Embedding path designs into a maximum packing of $K_n$ with 4-cycles

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Abstract

A packing of $K_n$ with copies of $C_4$ (the cycle of length 4), is an ordered triple $(V, C, L)$, where $V$ is the vertex set of the complete graph $K_n$, $C$ is a collection of edge-disjoint copies of $C_4$, and $L$ is the set of edges not belonging to a block of $C$. The number $n$ is called the order of the packing and the set of unused edges $L$ is called the leave. If $C$ is as large as possible, then $(V, C, L)$ is called a maximum packing $MPC(n, 4, 1)$. We say that a path design $P(v, k, 1) (W, P)$ is embedded into an $MPC(n, 4, 1) (V, C, L)$ if there is an injective mapping $f : P \rightarrow C$ such that $P$ is a subgraph of $f(P)$ for every $P \in P$. Let $SP(n, 4, k)$ denote the set of the integers $v$ such that there exists an $MPC(n, 4, 1)$ which embeds a $P(v, k, 1)$. If $n \equiv 1 \pmod{8}$ then an $MPC(n, 4, 1)$ coincides with a 4-cycle system of order $n$ and the related embedding problem is completely solved by Quattrocchi, *Discrete Math.*, 255 (2002).

The aim of the present paper is to determine $SP(n, 4, k)$ for every integer $n \not\equiv 1 \pmod{8}$, $n \geq 4$.

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1 Introduction

Let $G$ be a subgraph of $K_n$, the complete undirected graph on $n$ vertices. A $G$-design of $K_n$ is a pair $(V, B)$, where $V$ is the vertex set of $K_n$ and $B$ is an edge-disjoint decomposition

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of $K_n$ into copies of the graph $G$. If $B \in \mathcal{B}$ we say that $B$ is a block (or a $G$-block) of the $G$-design. The set $\mathcal{B}$ is called the block-set. A $G$-design of $K_n$ is also called a $G$-design of order $n$.

A path design $P(v, k, 1)$ [5] is a $P_k$-design of $K_v$, where $P_k$ is the simple path with $k-1$ edges ($k$ vertices) $[a_1, a_2, \ldots, a_k] = \{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}\}$. A $G$-design is balanced if each vertex belongs to the same number of blocks. Obviously not every $G$-design is balanced. An handcuffed design $H(v, k, 1)$ [4, 5] is a balanced $P_k$-design of order $v$ and block size $k$.

Clearly a $P(v, 2, 1) (V, \mathcal{P})$ exists for every $v \geq 2$ and it is always balanced. Hung and Mendelsohn [4] proved that an $H(v, 2h+1, 1) (h \geq 1)$ exists if and only if $v \equiv 1 \pmod{4h}$, and an $H(v, 2h, 1) (h \geq 2)$ exists if and only if $v \equiv 1 \pmod{2h-1}$. Tarsi [12] proved that the necessary conditions for the existence of a $P(v, k, 1)$, $v \geq k$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient.

A 4-cycle system of order $n$ is a $C_4$-design of $K_n$, where $C_4$ is the 4-cycle (cycle of length 4) $(a_1, a_2, a_3, a_4) = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}\}$. It is well-known [6] that the spectrum for 4-cycle systems is precisely the set of all $n \equiv 1 \pmod{8}$.

A packing of $K_n$ with copies of $C_4$ is an ordered triple $(V, \mathcal{C}, L)$, where $V$ is the vertex set of $K_n$, $\mathcal{C}$ is a collection of edge-disjoint copies of $C_4$, and $L$ is the set of edges not belonging to a block of $\mathcal{C}$. The number $n$ is called the order of the packing and the set of unused edges $L$ is called the leave. If $\mathcal{C}$ is as large as possible, then $(V, \mathcal{C}, L)$ is called a maximum packing $\text{MPC}(n, 4, 1)$.

An $\text{MPC}(n, 4, 1)$ can be considered as the natural generalization of a 4-cycle system of order $n$ when $n \not\equiv 1 \pmod{8}$. For, an $\text{MCP}(n, 4, 1)$ with $n \equiv 1 \pmod{8}$ coincides with a 4-cycle system of order $n$. The existence of an $\text{MPC}(n, 4, 1)$ has been proved by Schöneim and Bialostocki [11].

