On the Chromatic Index of Path Decompositions

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Abstract

If \( G = (V', E) \) is a graph and \( H = (V, H') \) is a graph whose edges can be decomposed into isomorphic copies of \( G \), then we define a \( k \)-block colouring of a \( G \)-decomposition of \( H \) to be an assignment of \( k \) colours to the copies of \( G \) so that no two copies of \( G \) having a vertex in common have the same colour. A \( G \)-decomposition of \( H \) has chromatic index \( \chi' \) if it is \( k \) block colourable and not \( k - 1 \) block colourable. We use the techniques of \( G \)-resolvable designs and \( G \)-frames to solve the Minimal Chromatic Index Problem: Given \( H \) determine the minimum \( \chi'(D) \) and exhibit a \( D \) that achieves this minimum, for the cases where \( H \) the complete graph on \( n \) vertices and \( G \) is the path of length 2 or 3.

1 Introduction

If \( G = (V', E) \) is a graph and \( H = (V, H') \) is a graph whose edges can be decomposed into isomorphic copies of \( G \), such a decomposition requires that \( |E| \)

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divides $|H|$, then we define a $k$-block colouring of a $G$–decomposition of $H$ to be an assignment of $k$ colours to the copies of $G$ so that no two copies of $G$ having a vertex in common have the same colour. A $G$–decomposition of $H$ has chromatic index $\chi' = k$, if it is $k$ block colourable and not $k-1$ block colourable. These kinds of colourings for general graphs have been studied by Brown and Corneli [2].

A special case which has been widely studied is when $|V'|$ divides $|V|$, and the colour classes form ‘resolutions’ of $V$, and $|V|$ divides $|H|$. When both $G$ and $H$ are complete graphs the problem goes back to Kirkman’s Schoolgirl problem [10]. The case where $G$ is a path of length $n$ and $H = K_m$ is a special case of the Oberwolfach problem [11], called the resolvable $m$-cycle system problem [12].

The case where $G$ is a cycle of length $n$ and $H = K_n$ is the Oberwolfach problem for $v = kn$ and it is the Walke problem for $v = kn + 1$, which is still open as far as we know (this nomenclature is explained in [13] with reference to the Dundas index). The term near-Oberwolfach refers to $C_n$–decompositions of the cocktail party graph. ($K_n \setminus$ a one – factor) [15].

When $G$ is not regular or when $|V(G)|$ does not divide $|E(G)|$, the problem is substantially more difficult; paths have both these difficulties. Denote by $P_k$ the path of length $k - 1$ ($k$ vertices). What we intend to do is give a technique for finding the chromatic index of a $P_3$–design and a $P_4$–design as a paradigm for solving the chromatic index for various graphs.

**Definition 1.**

1. A $G$–design $G = (v, \lambda)$, is a pair $(V, B)$ where $V$ is a $v$-set and $B$ is a collection of isomorphic copies of the graph $G$, called blocks, which partition the edges of $\lambda K_v$ on the vertex set $V$.

2. A $G$–design $G = (v, \lambda)$ with a hole of size $w$, is a triple $(V, W, B)$ where $V$ is a $v$-set, $W \subseteq V$ is a $w$-set and $B$ is a collection of isomorphic copies of the graph $G$, called blocks, which partition the edges of $\lambda (K_v \setminus K_w)$ on the vertex set $V$. 

The Minimal Chromatic Index Problem

Given $H$ determine

$$\min \{ \chi'(D) \mid D \text{ is a } G \text{– decomposition of } H \}$$

and exhibit a $D$ that achieves this minimum.

For $G = K_3$ and $H = K_v$, a $K_3$–decomposition with a minimum chromatic index is a Resolvable Triple System for $v = 6k + 3$ and a Hanani Triple System for $v = 6k + 1$ [4, 18]. When $G = C_n$ (the $n$-cycle), and $H = K_n$, it is the Oberwolfach problem for $v = kn$ and it is the Walke problem for $v = kn + 1$, which is still open as far as we know (this nomenclature is explained in [13] with reference to the Dundas index). The term near-Oberwolfach refers to $C_n$–decompositions of the cocktail party graph. ($K_n \setminus$ a one – factor) [15].

