Exact embedding of two $G$-designs into a $(G + e)$-design *

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Abstract

Let $G$ be a connected simple graph and let $S_G$ be the spectrum of integers $v$ for which there exists a $G$-design of order $v$. Put $e = \{x, y\}$, with $x \in V(G)$ and $y \notin V(G)$. Denote by $G + e$ the graph having vertex set $V(G) \cup \{y\}$ and edge set $E(G) \cup \{e\}$. Let $(X, D)$ be a $(G+e)$-design. We say that two $G$-designs $(V_i, B_i), i = 1, 2,$ are exactly embedded into $(X, D)$ if $X = V_1 \cup V_2, |V_1 \cap V_2| = 0$ and there is a bijective mapping $f : B_1 \cup B_2 \rightarrow D$ such that $B$ is a subgraph of $f(B)$, for every $B \in B_1 \cup B_2$. We give necessary and sufficient conditions so that two $G$-designs can be exactly embedded into a $(G + e)$-design. We also consider the following two problems: 1) determine the pairs \{\(v_1, v_2\)\} \subseteq S_G for which any two nontrivial $G$-designs $(V_i, B_i), |V_i| = v_i$ $i = 1, 2,$ can be exactly embedded into a $(G + e)$-design; 2) determine the pairs \{\(v_1, v_2\)\} \subseteq S_G for which there exists a $(G + e)$-design of order $v_1 + v_2$ exactly embedding two nontrivial $G$-designs $(V_i, B_i), |V_i| = v_i$, $i = 1, 2$. We study these problems for BIBDs, cycle systems, cube systems, path designs and star designs.

Keywords: Embedding, $G$-design, $T$-balanced.

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1 Introduction and Definitions

Let $G$ be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $e = \{x, y\}$, with $x \in V(G)$ and $y \notin V(G)$. Denote by $G + e$ the graph having vertex set $V(G) \cup \{y\}$ and edge set $E(G) \cup \{e\}$. We write $G + e_x$ instead of $G + e$ when we need to specify in which vertex $e$ is incident to $G$.

A $G$-decomposition of a graph $K$ is a set of subgraphs of $K$, each isomorphic to $G$, whose edge sets partition the edge set of $K$. A $G$-decomposition is denoted by a pair $(V, B)$ where $V$ is the vertex set of $K$ and $B$ (the $G$-block set) is the set of copies of $G$. A $G$-design of order $v$ is a $G$-decomposition of the complete graph on $v$ vertices which is denoted by $K_v$. A partial $G$-design of order $v$ is a $G$-decomposition of some subgraph of $K_v$. Denote by $\mathcal{S}_G$ the spectrum of integers $v$ for which there exists a $G$-design of order $v$. See [1] or [2] for results, definitions and terminology not explicitly given here.

Let $(V, B)$ be a partial $G$-design, $B = \{B_1, B_2, \ldots, B_b\}$. Let $r_i$ be the number of blocks of $B$ that contain $i \in V$. We say that $(V, B)$ is balanced if $r_i = r$ for every $i \in V$.

A $G$-design $(W, C)$ is called embedded into a $H$-design $(V, B)$ if $G$ is a subgraph of $H$, $W \subseteq V$ and there is an injective mapping $f : C \rightarrow B$ such that $B$ is a subgraph of $f(B)$, for every $B \in C$.

Recently, embeddings of $G$-designs into $H$-designs have been investigated in many papers. Some results in this direction are also interesting for their application to groomings for two-period optical networks (see [4] and the references therein). An interesting problem, useful also for groomings for two-period optical networks, is the construction of $H$-designs exactly embedding more than one $G$-design.

**Definition 1.1.** Let $(V, B)$ and $(W, C)$ be two vertex disjoint $G$-designs. We say that $(V, B)$ and $(W, C)$ are exactly embedded into a $H$-design $(V \cup W, D)$ if $G$ is a subgraph of $H$, $W \subseteq V$ and there is a bijective mapping $f : B \cup C \rightarrow D$ such that $B$ is a subgraph of $f(B)$, for every $B \in C$.

Necessary and sufficient conditions for the existence of two vertex disjoint $P_3$-designs exactly embedded into a $C_4$-design are given in [7] ($C_4$ denotes the 4-cycle). A generalization of this problem has been studied in [8].

There are many variants of colouring problem of graph designs. The problem we are dealing with, can be restated as a vertex colouring of a $H$-design $(X, D)$ with two colours, say black and cyan, such that: 1) each $H$-block contains exactly one monochromatic subgraph isomorphic to $G$, say it a $G$-block; 2) if $B$ is the set of black $G$-blocks and $C$ is the set of cyan $G$-blocks, then $(V, B)$ and $(W, C)$ are two vertex disjoint $G$-designs, where $V$ ($W$) is the set of vertices of $X$ which receive colour black (cyan, respectively).
In this paper we study the exact embedding when $H = G + e$. Our embeddings depend not only from $G$ and sizes $v$ and $w$, but also from the choice of the vertex $x$ of the pendent edge $e_x = \{x, y\}$. For example, take $G = P_3 = [p_0, p_1, p_2]$, the simple path of length 2. If $x \in \{p_0, p_2\}$ then $G + e_x$ is $P_4$, the simple path of length 3. While if $x = p_1$, $G + e_x$ is $S_3$, the star of center $p_1$ and 3 pendent edges. We will talk of exact embedding of two $P_3$-designs into a $P_3$-design in the former case, and of exact embedding of two $P_3$-designs into an $S_3$-design in the latter case.

Let $x_1, x_2 \in V(G)$. We say that $x_1 \sim x_2$ if and only if $G + e_{x_1}$ and $G + e_{x_2}$ are isomorphic. Clearly, $\sim$ is an equivalence relation. Denote by $\mathcal{T} = \{T_0, T_1, \ldots, T_{h-1}\}$ the partition of $V(G)$ induced by $\sim$. For example if $G = P_{k+1} = [p_0, p_1, \ldots, p_k]$, the simple path with $k + 1$ vertices and $k$ edges, then $V(G) = \{p_0, \ldots, p_k\}$, $E(G) = \{\{p_i, p_{i+1}\} \mid i = 0, \ldots, k - 1\}$, $\mathcal{T} = \{T_i = \{p_i, p_{i+1}\} \mid i = 0, \ldots, \lceil \frac{k}{2} \rceil\}$.