**Theorem 1.1** ([11]) *For every integer $n \geq 4$ there exists an $\text{MPC}(n, 4, 1) (V, \mathcal{B})$ and it is:*

- $|\mathcal{B}| = \left\lceil \frac{n}{4} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil$ if $n \not\equiv 5$ or $7 \pmod{8}$,
- $|\mathcal{B}| = \left\lceil \frac{n}{4} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil - 1$ otherwise.

*If $n \equiv 1 \pmod{8}$ then the leave $L$ does not contain any edge. If $n \not\equiv 1 \pmod{8}$ then the non-packed edges may be chosen so that the leave $L$ is isomorphic to a one-factor if $n$ is even, a 3-cycle if $n \equiv 3 \pmod{8}$, two 3-cycles having a common vertex if $n \equiv 5 \pmod{8}$ and a 5-cycle if $n \equiv 7 \pmod{8}$.*

Let $G_1$ be a subgraph of $G_2$. We say that a $G_1$-design $(V, \mathcal{P})$ of order $v$ is embedded into a $G_2$-design $(W, \mathcal{C})$ of order $n$ if there is an injective mapping $f : \mathcal{P} \rightarrow \mathcal{C}$ such that $P$ is a subgraph of $f(P)$ for every $P \in \mathcal{P}$. 

2
The following embedding problem arises: for every admissible integer \( n \) determine the set of the integers \( v \) such that there exists a \( G_2 \)-design of order \( n \) which embeds some \( G_1 \)-design of order \( v \).

This embedding problem has been investigated in many cases [1, 2, 3, 8, 10]. Quattrocchi [7] gives a complete answer to the embedding problem of a \( \mathcal{P}(v, k, 1) \) into a 4-cycle system of order \( n \).

In this paper we study the embedding problem of a \( \mathcal{P}(v, k, 1) \) into an \( \text{MPC}(n, 4, 1) \). This problem could be seen as the natural generalization of the embedding problem of a \( \mathcal{P}(v, k, 1) \) into a 4-cycle system of order \( n \).

**Definition 1.1** A \( \mathcal{P}(v, k, 1) \) is embedded into an \( \text{MPC}(n, 4, 1) \) if there is an injective mapping

\[
f : \mathcal{P} \to \mathcal{C}
\]

such that \( P \) is a subgraph of \( f(P) \) for every \( P \in \mathcal{P} \).

**Example 1.1** A \( \mathcal{P}(4, 3, 1) \) on vertex set \( V = \{0, 1, 2, 3\} \) embedded into an \( \text{MPC}(7, 4, 1) \) on vertex set \( W = V \cup \{5, 6, 7\} \). Let \( L = (4, 1, 5, 2, 6) \) and \( \mathcal{C} = \{(0, 1, 2, 4), (0, 2, 3, 5), (0, 3, 1, 6), (4, 5, 6, 3)\} \).

**Example 1.2** A \( \mathcal{P}(4, 3, 1) \) on vertex set \( V = \{1, 2, 3, 4\} \) embedded into an \( \text{MPC}(8, 4, 1) \) on vertex set \( W = V \cup \{5, 6, 7, 8\} \). Let \( L = \{[i, 4 + i] \mid i = 1, 2, 3, 4\} \) and \( \mathcal{C} = \{(1, 2, 3, 6), (1, 3, 4, 7), (1, 4, 2, 8), (5, 6, 7, 2), (5, 7, 8, 3), (5, 8, 6, 4)\} \).

**Example 1.3** A \( \mathcal{P}(8, 3, 1) \) on vertex set \( V = \{0, 1, \ldots, 7\} \) embedded into an \( \text{MPC}(13, 4, 1) \) on vertex set \( W = V \cup \{8, 9, \ldots, 12\} \). Let \( L = \{(9, 10, 8), (8, 11, 12)\} \) and \( \mathcal{C} = \{(0, 1, 2, 10), (0, 2, 3, 11), (0, 3, 1, 12), (4, 5, 6, 10), (4, 6, 7, 11), (4, 7, 5, 12), (4, 0, 5, 8), (4, 1, 5, 9), (6, 2, 7, 8), (6, 3, 7, 9), (0, 6, 1, 8), (0, 7, 1, 9), (2, 4, 3, 8), (2, 5, 3, 9), (11, 9, 12, 10), (10, 1, 11, 5), (11, 2, 12, 6), (10, 3, 12, 7)\} \).