When $G$ is not regular or when $|V(G)|$ does not divide $|E(G)|$, the problem is substantially more difficult; paths have both these difficulties. Denote by $P_k$ the path of length $k - 1$ ($k$ vertices). What we intend to do is give a technique for finding the chromatic index of a $P_3$–design and a $P_4$–design as a paradigm for solving the chromatic index for various graphs.
3. A $G$−GDD (Group Divisible Design) is a triple $(V, G, B)$, where $V$ is a finite set, $G = \{g_1, \ldots, g_n\}$ is a partition of $V$ into subsets, the elements of $G$ are called groups, and $B$ is a collection of isomorphic copies of $G$, called blocks, which partition the edges of $\lambda K_{g_1, \ldots, g_n}$, on the vertex set $V$. 

If for $i = 1, \ldots, t$, there are $k_i$ groups of size $n_i$ we say that the $G$−GDD is of type $n_i^{k_i}$, $i = 1, \cdots, t$.

Thus a $G$-design $G - (v, \lambda)$ is a $G$-decomposition of $\lambda K_v$, and a $G$−GDD is a $G$-decomposition of $\lambda K_{g_1, \ldots, g_n}$. Define $\alpha_v = \lfloor \frac{v}{|\alpha_v|} \rfloor$.

**Definition 2.**

1. A near-parallel class of a $G$-design $G - (v, \lambda)$ is a collection of $\alpha_v$ vertex disjoint blocks.

2. If $v \equiv 0 \pmod{V(G)}$ a near-parallel class is called a parallel class.

3. A partial parallel class is a collection of $n \leq \alpha_v$ vertex disjoint blocks.

**Definition 3.**

1. A $G$-design is $n$ block colourable if its blocks can be partitioned into $n$ partial parallel classes.

2. A $G$-design has chromatic index $\chi'$ if it is $\chi'$ (but not $\chi' - 1$) block colourable.

A simple counting argument shows that the minimum possible $\chi'$ colouring of a $G$-design $G - (v, \lambda)$ will partition the block set into $\lfloor \frac{w(v-1)}{2|E(G)|\alpha_v} \rfloor$ near-parallel classes and possibly one extra partial parallel class, called the leftover class; we call such a design a nearly resolvable $G$-design.

**Definition 4.** An $(n, m)$ block colouring of a $G$-design $G - (v, \lambda)$ with a hole of size $w$, is a partition of the block set into $n$ partial-parallel classes of $V$ and $m$ partial parallel classes of $V \setminus W$.

Given a $(n, m)$ block colouring of a $G$-design $G - (v, \lambda)$ with a hole of size $w$, $A$ say, and an $m$ block colouring of a $G$-design $G - (w, \lambda)$, $B$ say, we may ‘fill in the hole’ of size $w$ in $A$ with a copy of $B$ and associate the $m$ short classes of $A$ with the classes of $B$ (the leftover classes of each are also associated) to obtain an $n + m$ colouring of a $G$-design $G - (v, \lambda)$. We call this an $n + m$ block colourable $G$-design $G - (v, \lambda)$ with an $m$ block colourable subdesign on $w$ points. In particular if $m = \lfloor \frac{w(w-1)}{2|E(G)|\alpha_w} \rfloor$, together with a possible leftover class, and

$$n = \left\lfloor \frac{1}{\alpha_v} \left( \frac{(v-w)(v+w-1)}{2|E(G)|} \right) \alpha_v - \alpha_v \left( \frac{w(w-1)}{2|E(G)|\alpha_w} \right) \right\rfloor$$

we obtain the minimal $\lfloor \frac{w(v-1)}{2|E(G)|\alpha_v} \rfloor$ colouring of a $G$-design, together with a possible leftover class.
Definition 5.

1. A $G^{-}$\text{-}GDD$_\lambda$ is a resolvable $G^{-}$\text{-}GDD$_\lambda$ if the blocks may be partitioned into parallel classes.