Let $(V, \mathcal{B})$ be a partial $G$-design and let $T \in \mathcal{T}$. For every $B \in \mathcal{B}$ define $T(B) = \{y \mid y \in V(B), \phi_B(y) \in T\}$, where $\phi_B$ is an isomorphism between the graphs $B$ and $G$. For $y \in V$, put $R_y^T = \{B \mid B \in \mathcal{B}, y \in T(B)\}$ and $r^T_y = |R^T_y|$.

**Definition 1.2.** A $G$-design $(V, \mathcal{B})$ is called $T$-balanced if $|r^T_y - r^T_z| \leq 1$ for every $y, z \in V$, $y \neq z$.

As we show in the following examples, not every balanced $G$-design is also $T$-balanced and an unbalanced $G$-design could be $T$-balanced.

**Example 1.1.** Let $(V, \mathcal{B})$ be the balanced $P_5$-design defined as follows:

$V = \{a_0, a_1, \ldots, a_{10}\}$, $B_0 = [a_0, a_1, a_8, a_2, a_9, a_3]$, $B_1 = [a_1, a_2, a_0, a_3, a_{10}, a_9]$, $B_2 = [a_8, a_3, a_1, a_2, a_0, a_{10}]$, $B_3 = [a_3, a_4, a_2, a_5, a_1, a_6]$, $B_4 = [a_{10}, a_5, a_3, a_6, a_4, a_7]$, $B_5 = [a_5, a_6, a_2, a_7, a_9, a_8]$, $B_6 = [a_6, a_7, a_5, a_8, a_4, a_9]$, $B_7 = [a_7, a_8, a_6, a_5, a_4]$, $B_8 = [a_2, a_3, a_7, a_{10}, a_6, a_0]$, $B_9 = [a_4, a_{10}, a_8, a_0, a_7, a_1]$, $B_{10} = [a_5, a_0, a_9, a_1, a_{10}, a_2]$.

We have, for example, $T_0(B_0) = \{a_0, a_3\}$ and $R^T_{a_0} = \{B_0, B_3\}$. It is easy to check that $r^T_{a_0} = r^T_{a_{10}} = 2$, $i = 0, 1, \ldots, 10$. So $(V, \mathcal{B})$ is $T_0$-balanced. It is also $r^T_{a_j} = r^T_{a_{10}} = 2$ for $j = 1, 2$ and $i = 0, 1, 3, 5, 6, 7, 9$, $r^T_{a_2} = r^T_{a_{12}} = 0$, $r^T_{a_4} = r^T_{a_{14}} = 4$, $r^T_{a_8} = r^T_{a_{18}} = 1$ and $r^T_{a_7} = r^T_{a_{17}} = 3$. So $(V, \mathcal{B})$ is not $T_j$-balanced for $j = 1, 2$.

**Example 1.2.** Let $(V, \mathcal{B})$ be the unbalanced $P_6$-design defined as follows:

$V = \{a_0, a_1, \ldots, a_9\}$, $B_0 = [a_0, a_1, a_9, a_2, a_8, a_4]$, $B_1 = [a_1, a_2, a_0, a_3, a_9, a_4]$, $B_2 = [a_2, a_3, a_1, a_4, a_0, a_5]$, $B_3 = [a_3, a_4, a_2, a_5, a_1, a_6]$, $B_4 = [a_4, a_5, a_3, a_6, a_2, a_7]$, $B_5 = [a_4, a_6, a_7, a_5, a_8, a_3]$, $B_6 = [a_4, a_7, a_8, a_6, a_9, a_5]$, $B_7 = [a_8, a_9, a_7, a_0, a_6, a_5]$, $B_8 = [a_9, a_0, a_8, a_1, a_7, a_3]$.
It is easy to check that \((V, B)\) is \(T_2\)-balanced and it is not \(T_j\)-balanced for \(j = 0, 1\).

Denote by \(d_G(x)\) the degree of \(x \in V(G)\), i.e. the number of edges of \(E(G)\) which are incident to \(x\). If \(T \in \mathcal{T}\), then \(d_G(x_1) = d_G(x_2) = d^T\) for every \(x_1, x_2 \in T\).

**Theorem 1.1.** Let \((V, B)\) be a \(G\)-design of order \(n\) such that \(|\mathcal{T}| = 1\), i.e. \(\mathcal{T} = \{V(G)\}\). Then \((V, B)\) is \(T\)-balanced and \(r^T_y = \frac{n-1}{d^T}\) for every \(y \in V(G)\).

**Theorem 1.2.** Let \((V, B)\) be a balanced \(G\)-design of order \(n\) and let \(\mathcal{T} = \{T_0, T_1, \ldots, T_p\}\). If \(y \in V\) then

\[
\sum_{i=0}^{p} r_{y}^{T_i} = \frac{(n-1)|V(G)|}{2|E(G)|},
\]

and

\[
\sum_{i=0}^{p} d^{T_i} r_{y}^{T_i} = n - 1.
\]

**Corollary 1.3.** Let \((V, B)\) be a balanced \(G\)-design. If \(\mathcal{T} = \{T_0, T_1\}\) and \(d^{T_0} \neq d^{T_1}\), then \((V, B)\) is \(T_i\)-balanced, \(i = 0, 1\).

We determine necessary and sufficient conditions so that two given \(G\)-designs can be exactly embedded into a \((G + e)\)-design. Moreover we study the following two problems.

**Problem 1.1.** Let \(G\) be a simple connected graph and let \(T \in \mathcal{T}, x \in T\). Determine the set \(E_{G,T}^1\) of all the pairs \(\{v, w\}, v, w \in S_G\), such that any pair of nontrivial \(G\)-designs of order \(v\) and \(w\), respectively, can be exactly embedded into a \((G + e_x)\)-design. If \(\mathcal{T} = \{V(G)\}\), then write \(E_{G}^1\) instead of \(E_{G,T}^1\).