**Example 1.4** A \( \mathcal{P}(12, 3, 1) \) on vertex set \( V = \{0, 1, \ldots, 11\} \) embedded into an \( \text{MPC}(19, 4, 1) \) on vertex set \( W = V \cup \{a_0, a_1, \ldots, a_6\} \). Let \( L = (a_0, a_1, a_2) \) and \( \mathcal{C} = \{(0, 1, 2, a_4), (0, 2, 3, a_5), (0, 3, 1, a_6), (4, 0, 5, a_2), (4, 1, 5, a_3), (6, 2, 7, a_2), (6, 3, 7, a_3), (0, 6, 1, a_0), (0, 7, 1, a_1), (2, 4, 3, a_0), (2, 5, 3, a_1), (4, 5, 6, a_4), (4, 6, 7, a_5), (4, 7, 5, a_6), (8, 0, 9, a_0), (8, 1, 9, a_1), (10, 2, 11, a_0), (10, 3, 11, a_1), (0, 10, 1, a_2), (0, 11, 1, a_3), (2, 8, 3, a_2), (2, 9, 3, a_3), (8, 9, 10, a_4), (8, 10, 11, a_5), (8, 11, 9, a_6), (8, 4, 9, a_2), (8, 5, 9, a_3), (10, 6, 11, a_2), (10, 7, 11, a_3), (4, 10, 5, a_0), (4, 11, 5, a_1), (6, 8, 7, a_0), (6, 9, 7, a_1), (a_0, a_3, a_1, a_4), (a_0, a_5, a_1, a_6), (a_2, a_3, a_4, a_5), (a_4, a_2, a_6, 3), (a_5, a_3, a_6, 2), (a_4, a_6, a_5, 1), (a_4, 5, a_5, 9), (a_5, 6, a_6, 10), (a_4, 7, a_6, 11)\} \).

**Definition 1.2** Denote by \( \mathcal{SP}(n, 4, k) \) the set of the integers \( v, v \geq k \), such that there exists a \( \mathcal{P}(v, k, 1) \) embedded into an \( \text{MPC}(n, 4, 1) \).

Quattrocchi [9] determined the spectrum \( \mathcal{SH}(n, 4, k), k = 2, 3 \), of the integers \( v \geq k \) such that there exists an handcuffed design of order \( v \) and block size \( k \) embedded into an \( \text{MPC}(n, 4, 1) \). The aim of the present paper is to determine \( \mathcal{SP}(n, 4, k) \) for every integer \( n \geq 4 \). We resume in the next two theorems the known results about \( \mathcal{SP}(n, 4, k) \) .
Theorem 1.2 ([7]) For every \( n \equiv 1 \pmod{8}, \ n \geq 9, \ SP(n, 4, 3) = \{v \mid 4 \leq v \leq \frac{2n-t(n)}{3}, \ v \equiv 0, 1 \pmod{4}\}, \) where \( t(n) = 2 \) if \( n \equiv 1 \pmod{24}, \ t(n) = 3 \) if \( n \equiv 9 \pmod{24} \) and \( t(n) = 7 \) if \( n \equiv 17 \pmod{24} \).

Theorem 1.3 ([9]) For every even integer \( n, \ n \geq 4, \ \cal{SH}(n, 4, 2) = \{v \mid 2 \leq v \leq \frac{n}{2}\}. \) For every odd \( n, \ n \geq 5, \ \cal{SH}(n, 4, 2) = \{v \mid 2 \leq v \leq \frac{n+1}{2}\}. \) Let

\[
\theta(n) = \begin{cases} 
\frac{n-6}{2} & \text{if } n \equiv 0 \pmod{8}, \ n \geq 16; \\
\frac{n}{2} & \text{if } n \equiv 2 \pmod{8}, \ n \geq 10; \\
\frac{n-2}{2} & \text{if } n \equiv 4 \pmod{8}, \ n \geq 12; \\
\frac{n-4}{2} & \text{if } n \equiv 6 \pmod{8}, \ n \geq 14; \\
\frac{2n-11}{3} & \text{if } n \equiv 1 \pmod{6}, \ n \geq 13; \\
\frac{2n-3}{3} & \text{if } n \equiv 3 \pmod{6}, \ n \geq 9; \\
\frac{2n-7}{3} & \text{if } n \equiv 5 \pmod{6}, \ n \geq 11.
\end{cases}
\]