2. A $G^{-}$\text{-}GDD$_\lambda$, $(V, G, B)$, is a $G^{-}$\text{-}frame$_\lambda$ if its blocks can be partitioned into partial parallel classes, each of which is a resolution of $V \setminus g_i$ for some group $g_i \in G$. These partial parallel classes are called holey classes with hole $g_i$.

We now concentrate on the case where $G = P_k$, $k = 3$ or 4. We note that when $\alpha_v \left| \frac{v(v-1)}{2|V(G)|} \right|$ the resolvable case ($v \equiv 0 \text{ mod } |V(G)|$) has been solved in [1, 7] and the almost resolvable case ($v \equiv 1 \text{ mod } |V(G)|$) has been solved in [16]. For $P_3$ and $P_4$ these correspond to the cases $v \equiv 4, 9 \text{ mod } 12$. We include these cases in our exposition, partly to provide an alternate proof, but mostly to show how the ‘frame’ methods apply to these cases. For the remainder of this paper $\lambda = 1$ and will be omitted.

We now introduce the required building blocks for the construction.

Lemma 6 ([6, 7]).

1. If $k$ is even, then a $K_k$ parallel class is the union of $\frac{k}{2} P_k$ parallel classes [6].

2. If $k$ is odd, then a pair of $K_k$ parallel classes on the same vertex-set and disjoint edge-sets, is the union of $k P_k$ parallel classes [7].

We will use the following special case of the resolvable $P_k$-GDDs, found by Yu [17]. We note that these can also be found by applying Lemma 6 to a resolvable $K_3$-GDD, which may be found in [14].

Theorem 7 ([17]).

1. For $t \geq 2$ there is a resolvable $P_3$-GDD of type $12^t$ with $9(t - 1)$ parallel classes.

2. For $t \geq 2$ there is a resolvable $P_4$-GDD of type $12^t$ with $8(t - 1)$ parallel classes.

Theorem 8.

1. For $t \geq 3$ there is a $P_3$-frame of type $12^t$ with $9t$ holey parallel classes, 9 of which miss each group.

2. For $t \geq 3$ there is a $P_4$-frame of type $12^t$ with $8t$ holey parallel classes, 8 of which miss each group.

Proof. The Handbook of Combinatorial Designs [14] (and its online updates <www.emba.uwm.edu/~dinitz/newresults.html>) give $K_3$-frames of type $12^t$ for all $t > 3$ and $K_4$-frames of type $12^t$ for all $t > 4$. We may use Lemma 6
with \( k = 3 \) and 4 to decompose the \( K_k \)'s and obtain \( P_k \)-frames of type 12\( t \) for all \( t > k \).

A \( P_k \)-frame of type 12\( k \) may be obtained by taking a resolvable \( P_k \)-GDD of type 12\( 2 \) on each pair of groups in turn.

We present a \( P_4 \)-frame of type 12\( 4 \), with point set \( \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \), groups \( \mathbb{Z}_3 \times \mathbb{Z}_4 \times \{ i \} \), \( i \in \mathbb{Z}_4 \).

Develop the following 8 classes mod (−, 4, −):

\[
\| (031)(012)(023)(001), (021)(133)(232)(113), (011)(102)(203)(122) \mod (3, -,-) \|
\]
\[
\| (032)(233)(001)(202), (003)(211)(012)(121), (033)(231)(013)(222) \mod (3, -,-) \|
\]
\[
\| (033)(032)(030)(212), (010)(103)(220)(022), (002)(113)(000)(123) \mod (3, -,-) \|
\]
\[
\| (033)(000)(122)(130), (013)(010)(202)(223), (020)(112)(103)(232) \mod (3, -,-) \|
\]
\[
\| (013)(000)(221)(223), (033)(230)(103)(201), (010)(111)(020)(131) \mod (3, -,-) \|
\]
\[
\| (013)(001)(033)(010), (000)(211)(103)(231), (030)(223)(020)(221) \mod (3, -,-) \|
\]
\[
\| (020)(011)(022)(010), (012)(030)(001)(000), (021)(202)(031)(032) \mod (3, -,-) \|
\]
\[
\| (021)(000)(212)(010), (032)(220)(111)(030), (001)(122)(131)(202) \mod (3, -,-) \|
\]

\[
\square
\]

2 Constructions and Exceptional values

2.1 Constructions

We outline here the main constructions which we shall use. These are adaptations to graph decomposition of standard frame and resolvable design techniques [14].