**Problem 1.2.** Let \(G\) be a simple connected graph and let \(T \in \mathcal{T}, x \in T\). Determine the set \(E_{G,T}^2\) of all the pairs \(\{v, w\}, v, w \in S_G\), such that there exists a pair of nontrivial \(G\)-designs of orders \(v\) and \(w\), respectively, exactly embedded into a \((G + e_x)\)-design.

We solve Problem 1.1 for any \(G\)-design such that \(|\mathcal{T}| = 1\) as, for example, BIBDs, \(k\)-cycle systems and \(k\)-cube systems. When \(|\mathcal{T}| > 1\), it is not sure that two \(G\)-designs can be exactly embedded into a \((G + e_x)\)-design (see Examples 1.3 and 1.4). In such cases, the problem doesn't depend only on the graph \(G\), the set \(T \in \mathcal{T}\) and orders \(v\) and \(w\), but also from the specific designs we wish to embed. So only Problem 1.2 makes sense. For the sake of brevity we give a complete answer to it only when \(G\) is a path or a star.
Example 1.3. Let $G = S_3 = [p_0; p_1, p_2, p_3]$ be the $3$-star with vertices \{p_0, p_1, p_2, p_3\} and edges \{p_0, p_1\}, \{p_0, p_2\}, \{p_0, p_3\}. Set $V = \{a_0, a_1, a_2, a_3, a_4\}$, $W = \{b_0, b_1, \ldots, b_{18}\} \cup \{\infty\}$, $B = \{a_1, a_0, a_4\}, \{a_2, a_1, a_4\}, \{a_0, a_2, a_4\}, \{a_0, a_3, a_4\}, \{a_2, a_3, a_4\})$. $C_1 = \{(\infty, b_i, b_{i+1}) \mid i \in \mathbb{Z}_{19}\}$ and $C_2 = \{(b_{2j+i}, b_i, b_{2j+i+1}) \mid j = 1, 2, 3, 4, i \in \mathbb{Z}_{19}\}$ (the indices of the elements of $W$ are $\pmod{19}$). Set $C = C_1 \cup C_2$. Then $(V, B)$ and $(W, C)$ are two vertex disjoint $P_3$-designs of order $v = 5$ and $w = 20$, respectively. It is easy to see that $(V, B)$ and $(W, C)$ cannot be exactly embedded into a $S_3$-design. Note that $(V, B)$ and $(W, C)$ are not $T_1$-balanced.

Example 1.4. Let $(V, B)$ and $(W, C)$ be the two $P_3$-designs given in Example 1.3. Replace in $C$ the two $P_3$s $[\infty, b_0, b_1]$ and $[\infty, b_1, b_2]$ with $[b_0, \infty, b_1]$ and $[b_0, b_1, b_2]$. The result is a $P_3$-design $(V, \overline{C})$.

Let $D_1 = \{[a_i; \infty, a_1, a_4], [a_1; \infty, a_2, a_4], [a_2; \infty, a_0, a_4], [a_3; \infty, a_0, a_4], [a_0; a_1, a_2, b_0]\}$, $D_2 = \{[\infty, b_0, b_1, a_1], [b_1, b_0, b_2, a_3] \mid i = 2, 3, \ldots, 18\}, D_3 = \{[b_0, b_2, b_9, a_4] \cup \{[b_i; b_{2j+i}, b_{2j+i+1}, a_{j+1}] \mid i = 1, 2, 3, i = 0, 1, \ldots, 18\}$. Put $D = \cup_{i=1}^4 D_i$. It is easy to check that $(V, B)$ and $(W, \overline{C})$ are vertex disjoint $P_3$-designs exactly embedded into the $S_3$-design $(V \cup W, D)$. Note that $(V, B)$ and $(W, \overline{C})$ are not $T_1$-balanced.

Let $V = \{0, 1, \ldots, v - 1\}$, $B = \{B_1, B_2, \ldots, B_b\}$ and $T \in \mathcal{T}$. A $T$-headset of a partial $G$-design $(V, B)$ is a multiset $\mathcal{H}_V = \{x_1, \ldots, x_b\}$ so that $x_\mu \in T(B_\mu)$ for $\mu = 1, 2, \ldots, b$. We say that $x_\mu$ is marked in $B_\mu$ or, also, that $B_\mu$ is marked in $x_\mu$. An equitable edge $c$-colouration of a graph is a colouration of its edges with $c$ colours such that for each vertex $x$, we have $|f_p(x) - f_q(x)| \leq 1$ for all $p, q \in \{1, \ldots, c\}$, where $f_p(x)$ denotes the number of edges with colour $p$ which are incident to $x$.

Lemma 1.4. [5] A bipartite graph has an equitable edge $c$-colouring for any $c$.

The following theorem generalizes a result on headsets for partial triple systems given in [3].

Theorem 1.5. For each $T \in \mathcal{T}$, the partial $G$-design $(V, B)$ has a $T$-headset $\mathcal{H}_V = \{x_1, \ldots, x_b\}$ such that for $0 \leq x \leq |V| - 1$ the number of occurrences of $x$ in $\mathcal{H}_V$ is $\left\lfloor \frac{x}{|T|} \right\rfloor$ or $\left\lceil \frac{x}{|T|} \right\rceil$.

Proof. For $|T| = 1$, the proof is straightforward. Suppose $|T| \geq 2$. Form a bipartite graph $\Psi$ with vertex set $V \cup B$, and an edge $\{y, B\}$ for $y \in V$ and $B \in B$ if and only if $y \in T(B)$. By Lemma 1.4, $\Psi$ admits an equitable edge $|T|$-colouring with colours $\{c_1, \ldots, c_{|T|}\}$. Then for $1 \leq \mu \leq b$, $B_\mu$ is incident
to exactly $|T|$ edges, and hence to exactly one edge $\{y_\mu, B_\mu\}$ that is coloured $c_1$; set $x_\mu = y_\mu$. Then $\mathcal{H}_V = \{x_1, \ldots, x_b\}$ forms the required $T$-headset. \hfill $\square$

## 2 Necessary and sufficient conditions

Let $G$ be a simple connected graph with $k$ edges and let $T \in T$, $x \in T$. Suppose there exists a $(G + e_x)$-design exactly embedding two $G$-designs $(V, B)$ and $(W, C)$ with $|V| = v$ and $|W| = w$. Then

$$\frac{1}{k} \binom{v}{2} + \frac{1}{k} \binom{w}{2} = vw.$$  \hfill (3)

Denote by $\mathcal{I}_k$ the set of pairs $(v, w)$ with $v, w$ nonnegative integers satisfying (3).