Then for every integer \( n, \ n \geq 9, \ \cal{SH}(n, 4, 3) = \{v \mid 5 \leq v \leq \theta(n), \ v \equiv 1 \pmod{4}\}. \)

By Theorem 1.3, \( SP(n, 4, 2) = \cal{SH}(n, 4, 2) \) (note that every \( P(v, 2, 1) \) is balanced). Moreover \( \cal{SH}(n, 4, 3) \subseteq \cal{SP}(n, 4, 3) \). In next section we completely determine \( \cal{SP}(n, 4, 3) \).

2 \( \cal{SH}(n, 4, 3) \)

Let

\[
\tau(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \equiv 0, 2 \pmod{8}, \ n \geq 8; \\
\frac{n}{2} & \text{if } n \equiv 4 \pmod{8}, \ n \geq 12; \\
\frac{n-4}{2} & \text{if } n \equiv 6 \pmod{8}, \ n \geq 14; \\
\frac{2n-2}{3} & \text{if } n \equiv 1 \pmod{6}, \ n \geq 7; \\
\frac{2n-3}{3} & \text{if } n \equiv 3 \pmod{6}, \ n \geq 9; \\
\frac{2n-7}{3} & \text{if } n \equiv 5 \pmod{6}, \ n \geq 11.
\end{cases}
\]

Lemma 2.1 \( SP(n, 4, 3) \subseteq \{v \mid 4 \leq v \leq \tau(n), \ v \equiv 0, 1 \pmod{4}\}. \)

Proof. Let \( (V, W) \) be a \( P(v, 3, 1), \ v \geq 4, \) embedded into an \( MPC(n, 4, 1) \ (W, C, L). \)

Let \( n \) be even. The leave \( L \) is a one-factor not covering any edge of \( K_v. \) Then \( n \geq 2v \) and the proof follows from the condition \( v \equiv 0, 1 \pmod{4}. \)

Let \( n \) be odd. Then \( v(n-v) \geq \binom{v}{2} \) and the proof follows from Theorems 1.2, 1.3 and the condition \( v \equiv 0, 1 \pmod{4}. \)

Lemma 2.2 Let \( (W, C, L) \) be an \( MPC(n, 4, 1) \) with \( n \) even, \( n \geq 4. \) Then there exists an \( MPC(n + 2, 4, 1) \ (W, \bar{C}, \bar{L}) \) such that \( W \subset \bar{W}, \ C \subset \bar{C} \) and \( L \subset \bar{L}. \)

Proof. Let \( \bar{W} = W \cup \{\infty_1, \infty_2\}, \ \bar{L} = L \cup \{\infty_1, \infty_2\}. \) Let \( B \) be an edge-disjoint decomposition of the complete bipartite graph \( K_{\bar{W}, \{\infty_1, \infty_2\}} \) into subgraphs isomorphic to a 4-cycle. Let \( \bar{C} = C \cup B. \)

4
Lemma 2.3 Let \( v \equiv 0 \pmod{4} \). If \( v \in \mathcal{SP}(2v, 4, 3) \) then \( v + 4 \in \mathcal{SP}(2v + 8, 4, 3) \).

**Proof.** Let \((V, \mathcal{P})\) be an \( P(v, 3, 1) \) embedded into an \( MPC(2v, 4, 1) \) \((W, \mathcal{C}, L)\). Let \( v = 4k, k \geq 1 \). Let \( A_i = \{a^1_i, a^2_i, a^3_i, a^4_i\} \), \( B_i = \{b^1_i, b^2_i, b^3_i, b^4_i\} \), \( V = \bigcup_{i=1}^{k} A_i \), \( B = \bigcup_{i=1}^{k} B_i \), \( D = \bigcup_{i=1}^{k} \{b^3_i, b^4_i\} \), \( W = V \cup B \). Let \((X, \mathcal{D})\) be a \( P(4, 3, 1) \) embedded into an \( MPC(8, 4, 1) \) \((X \cup Y, B, L_1)\) with \( X = \{1, 2, 3, 4\}, Y = \{5, 6, 7, 8\} \) (see Example 1.2). In order to construct an \( MPC(2v + 8, 4, 1) \), put \( \overline{W} = W \cup X \cup Y \), \( \overline{L} = L \cup L_1 \) and assign to \( \overline{\mathcal{C}} \) the following 4-cycles:

