) and for $\lambda = 4$ if $v \equiv 2$ or $3$ (mod 4).

Case (1). Let $v \equiv 0$ or $1$ (mod 8) and $\lambda = 1$.

A $\tilde{D}$-design $N(9, 2)$ is given in Example (3.1) and a $\tilde{D}$-design $N(v, 2)$ for $v = 8, 16, 24, 25, 32, 33, 56$ there is by Theorem 3.

The existence of a $\tilde{D}$-design $N(v, 2)$ for every admissible $v$ except possibly if $v = 57, 64, 65$ follows from Lemma 6 (a $2K_{2,2,2,2,2} \rightarrow \tilde{D}$ is given in Example 4), Lemma 10, and the existence of following GDDs ([7, 8, 19]): a $GD[5, 1, \{8\}; 48]$; a $GD[5, 1, \{4\}; v]$ for every $v \equiv 0$ or $4$ (mod 20); a $GD[5, 1, \{4, 8\}; v]$ for every $v \equiv 8$ or $16$ (mod 20), $v \geq 36$, except possibly if $v = 48$; a $GD[5, 1, \{4, 12\}; v]$ for every $v \equiv 12$ (mod 20), $v \geq 52$.

To prove the cases $v = 57, 64, 65$ use Lemma 13.

Case (2). Let $v \equiv 4$ or $5$ (mod 8), $v \geq 5 v \neq 124, 212$ and $\lambda = 2$.

By the above Case (1), Theorem 3, Lemma 7 and Lemma 8 (Table (1.4)), there is a $\tilde{D}$-design $N(v, 4)$ for every admissible $v \notin \{20, 44, 60, 68, 92, 100, 108, 116, 124, 132, 140, 156, 172, 173, 204, 212, 228, 244, 252, 268, 300, 308\}.

The cases $v = 20, 60, 68, 100, 108, 140, 228, 268, 300, 308$ follow from Lemma 11 and the existence of a $2K_v \rightarrow K_5$ for every $v \equiv 1$ or $5$ (mod 10), $v \neq 15$ [8].

The cases $v = 44, 173$ follow from Lemma 7 and the existence of a $(45, 2, 9)$-PBD [12] and a $(173, 1, \{5, 13\})$-PBD [7].

The cases $v = 92, 116, 156, 172, 204, 244$ follow from Lemma 13.

The cases $v = 132, 252$ follow from Lemma 10 and the existence of a $GD[5, 1, \{6\}; v]$ for $v = 66, 126, [19]$.

Case (3). Let $v \equiv 2$ or $3$ (mod 4), and $\lambda = 4$.

For $v = 7, 10, 11, 14, 15, 22, 23, 30, 34, 42$ a $\tilde{D}$-design $N(v, 8)$ there is by Theorem 3.

For any $v \in \{94, 95, 98, 99, 110, 138, 139, 142, 143, 146, 150, 154, 162, 163, 170, 172, 243\}$ there is a $(v, 1, \{8, 9, 10\})$-PBD [1]. Since for $v = 5, 7, 8, 9$ there is a $N(v, 8)$, then by Lemma 7 and Lemma 8 (Table (1.3)) there exists a $\tilde{D}$-design $N(v, 8)$ for each $v \equiv$
2 or 3 (mod 4) such that \( v \notin \{6, 18, 19, 26, 27, 31, 38, 39, 43, 46, 51, 102, 106, 107, 111, 158, 166, 167, 174, 206\} \).

By truncation of groups of the transversal design \( TD[17,16] \) (Lemma 9) construct a GDD of type \( 10^49^67^6 \). Then by Lemma 7 there is a \( \tilde{D} \)-design \( N(206,8) \). Similarly we can prove the theorem for \( v = 102, 107, 158, 167, 174, 206 \).

By Lemma 11 with \( \lambda_1 = 4 \) and \( \lambda_2 = 1 \) and the existence of a \((20, 5, 4)\)-PBD \[12\], we obtain the proof for \( v = 38, 39 \).

The cases \( v = 18, 19, 26, 27, 31, 43 \) follow from lemma 7 and the existence of a \((19, 9, 4)\)-PBD, a \((27, 9, 4)\)-PBD, \((31, 5, 2)\)-PBD and a \((43, 8, 4)\)-PBD.

To prove cases \( v = 46, 51, 106, 111, 166 \) use Lemma 13. \[\square\]

**Remark 3.** It is easy to verify that there is not a \( \tilde{D} \)-design \( N(6, 2\lambda) \) for \( \lambda = 4 \). But we are unable to prove the nonexistence of these designs for every \( \lambda \equiv 0 \pmod{4} \), \( \lambda \geq 8 \).

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**References**