**Theorem 2.1.** Let $x_n, n \in \mathbb{N}$, be the sequence

$$x_0 = 0, \quad x_{n+1} = \frac{1 + 2kx_n + \sqrt{4(k^2 - 1)x_n^2 + 4(k + 1)x_n + 1}}{2},$$  \hfill (4)

and let $\mathcal{X}_k = \{(x_n, x_{n+1}), (x_{n+1}, x_n) \mid n \in \mathbb{N}\}$. Then $\mathcal{I}_k = \mathcal{X}_k$ for every integer $k \geq 1$.

**Proof.** By (3), $(v, w) \in \mathcal{I}_k \iff (w, v) \in \mathcal{I}_k$. Write (3) as follows

$$w^2 - (1 + 2kv)w + v^2 - v = 0.$$  \hfill (5)

Solving for $w$ we obtain $(w - w_1)(w - w_2) = 0$, with

$$w_1 = \frac{1 + 2kv - \sqrt{4(k^2 - 1)v^2 + 4(k + 1)v + 1}}{2},$$

and

$$w_2 = \frac{1 + 2kv + \sqrt{4(k^2 - 1)v^2 + 4(k + 1)v + 1}}{2}.$$  \hfill (6)

Note that

$$(v, w_1) \in \mathcal{I}_k \iff (v, w_2) \in \mathcal{I}_k.$$  \hfill (7)

Since $(0, 0), (0, 1) \in \mathcal{I}_k$, it is $(x_n, x_{n+1}) \in \mathcal{I}_k$ for every $n \in \mathbb{N}$.

Now we must prove that $\mathcal{I}_k \subseteq \mathcal{X}_k$. Take $(\alpha, \beta) \in \mathcal{I}_k$. Put $f(t) = \frac{1 + 2kt - \sqrt{4(k^2 - 1)t^2 + 4(k + 1)t + 1}}{2}$. Let $\beta_0 = \alpha$ and $\beta_{n+1} = f(\beta_n)$, $n \in \mathbb{N}$. By (6), $(\beta_n, \beta_{n+1}) \in \mathcal{I}_k$ and so, being (5) a second degree equation,

$$\beta_n \beta_{n+2} = \beta_{n+1}(\beta_{n+1} - 1).$$  \hfill (7)
By (7), if $\beta_n > 1$ and $\beta_{n+1} < 0$ then $\beta_{n+2} > 0$. It follows a contradiction since $\beta_n \geq \beta_{n+1}$ and the equality holds only for $\beta_n = 0$. If $\beta_n = 0, 1$ then $\beta_{n+1} = 0$. It follows that there exists a $n^* \in \mathbb{N}$ such that $\beta_{n^*} = 1 = x_1$ and $\beta_n = 0$ for $n > n^*$ and so $(\alpha, \beta) \in X_k$, since $\beta_{n^*} = x_1$ implies $\beta_{n^*+1} = x_2$ and so on.

\textbf{Theorem 2.2.} Let $x_n$ be the sequence (4). Then $x_n \equiv 0 \pmod{2k}$ for $n \equiv 0, 3 \pmod{4}$, and $x_n \equiv 1 \pmod{2k}$ for $n \equiv 1, 2 \pmod{4}$.

\textbf{Proof.} The proof is straightforward for $n = 0, 1$. Let $\rho$ be a nonnegative integer. To complete the proof it is sufficient to prove that if $x_{4\rho} \equiv 0 \pmod{2k}$ and $x_{1+4\rho} \equiv 1 \pmod{2k}$, then $x_{2+4\rho} \equiv 1 \pmod{2k}$, $x_{3+4\rho} \equiv 0 \pmod{2k}$, $x_{4+4\rho} \equiv 0 \pmod{2k}$ and $x_{5+4\rho} \equiv 1 \pmod{2k}$. Note that (3) for $v = x_{n+1}$ has $x_n$ and $x_{n+2}$ as roots. Then

$$x_n + x_{n+2} = 1 + 2kx_{n+1}.$$  

By (8) we have $x_{4\rho} + x_{2+4\rho} \equiv 1 + 2kx_{1+4\rho}$. If $x_{1+4\rho} \equiv 0 \pmod{2k}$ and $x_{1+4\rho} \equiv 1 \pmod{2k}$, then $x_{2+4\rho} \equiv 1 \pmod{2k}$. Applying $n = 1 + 4\rho, 2 + 4\rho, 3 + 4\rho, 4 + 4\rho$, to (8) it follows $x_{3+4\rho} \equiv 0 \pmod{2k}$, $x_{4+4\rho} \equiv 0 \pmod{2k}$ and $x_{5+4\rho} \equiv 1 \pmod{2k}$, respectively.

Let $(V, B)$ and $(W, C)$ be two nontrivial $G$-designs with $V = \{a_1, a_2, \ldots, a_v\}$ and $W = \{b_1, b_2, \ldots, b_w\}$. Suppose that $H_V (H_W)$ is a $T$-headset of $(V, B)$ ($(W, C)$, respectively). For $x \in V \ (y \in W)$, denote by $\pi_V (x)$ ($\pi_W (y)$) the number of occurrences of $x$ in $H_V \ (y \in H_W)$.

\textbf{Theorem 2.3.} $(V, B)$ and $(W, C)$ can be exactly embedded into a $(G + e_x)$-design, $x \in T$, if and only if there exist two $T$-headsets $H_V$ and $H_W$ verifying the following properties:

(1) $0 \leq \pi_V (a_i) \leq w, \ i = 1, 2, \ldots, v$;

(2) $0 \leq \pi_W (b_i) \leq v, \ i = 1, 2, \ldots, w$;

(3) there exists a simple bipartite graph $\Theta$ on vertex set $V \cup W$ such that the degree of $a_i$ ($b_j$) is $m_i = \pi_V (a_i), \ i = 1, 2, \ldots, v$ ($s_j = v - \pi_W (b_j), \ j = 1, 2, \ldots, w$).