- the blocks of \( \mathcal{C} \cup \mathcal{B} \);
- \((1, a^1_i, 2, b^1_i), (1, a^2_i, 2, b^2_i), (3, a^3_i, 4, b^3_i), (3, a^4_i, 4, b^4_i), (a^3_i, 1, a^4_i, 5), (a^3_i, 2, a^4_i, 6), (a^1_i, 3, a^2_i, 5), (a^1_i, 4, a^2_i, 6) \) for \( i = 1, 2, \ldots, k \);

\((\overline{W}, \overline{\mathcal{C}}, \overline{L})\) is an \( MPC(2v + 8, 4, 1) \) which embeds an \( H(v + 4, 3, 1) \) on vertex set \( V \cup \{1, 2, 3, 4\} \).

Lemma 2.4 Let \( n \equiv 3, 5, 7 \pmod{8} \) and \( v \equiv 0 \pmod{4} \). If \( v \in \mathcal{SP}(n, 4, 3) \) then \( v + 4 \in \mathcal{SP}(n + 4, 4, 3) \).

**Proof.** Let \((V, \mathcal{P})\) be a \( P(v, 3, 1) \) embedded into an \( MPC(n, 4, 1) \) \((V \cup W, \mathcal{C}, L)\). Let \( v = 4k, k \geq 1 \), \( A_i = \{a^1_i, a^2_i, a^3_i, a^4_i\} \), \( V = \bigcup_{i=1}^{k} A_i \), \( W = \{\infty\} \cup \{b^i_j \mid j = 1, 2, \ldots, n - 4k - 1\} \) (note that \( n - 4k - 1 \geq 2k \)). By Theorem 1.2, there exists a \( P(4, 3, 1) \) \((V_1, \mathcal{P}_1)\) embedded into a 4-cycle system of order 9 \((W_1, \mathcal{C}_1)\). Let \( V_1 = \{1, 2, 3, 4\} \) and \( W_1 = V_1 \cup \{\infty, 5, 6, 7, 8\} \). Now we construct an \( MPC(n + 8, 4, 1) \). Let \( \overline{V} = V \cup V_1 \), \( \overline{W} = W \cup \overline{V} \cup W_1 \), \( \overline{L} = L \). Assign to \( \overline{\mathcal{C}} \) the following 4-cycles:

- the blocks of \( \mathcal{C} \cup \mathcal{C}_1 \);
- \((1, a^1_i, 2, b^1_{2i-1}), (1, a^2_i, 2, b^2_{2i}), (3, a^3_i, 4, b^3_{2i-1}), (3, a^4_i, 4, b^4_{2i}), (a^3_i, 1, a^4_i, 5), (a^3_i, 2, a^4_i, 6), (a^1_i, 3, a^2_i, 5), (a^1_i, 4, a^2_i, 6) \) for \( i = 1, 2, \ldots, k \);

the blocks of an edge-disjoint decomposition of \( K_{A_i(7,8)}, K_{W \backslash \{\infty\}, \{5,6,7,8\}} \) and, if \( n - 4k - 1 > 2k \), \( K_{\{b_{2i+1}, b_{2i+2}, \ldots, b_{n-4k-1}\}, \{1,2,3,4\}} \), into 4-cycles.

It is easy to see that \((\overline{W}, \overline{\mathcal{C}}, \overline{L})\) is an \( MPC(n + 8, 4, 1) \) which embeds a \( P(v + 4, 3, 1) \) on vertex set \( \overline{V} \).

Lemma 2.5 Let \( n \equiv 3, 5, 7 \pmod{8} \) and \( v \equiv 0 \pmod{4} \). If \( v \in \mathcal{SP}(n, 4, 3) \) then \( v + 16 \in \mathcal{SP}(n + 24, 4, 3) \).
Proof. Let \((V, \mathcal{P})\) be a \(P(v, 3, 1)\) embedded into an \(MPC(n, 4, 1)\) \((WC, L)\). Let \(v = 4k, V = \{0, 1, \ldots, 4k - 1\}\) and \(W = V \cup \{4k, 4k + 1, \ldots, n - 1\}\) (note that \(n - 4k \geq 2k + 1\)).

Let \(A_i = \{a_i^1, a_i^2, a_i^3, a_i^4\}\), \(B_i = \{a_i^1, A_i, B = \cup_{i=1}^{\rho} B_i\} \text{ and } W = W \cup A \cup B\). In order to construct an \(MPC(n + 24, 4, 1)\) assign to \(C\) the following 4-cycles:

- the blocks of \(C\);
- \((a_i^1, a_i^2, a_i^3, 4k), (a_i^1, a_i^2, a_i^4, 4k + 1), (a_i^1, a_i^3, a_i^4, 4k + 2), (a_i^1, 0, a_i^2, a_i^4), (a_i^1, 1, a_i^2, a_i^4), (a_i^2, 0, a_i^4, a_i^1), (a_i^2, 3, a_i^4, 4k + 2), (2, a_i^3, 3, a_i^1), (2, a_i^3, 2, a_i^4), (0, a_i^3, 1, a_i^4), (0, a_i^3, 1, a_i^2)\) for every \(i = 1, 2, 3, 4\);
- \((a_i^1, a_i^2, a_i^3, 4k), (a_i^1, a_i^2, a_i^4, a_i^3), (a_i^2, a_i^3, a_i^4, a_i^1), (a_i^2, a_i^3, a_i^4, a_i^2), (a_i^3, a_i^4, a_i^1, a_i^2), (a_i^3, a_i^4, a_i^2, a_i^1)\) for every \(i, j = 1, 2, 3, 4, i < j\);
- if \(k \geq 2\) then, for every \(i = 1, 2, 3, 4, j = 2, 3, \ldots, k\), add the blocks of an edge-disjoint decomposition of \(K_{\{a_i^1, a_i^2\}}\) into 4-cycles and the following 4-cycles \((a_i^1, 4(j - 1), a_i^2, 4k + 2j - 1), (a_i^1, 1 + 4(j - 1), a_i^2, 4k + 2j), (a_i^1, 2 + 4(j - 1), a_i^2, 4k + 2j - 1), (a_i^1, 3 + 4(j - 1), a_i^2, 4k + 2j), (2 + 4(j - 1), a_i^4, 3 + 4(j - 1), a_i^1)\);
- if \(n > 6k + 1\) then, for every \(i = 1, 2, 3, 4\), add the blocks of an edge-disjoint decomposition into 4-cycles of \(K_{\{6k + 1, 6k + 2, \ldots, n - 1\}}\) and \(K_{\{6k + 1, 6k + 2, \ldots, n - 1\}}\) \(\{a_i^1, a_i^2\}\);

\[
(a_i^1, a_i^2, a_i^3, a_i^4), (a_i^1, a_i^3, a_i^2, a_i^4), (a_i^2, a_i^4, a_i^1, a_i^3), (a_i^2, a_i^4, a_i^1, a_i^3), (a_i^3, a_i^4, a_i^1, a_i^2), (a_i^3, a_i^4, a_i^1, a_i^2), (4k, a_i^4, a_i^1, a_i^3), (4k, a_i^4, a_i^1, a_i^3), (4k + 1, a_i^3, a_i^2, a_i^1), (4k + 1, a_i^3, a_i^2, a_i^1), (4k + 1, a_i^3, a_i^2, a_i^1), (4k + 1, a_i^3, a_i^2, a_i^1). 
\]

It is easy to check that \((W, C, L)\) is an \(MPC(n + 24, 4, 1)\) which embeds a \(P(v + 16, 3, 1)\) on vertex set \(V \cup A\).

Remark 2.1 If in Lemma 2.5 we suppose that the path design \((V, \mathcal{P})\) embeds a \(P(4\rho, 3, 1)\) \((V, \mathcal{P})\) for every \(\rho = 1, 2, \ldots, \frac{v - 4}{4}\), then the produced \(P(v + 16, 3, 1)\) on vertex set \(V \cup A\) embeds a path design of order \(4\rho\) for every \(\rho = 1, 2, \ldots, \frac{v - 12}{4}\).