\textbf{Proof.} Suppose there is a $(G + e_x)$-design $(V \cup W, D)$ exactly embedding $(V, B)$ and $(W, C)$. Denote by $D_1$ ($D_2$) the set of $D \in D$ having subgraph a $B \in B \ (C \in C)$. Each $D \in D_2$ contains one edge, say $\{a_i, b_j\}$, of $K_{V, W}$. Mark the subgraph $B$ of $D$ in $a_i$. The result is a $T$-headset $H_V$. Analogously, starting from the blocks of $D_2$, construct the $T$-headset $H_W$. Conditions (1) and (2) are trivially satisfied. Let $\Theta$ be the simple bipartite graph having
vertex set $V \cup W$ and edge set $E(\Theta)$, where $E(\Theta)$ is the set of edges of $K_{V,W}$ covered by the blocks of $D_1$.

Now suppose that $H_V$ and $H_W$ are two $T$-headsets of $(V,B)$ and $(W,C)$ verifying the conditions (1) – (3). Let $\{a_i, b_j\} \in K_{V,W}$. If $\{a_i, b_j\} \in E(\Theta)$, then form a $(G + e)$-block by attaching it to a $B \in B$ having $a_i$ as marked vertex. If $\{a_i, b_j\} \not\in E(\Theta)$, then form a $(G + e)$-block by attaching it to a $C \in C$ having $b_j$ as marked vertex.

**Definition 2.1.** ([6]) Let $m_1, m_2, \ldots, m_v$ be a nonnegative integer sequence. For every positive integer $j$ let $M_j = \{m_i \mid m_i \geq j\}$. The sequence $m_j^* = |M_j|$ is called the dual sequence of $m_i$, $i = 1, 2, \ldots, v$.

**Lemma 2.4.** ([6]) Let $m_1 \geq m_2 \geq \ldots \geq m_v$ and $s_1 \geq s_2 \geq \ldots \geq s_w$ be two nonnegative integer sequences. There exists a simple bipartite graph having the two sequences as degree of the two parts if and only if

1. $\sum_{i=1}^v m_i = \sum_{j=1}^w s_j$;
2. $\sum_{i=1}^h m_i^* \geq \sum_{j=1}^h s_j$, $h = 1, 2, \ldots, w - 1$.

**Corollary 2.5.** Let $m_1 \geq m_2 \geq \ldots \geq m_v$ and $s_1 \geq s_2 \geq \ldots \geq s_w$ be two nonnegative integer sequences. If $m_1 \leq w$, $s_1 \leq v$, $\sum_{i=1}^v m_i = \sum_{j=1}^w s_j$ and $|m_i - m_j| \leq 1$ for every $i, j = 1, 2, \ldots, v$, then there exists a simple bipartite graph having the two sequences as degree of the two parts.

**Proof.** For $m_1 = 1$ the proof is trivial. Suppose $m_1 \geq 2$ and put $p = m_1 - 1$. Then $m_1 = \ldots = m_h = m_{h+1} + 1 = \ldots = m_v + 1 = p + 1$ for some $1 \leq h \leq v$. It follows $m_1^* = m_2^* = \ldots = m_v^* = v$ and $\sum_{i=1}^{p+1} m_i^* = \sum_{i=1}^v m_i = \sum_{j=1}^w s_j$.

Lemma 2.4 completes the proof.

**Theorem 2.6.** Suppose that $(V,B)$ and $(W,C)$ satisfy the following conditions:

(a) $(v,w) \in \mathcal{I}_k$, $v, w \geq 1 + 2k$;
(b) there exist two $T$-headsets $H_V$ and $H_W$ such that $\pi_W(b_j) \leq v$, $j = 1, 2, \ldots, w$ and $|\pi_V(a_i) - \pi_V(a_j)| \leq 1$, $i, j = 1, 2, \ldots, v$.

Then $(V,B)$ and $(W,C)$ can be exactly embedded into a $(G + e_x)$-design, $x \in T$.

**Proof.** Rename the elements of $V$ and $W$ so that the sequences $m_i = \pi_V(a_i)$ and $s_j = v - \pi_W(b_j)$ are decreasing. By Theorem 2.3 and Corollary 2.5, it is sufficient to prove that $\pi_V(a_i) \leq w$, $i = 1, 2, \ldots, v$. If $\pi_V(a_1) \geq w + 1$, then $\pi_V(a_i) \geq \pi_V(a_1) - 1 = w$ for $i = 2, 3, \ldots, v$. It follows $|B| \geq w + 1 + w(v - 1) = vw + 1$. This contradicts the condition $(v,w) \in \mathcal{I}_k$. \qed
3 Problems 1.1 and 1.2

We devote this section to study the Problems 1.1 and 1.2. Let $I_k$ be the set defined at the beginning of Section 2.

**Theorem 3.1.** It is

$$\mathcal{E}^1_{G,T} \subseteq \mathcal{E}^2_{G,T} \subseteq I_k.$$ 

**Proof.** Suppose there is a $(G + e)$-design $(V \cup W, D)$ exactly embedding the $G$-designs $(V, B)$ and $(W, C)$ of order $v$ and $w$ respectively. Then every edge of the bipartite graph $K_{v,w}$ on $V \cup W$ is covered by one block of $D$ and it is not covered by any block of $B \cup C$. \hfill $\Box$

**Lemma 3.2.** Let $(U, U)$ be a nontrivial $T$-balanced $G$-design of order $u$. Then $(U, U)$ contains a $T$-headset $H_U$ such that

$$|\pi_U(t) - \pi_U(y)| \leq 1 \text{ for every } t, y \in U, \ t \neq y.$$ 

**Proof.** Suppose at first $|V(G)| = 2$. Then $|E(G)| = k = 1$ and $T = V(G)$. It follows $r^T_y = u - 1$ for every $y \in U$. By Theorem 1.5, $(U, U)$ has a $T$-headset $H_U$ such that $\pi(y) \in \{\lfloor \frac{u-1}{2} \rfloor, \lceil \frac{u-1}{2} \rceil\}$ for every $y \in U$.