Lemma 2.6 Let \(n \neq 1 \pmod{8}, n \geq 7\). Then

- \(\{v \mid 4 \leq v \leq \frac{n - 2u}{2}, v \equiv 0 \pmod{4}\} \subseteq \mathcal{SP}(n, 4, 3)\) if \(n \equiv 2\mu \pmod{8}\), \(\mu = 0, 1, 2, 3\);
- \(\{v \mid 4 \leq v \leq \frac{2n - 6}{3}, v \equiv 0 \pmod{4}\} \subseteq \mathcal{SP}(n, 4, 3)\) if \(n \equiv 3, 15, 21 \pmod{24}\);
- \(\{v \mid 4 \leq v \leq \frac{2n - 10}{3}, v \equiv 0 \pmod{4}\} \subseteq \mathcal{SP}(n, 4, 3)\) if \(n \equiv 5, 11, 23 \pmod{24}\);
- \(\{v \mid 4 \leq v \leq \frac{2n - 2}{3}, v \equiv 0 \pmod{4}\} \subseteq \mathcal{SP}(n, 4, 3)\) if \(n \equiv 7, 13, 19 \pmod{24}\).
Proof. If \( n \) is even then apply Lemmas 2.2 and 2.3 to the Example 1.2. If \( n \equiv 3, 5, 7 \) (mod 8) then apply Lemmas 2.4, 2.5 and Remark 2.1 to the Examples 1.1, 1.3 and 1.4.

From Theorems 1.2, 1.3 and Lemmas 2.1, 2.6 it follows our main result.

**Theorem 2.1**

\[ \mathcal{S}\mathcal{P}(n, 4, 3) = \{ v \mid 4 \leq v \leq \tau(n), v \equiv 0, 1 \text{ (mod 4)} \} \text{ for every integer } n \geq 7. \]

### 3 Concluding remark

Let \((V, C, L)\) be an \(MPC(n, 4, 1)\) with \(n \not\equiv 1 \text{ (mod 8)}\). Clearly \((V, C, L)\) induces an edge-disjoint graph decomposition \((V, C \cup L)\) of \(K_n\) into

- \( \frac{n(n-2)}{8} \) copies of \(C_4\) and \( \frac{n}{2} \) copies of \(P_{2n}\) if \(n \equiv 0 \text{ (mod 2)}\);
- \( \frac{1}{3} \left( \binom{n}{2} - 3 \right) \) copies of \(C_4\) and one copy of \(C_5\) if \(n \equiv 3 \text{ (mod 8)}\);
- \( \frac{1}{3} \left( \binom{n}{2} - 6 \right) \) copies of \(C_4\) and either two copies of \(C_3\) or one copy of a bowtie \(B (V(B) = \{a_0, a_1, \ldots, a_4\}, E(B) = \{\{a_0, a_1\}, \{a_1, a_2\}, \{a_0, a_2\}, \{a_0, a_3\}, \{a_3, a_4\}, \{a_0, a_4\}\}) \)
  if \(n \equiv 5 \text{ (mod 8)}\);
- \( \frac{1}{3} \left( \binom{n}{2} - 5 \right) \) copies of \(C_4\) and one copy of \(C_5\) if \(n \equiv 7 \text{ (mod 8)}\).

It is possible to change the embedding definition given in Definition 1.1 into the following definition.

**Definition 3.1**

A \(P(v, k, 1) (V, \mathcal{P})\) is embedded into the graph decomposition induced by an \(MPC(n, 4, 1) (W, C, L)\) if there is an injective mapping

\[ f : \mathcal{P} \rightarrow C \cup L \]

such that \(P\) is a subgraph of \(f(P)\) for every \(P \in \mathcal{P}\).

**Example 3.1**

A \(P(7, 3, 1)\) on vertex set \(V = \{0, 1, \ldots, 4\}\) embedded into the graph decomposition induced by an \(MPC(7, 4, 1)\) on vertex set \(W = V \cup \{a_0, a_1\}\) where \(L = (0, 1, 4, a_1, a_0), C = \{(1, 2, 0, a_1), (2, 3, 1, a_0), (3, 4, 2, a_1), (4, 0, 3, a_0)\}\).