Let now $|V(G)| \geq 3$. Then $|T| \leq |V(G)| < 2k$,

$$0 < \frac{|T|}{2k} < 1.$$ \hfill (9)

Being $(U, U)$ $T$-balanced, there is a nonnegative integer $\alpha$ such that $r^T_y \in \{\alpha, \alpha + 1\}$ for every $y \in U$. Then

$$\begin{cases}
\alpha \chi_0 + (\alpha + 1) \chi_1 = \frac{u(u-1)}{2k} |T|, \\
\chi_0 + \chi_1 = u
\end{cases},$$

$$\alpha = \frac{u - 1}{2k} |T| - \frac{\chi_1}{u}. \hfill (10)$$

Let $u = 2k\rho$. Then $\alpha = \rho|T| - \frac{|T|}{2k} - \frac{\chi_1}{2k\rho}$ By (9),

$$-2 < - \frac{|T|}{2k} - \frac{\chi_1}{2k\rho} < 0.$$ 

It follows $\alpha = \rho|T| - 1$. So $\{r^T_y \mid y \in U\} = \{\rho|T| - 1, \rho|T|\}$ and, by Theorem 1.5, $(U, U)$ has a $T$-headset $H_U$ such that $\pi(y) \in \{\rho - 1, \rho\}$.

Let $u = 1 + 2k\rho$. By (10), $\alpha = \rho|T| - \frac{\chi_1}{1+2kp}$: Then $\alpha \in \{\rho|T| - 1, \rho|T|\}$, and $\{r^T_y \mid y \in U\} = \{\rho|T| - 1, \rho|T|\}$ or $\{r^T_y \mid y \in U\} = \{\rho|T|, \rho|T| + 1\}$. So by Theorem 1.5, $(U, U)$ has a $T$-headset $H_U$ such that either $\pi(y) \in \{\rho - 1, \rho\}$ or $\pi(y) \in \{\rho, \rho + 1\}$. \hfill $\Box$

By Lemma 3.2 and Theorem 2.6, we obtain the following result.
Theorem 3.3. Let \((V, \mathcal{B})\) and \((W, \mathcal{C})\) be two nontrivial vertex disjoint \(G\)-designs of order \(v\) and \(w\) respectively, with \((v, w) \in \mathcal{I}_k, v \leq w\). The following statements hold:

1. If \((W, \mathcal{C})\) is \(T\)-balanced then \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \((G + e_x)\)-design with \(x \in T\).

2. If \((V, \mathcal{B})\) is \(T\)-balanced and \((W, \mathcal{C})\) has a \(T\)-headset \(H_W\) such that \(\pi_W(t) \leq v\) for every \(t \in W\), then \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \((G + e_x)\)-design with \(x \in T\).

Let \(G\) be a graph such that \(T = \{V(G)\}\). By Theorems 1.1, 2.1 and 3.3, Problem 1.1 is solved by all the pairs of integers \(v, w \in \mathcal{S}_G\) such that \((v, w) \in \mathcal{I}_k = \mathcal{X}_k, v, w \geq 1 + 2k\). So, Problem 1.1 is completely solved for BIBDs, cycle systems and cube systems. In order to explicitly determine the set \(\mathcal{S}_G\), Theorem 2.2 is useful. As example, put \(G = K_3\). It is well-known that \(\mathcal{S}_{K_3} = \{n \mid n \equiv 1, 3 \pmod{6}\}\). By Theorem 2.2, \(\mathcal{E}_{k_3}\) is given by the pairs \((v, w)\) such that \(\{v, w\} = \{x_n, x_{n+1}\}\) for every \(n \equiv 1 \pmod{4}, n \geq 5\).

Let \(G = P_{k+1} = [p_0, p_1, \ldots, p_k]\) be the simple path with \(k + 1\) vertices and \(k\) edges. Put \(T = \{T_i = \{p_i, p_{k-i}\} \mid i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}\).

Theorem 3.4. Let \(k \geq 2\). Let \((V, \mathcal{B})\) be a \(P_{k+1}\)-design of order \(v\) and let \((W, \mathcal{C})\) be a balanced \(P_{k+1}\)-design of order \(w\), \(|V \cap W| = 0\).

If \(v \leq w\) then \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \(P_{k+2}\)-design if and only if \((v, w) \in \mathcal{X}_k, v \geq 1 + 2k\).

If \(v > w\) then \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \(P_{k+2}\)-design if and only if \((v, w) \in \mathcal{X}_k, w \geq 1 + 2k\); and \((V, \mathcal{B})\) has a \(T_0\)-headset \(H_V\) such that \(\pi_V(z) \leq w\) for every \(z \in V\).

Proof. It is \(d_{i}^{G} = 2, i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\). So, by Theorem 1.2, a balanced \(P_{k+1}\)-design is also \(T_0\)-balanced. Theorems 3.1, 2.1 and 3.3 complete the proof.

By Corollary 1.3, a balanced \(P_{k+1}\)-design, with \(k = 2, 3\), is \(T_1\)-balanced. So we have the following result.

Theorem 3.5. Let \(k = 2, 3\). Let \((V, \mathcal{B})\) be a nontrivial \(P_{k+1}\)-design of order \(v\) and let \((W, \mathcal{C})\) be a nontrivial balanced \(P_{k+1}\)-design of order \(w\), \(|V \cap W| = 0\).

Let \(x \in T_1\). If \(v \leq w\) then \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \((G + e_x)\)-design if and only if \((v, w) \in \mathcal{X}_k\).

If \(v > w\) then \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \((G + e_x)\)-design if and only if \((v, w) \in \mathcal{X}_k\) and \((V, \mathcal{B})\) has a \(T_1\)-headset \(H_V\) such that \(\pi_V(z) \leq w\) for every \(z \in V\).
Note that a balanced \( P_{k+1} \)-design with \( k \geq 4 \) could be not \( T_i \)-balanced for any \( i = 1, \ldots, \lfloor \frac{k}{2} \rfloor \) (see Example 1.1). So Theorem 3.5 cannot be generalized to every \( k \geq 5 \). In order to study Problem 1.2 for path designs we recall the following well-known construction.