The related embedding problem is to determine the set \(\mathcal{I}\mathcal{S}(n, 4, k)\) of the integers \(v\) such that there exists a \(P(v, k, 1)\) embedded into the graph decomposition induced by an \(MPC(n, 4, 1)\). Next theorem gives a complete solution to this problem.

**Theorem 3.1**

\(\mathcal{I}\mathcal{S}(n, 4, 2) = \mathcal{S}\mathcal{P}(n, 4, 2)\) for every integer \(n \geq 4\). \(\mathcal{I}\mathcal{S}(n, 4, 3) = \mathcal{S}\mathcal{P}(n, 4, 3)\) for every integer \(n \geq 8\). \(\mathcal{I}\mathcal{S}(7, 4, 3) = \{4, 5\}\).
Proof. Clearly \( \mathcal{SP}(n, 4, k) \subseteq \mathcal{IS}(n, 4, k) \). Let \((V, \mathcal{P})\) be a \( P(v, k, 1) \) embedded into the graph decomposition induced by an \( MPC(n, 4, 1) \) \((W, \mathcal{C}, L)\). Denote by \( E(G) \) the edge set of the graph \( G \).

Case \( k = 2 \) and \( n \equiv 0 \pmod{2} \). Suppose \(|E(L) \cap E(K_V)| = \mu, 0 \leq \mu \leq n/2 - 2\). Then \(|E(L) \cap E(K_{V,W})| = v - 2\mu\) and \(|E(L) \cap E(K_{W,V})| = \frac{n}{2} - v + \mu\). Let \( \mathcal{D} \) denote the set of blocks \( C \in \mathcal{C} \) such that \(|E(C) \cap E(K_V)| = 1\). Since \(|E(C) \cap E(K_{W,V})| = 1\) for every \( C \in \mathcal{D} \), \( \binom{n-v}{2} = |E(K_{W,V})| \geq |\mathcal{D}| + |E(L) \cap E(K_{W,V})| \geq \binom{v}{2} - v\). So \( n \geq 2v \) and, by \( v \equiv 0, 1 \pmod{4}, \mathcal{IS}(n, 4, 2) \subseteq \mathcal{SP}(n, 4, 2) \).

Case \( k = 3 \). Let \( \mathcal{L} \) be the set of \( P \in \mathcal{P} \) such that \( P \) is a subgraph of \( L \). It is easy to see that \( |\mathcal{L}| = 0 \). If \(|\mathcal{L}| = 0\), then \( \mathcal{IS}(n, 4, 3) \subseteq \mathcal{SP}(n, 4, 3) \). Let \(|\mathcal{L}| = 1\). Then \( n \equiv 5, 7 \pmod{8} \) and \( L \) is a bowtie for \( n \equiv 5 \pmod{8} \) and a \( C_5 \) for \( n \equiv 7 \pmod{8} \). It is easy to see that \(|E(K_{V,W})| \geq 2(|\mathcal{P}| - 1) + |E(L) \cap E(K_{V,W})|\) and \(|E(L) \cap E(K_{W,V})| = x\) with \( x = 2\) if \( L \) is a \( C_5 \) or \( x = 4\) if \( L \) is a bowtie. We obtain \( v \leq \frac{2n+1}{2} \). Suppose \( v = \frac{2n+1}{2} \). Let \( \mathcal{D} \) denote the set of blocks of \( \mathcal{C} \) having a path of \( \mathcal{P} \) as a subgraph. Then \((W \setminus V, \mathcal{C} \setminus \mathcal{D}, E(L) \cap E(K_{W,V}))\) is a packing of \( K_{W,V} \). Since \(|E(L) \cap E(K_{W,V})| = 1\), then \(|\mathcal{C} \setminus \mathcal{D}| = 0\) and \( n - v = |W \setminus V| = 2\). It follows \( n = 7 \) and \( v = 5\). So, for \( n \equiv 5, 7 \pmod{8} \) and \( n \neq 7\), \( v < \frac{2n+1}{2} \) and, by \( v \equiv 0, 1 \pmod{4}, \mathcal{IS}(n, 4, 2) \subseteq \mathcal{SP}(n, 4, 2) \). Examples 1.1 and 3.1 imply \( \mathcal{IS}(7, 4, 3) = \{4, 5\} \).

References