**Construction 3.1.** Construction of a \( P_{k+1} \)-design \((X, \mathcal{X})\) of order \( n \equiv 0 \) \((\mod 2k)\). Let \( X = \mathbb{Z}_{n-1} \cup \{\infty\} \). By difference method, \( \mathcal{X} \) can be easily obtained by developing \((\mod n-1)\) the base blocks \( B_i = [x^i_0, x^i_1, \ldots, x^i_k] \), \( i = 0, 1, \ldots, \frac{n-2k}{2k} \), where \( x^i_j \) are defined as follows:

Case 1: \( k \) even, \( k \geq 2 \). Put \( x^0_0 = \infty, x^0_{2j} = j \) for \( j = 1, \ldots, \frac{k}{2}, x^0_{2j+1} = -j \) for \( j = 0, \ldots, \frac{k-2}{2} \). If \( n \geq 4k \) then, for \( i = 1, \ldots, \frac{n-2k}{2k} \), put \( x^i_{2j} = -j \) for \( j = 0, \ldots, \frac{k}{2}, x^i_{2j+1} = j + ki \) for \( j = 0, \ldots, \frac{k-2}{2} \).

Case 2: \( k \) odd, \( k \geq 3 \). Put \( x^0_0 = \infty, x^0_2 = j \) for \( j = 1, \ldots, \frac{k-1}{2}, x^0_{2j+1} = -j \) for \( j = 0, \ldots, \frac{k-1}{2} \). If \( n \geq 4k \) then, for \( i = 1, \ldots, \frac{n-2k}{2k} \), put \( x^i_{2j} = -j \) for \( j = 0, \ldots, \frac{k-1}{2}, x^i_{2j+1} = j + ki \) for \( j = 0, \ldots, \frac{k-1}{2} \).

Note that the \( P_{k+1} \)-design \((X, \mathcal{X})\) given by Construction 3.1 is well-known as a 1-rotational \( P_{k+1} \)-design. Let \( y \in \mathbb{Z}_{n-1} \). When we develop the base blocks we will put \( B_i + y = [x^i_0 + y, x^i_1 + y, \ldots, x^i_k + y] \) where the sum is \((\mod n-1)\) if \( x^i_j \in \mathbb{Z}_{n-1} \). Moreover we put \( \infty + y = \infty \).

**Theorem 3.6.** Let \( k \geq 2 \). Then

\[
\mathcal{E}^2_{P_{k+1}, T_0} = \{(v, w) \mid (v, w) \in \mathcal{X}_k \text{ and } v, w \geq 1 + 2k\}.
\]

**Proof.** The necessity follows from Theorems 3.1 and 2.1. Put \( v = x_n, w = x_{n+1}, n \geq 2 \). Suppose at first \( n \equiv 0, 1 \) \((\mod 4)\). By Theorem 2.2, \( x_{n+1} \equiv 1 \) \((\mod 2k)\). Let \((W, \mathcal{C})\) be a balanced \( P_{k+1} \)-design of order \( w = x_{n+1} \) and let \((V, \mathcal{B})\) be a \( P_{k+1} \)-design of order \( v = x_n \) with \( |V \cap W| = 0 \). By Theorem 3.4, \((V, \mathcal{B})\) and \((W, \mathcal{C})\) can be exactly embedded into a \( P_{k+2} \)-design on \( V \cup W \). Now let \( n \equiv 2, 3 \) \((\mod 4)\). Then \( x_{n+1} \equiv 0 \) \((\mod 2k)\). Say \((W, \mathcal{C})\) is the \( P_{k+1} \)-design of order \( w = x_{n+1} \) given by Construction 3.1. It is \( v^\infty_T = w - 1 \) and \( v^\infty_0 = \frac{w}{k} - 1 \) for every \( y \in W \setminus \{\infty\} \). Mark \( \infty \) in \( \frac{w}{2k} - 1 \) blocks of \( \mathcal{C} \). Now it is easy to mark the remaining blocks so that every \( y \in W \setminus \{\infty\} \) is marked in \( \frac{w}{2k} - 1 \) or in \( \frac{w}{2k} \) blocks. By Theorem 2.6, the proof follows. \( \square \)

**Construction 3.2.** Let \( k \geq 2 \) and \( h \in \{1, \ldots, \lfloor \frac{k}{2} \rfloor \} \). Construction of a \( P_{k+1} \)-design of order \( n \equiv 0 \) \((\mod 2k)\), \( n > 2k \), on \( X = \mathbb{Z}_{n-1} \cup \{\infty\} \) with exactly one block having \( \infty \in T_h \). Denote by \((X, \mathcal{X}_h)\) the \( P_{k+1} \)-design of order \( n \equiv 0 \) \((\mod 2k)\), \( n > 2k \), constructed as follows: let \((X, \mathcal{X})\) be the \( P_{k+1} \)-design of order \( n > 2k \) given in Construction 3.1. Replace the vertex \( x^1_h \) of \( B_1 \in \mathcal{X} \) by putting \( x^1_h = \infty \). If \( h = 2j + 1 \), then replace in \( B_0 - j - 1, B_0 - j \in \mathcal{X} \) the
vertices \( x_0^0 - j - 1 = x_0^0 - j = \infty \) by putting \( x_0^0 - j - 1 = x_0^0 - j = k + j \).

If \( h = 2j \), then replace in \( B_0 + k + j - 1, B_0 + k + j \in \mathcal{X} \) the vertices \( x_0^k + k + j - 1 = x_0^k + k + j = -j \).

**Theorem 3.7.** Let \( x \in T_h \), \( h = 1, \ldots, \lfloor \frac{k}{2} \rfloor \). Let \( k \geq 2 \). Then

\[
\mathcal{E}_{h+1}^2 T_h = \{ (v, w) \mid (v, w) \in \mathcal{X}_h \text{ and } v, w \geq 1 + 2k \}.
\]

**Proof.** The necessity follows from Theorems 3.1 and 2.1. Put \( V = \{ a_1, a_2, \ldots, a_v \} \) and \( W = \{ b_1, b_2, \ldots, b_w \} \) with \( v = x_n \) and \( w = x_{n+1} \), \( n \geq 2 \). By Theorem 2.2 it is \( v, w \equiv 0, 1 \pmod{2k} \). If \( w \equiv 1 \pmod{2k} \), let \((W, C)\) be a cyclic \( P_{h+1} \)-design of order \( w \). \((W, C)\) is \( T_h \)-balanced for every \( h \) so, by Theorem 3.3, the proof follows. Now suppose \( w \equiv 0 \pmod{2k} \) and \( v \equiv 1 \pmod{2k} \). Let \((V, B)\) be a cyclic \( P_{h+1} \)-design of order \( v \). Construct \((W, C)\) as in Construction 3.1, where we put \( b_w \) instead of \( \infty \). It is easy to find in it a \( T_h \)-headset so that no any block is marked in \( b_w \) and every \( b_i \), \( i = 1, \ldots, v - 1 \), is marked in \( \frac{v}{2k} \) blocks. Being \( v \geq x_2 = 2k + 1 \), we obtain that \( \frac{w}{2k} \leq v \). By Theorem 3.3, we obtain the proof.

Now let \( v, w \equiv 0 \pmod{2k} \). Put \( \{ a_1, a_2, \ldots, a_{v-1} \} = \{ 0, 1, \ldots, v - 2 \}, \quad a_v = \infty, \quad \{ b_1, b_2, \ldots, b_{w-1} \} = \{ 0, 1, \ldots, w - 2 \}, \quad b_w = \infty \). Construct \((V, B)\) as in Construction 3.2 where we suppose that \( x_h^1 = a_1 \). Moreover construct \((W, C)\) as in Construction 3.1. It is easy to find a \( T_h \)-headset \( \mathcal{H}_V \) of \((V, B)\) such that \( a_v \) is marked exactly in one block (precisely in \( B_1 \)), \( a_1 \) is marked in \( \frac{v - 4k}{2k} \) blocks and \( a_i, \quad i = 2, \ldots, v - 1 \), is marked in \( \frac{v - 2k}{2k} \) blocks. Analogously there is a \( T_h \)-headset \( \mathcal{H}_W \) of \((W, C)\) such that \( b_1, \quad i = 1, 2, \ldots, w - 1 \), is marked in \( \frac{w - 2k}{2k} \) blocks. Put \( m_i = \pi_V(a_{i+1}) \), \( m_v = \pi_V(a_v) \), \( s_1 = v \), \( s_j = v - \frac{w}{2k} j = 2, 3, \ldots, w \). By Theorem 2.3 and Corollary 2.5, we obtain the proof. \( \square \)

For \( k \geq 3 \), let \( S_k = [s_0; s_1, s_2, \ldots, s_k] \) be the \( k \)-star, that is, the complete bipartite graph \( K_{1,k} \) on \( \{ s_0 \} \cup \{ s_1, s_2, \ldots, s_k \} \). The vertex \( s_0 \) is called the center of the star. Put \( T = \{ T_0, T_1 \} \) with \( T_0 = \{ s_1, s_2, \ldots, s_k \} \) and \( T_1 = \{ s_0 \} \). By Corollary 1.3, a balanced \( S_k \)-design is also \( T_i \)-balanced for \( i = 0, 1 \). Then we have the following result.

**Theorem 3.8.** Let \( k \geq 3 \). Let \((V, B)\) be a \( S_k \)-design of order \( v \) and let \((W, C)\) be a balanced \( S_k \)-design of order \( w \), \( |V \cap W| = 0 \). Let \( x \in T_i \) for a fixed \( i \in \{ 0, 1 \} \).

If \( v \leq w \) then \((V, B)\) and \((W, C)\) can be exactly embedded into a \((S_k + e_x)\)-design if and only if \( (v, w) \in \mathcal{X}_h \), \( v \geq 1 + 2k \).

If \( v > w \) then \((V, B)\) and \((W, C)\) can be exactly embedded into a \((S_k + e_x)\)-design if and only if \( (v, w) \in \mathcal{X}_h \), \( w \geq 1 + 2k \), and \((V, B)\) has a \( T_i \)-headset \( \mathcal{H}_V \) such that \( \pi_V(z) \leq w \) for every \( z \in V \).
Construction 3.3. Construction of a $S_k$-design $(X, \mathcal{X})$ of order $n \equiv 0 \pmod{2k}$ on $X = \mathbb{Z}_{n-1} \cup \{\infty\}$. Put $\alpha = \frac{n - 2k}{2k}$. Develop (mod $n-1$) the base blocks $B_i = [x_i^0; x_i^1, \ldots, x_i^k]$ where $x_i^j$ are defined as follows: $x_i^0 = 0$, $x_i^j = j+ik$ for $i = 0, 1, \ldots, \alpha$ and $j = 1, 2, \ldots, k$ if $i \leq \alpha - 1$ or $j = 1, 2, \ldots, k - 1$ if $i = \alpha$; $x_i^k = \infty$.

Construction 3.4. Construction of a $S_k$-design $(X, \mathcal{X})$ of order $n \equiv 0 \pmod{2k}$ on $X = \mathbb{Z}_{n-1} \cup \{\infty\}$ with exactly one block having $\infty$ as center and $k \geq 3$. Construct a $S_k$-design as in Construction 3.3. Replace the blocks $[0; 1+ak, 2+ak, \ldots, k-1+ak, \infty], [1+j+ak; 2+j+2ak, 3+j+2ak, \ldots, k+j+2ak, \infty], j = 0, 1, \ldots, k-2$, with the following stars: $[\infty; 0, 1+ak, 2+ak, \ldots, k-1+ak], [1+j+ak; 0, 2+j+2ak, 3+j+2ak, \ldots, k+j+2ak], j = 0, 1, \ldots, k-2$. The result is the required $S_k$-design.

Based on the Constructions 3.3 and 3.4 and proceeding as in Theorems 3.6 and 3.7 we obtain the following result.

Theorem 3.9. Let $x \in T_h$, $h = 0, 1$. Let $k \geq 3$. Then

$$\mathcal{E}_{S_k,T_h}^g = \{(v, w) \mid (v, w) \in \mathcal{X}_k \text{ and } v, w \geq 1 + 2k\}.$$ 

Note that, when $x \in T_1$, the $(S_k + e_x)$-design is a $S_{k+1}$-design.

References