The RGDD Construction. We begin with a \( G \)-RGDD of type \( g^t \) with \( r \) parallel classes and an \( n \) coloured \( G \)-design on \( g \) points, \( (V, \mathcal{B}) \), with colour classes \( R_i \) and leftover class \( L \). We construct a \( G \)-design on \( gt \) points as follows:

1. We place an isomorphic copy of \( (V, \mathcal{B}) \) on each group \( g_j \) calling the colour classes \( R_{ij} \) and the leftover class \( L_j \).

2. The resulting colour classes are
   
   (a) the resolution classes of \( G \)-RGDD and
   
   (b) new colour classes \( \bigcup_{j=1}^r R_{ij} \)
   
   (c) the leftover class \( \bigcup_{j=1}^r L_j \).

The result is a \( n + r \) colouring of a \( G \)-design on \( gt \) points, together with a possible leftover class.

The Frame Construction. We begin with

- A \( G \)-frame of type \( g^t \) with \( n \) parallel classes missing each group.
• An \((n, m)\) colourable \(G-\)design on \(g + k\) points with a hole of size \(k\), \((V, W, B')\), with \(n\) near parallel classes, \(P_i\), of \(V\) and \(m\) near parallel classes, \(Q_i\), of \(V \setminus W\) and a possible leftover class \(L\).

• An \(n + m\) colourable \(G-\)design on \(g + k\) points \((V, B)\) with colour classes \(R_i\) and a possible leftover class \(M\).

We also require that the blocks of \(L\) contain no points of the hole. Note that the existence of an \(n + m\) block colourable \(G-\)design on \(g + k\) points \((V, B)\) with an \(m\) colourable subdesign on \(k\) points gives both the latter two objects. We construct a \(G-\)design on \(gt + k\) points as follows:

1. The vertex set is the vertices of the frame together with \(k\) infinite points \(\infty = \{\infty_1, \infty_2, \ldots, \infty_k\}\).
2. On the first group \(g_1 \cup \infty\) we place a copy of the \(n + m\) colourable \(G-\)design on \(g + k\) points with colour classes \(R_i\), \(i = 1, \ldots, n+m\).
3. On the each of the remaining groups \(g_j \cup \infty\), \(j = 2, \ldots, t\), we place a copy of a resolvable \(G-\)design on \(g + k\) points with a hole of size \(k\), placing the hole over the points \(\infty\). We call the resulting colour classes \(P_{ji}\), \(i = 1, \ldots, n\), \(Q_{ji}\), \(i = 1, \ldots, m\) and the leftover class \(L_j\).
4. If \(F_{ji}, i = 1, \ldots, n, j = 1, \ldots, t\), are the \(n\) holey classes of the frame with hole \(g_j\), then the colour classes the new design are:
   \(a\) \(F_{ji} \cup R_i\), \(i = 1, \ldots, n\).
   \(b\) \(F_{ji} \cup P_{ji}\), \(i = 1, \ldots, n, j = 2, \ldots, t\).
   \(c\) \(R_{i+n} \cup \bigcup_{j=2}^{t} Q_{ji}\), \(i = 1, \ldots, m\).
   \(d\) The leftover class \(M \cup \bigcup_{j=2}^{t} L_j\).

The result is an \(nt + m\) colouring of a \(G-\)design on \(gt + k\) points together with a possible leftover class.

**Hill Climbing with Divide and Conquer**

**Hill Climbing** We may find small cases by using a pseudo hill climbing algorithm for computer searches.

We start with the block set \(B = \emptyset\) and the resolution classes, \(R_i = \emptyset\). A resolution class is live if it does not contain \(\alpha_0\) blocks, the leftover class is live if it does not contain the number of blocks in the leftover class. We proceed as follows:

While there is a live class

Randomly pick a live class \(R\)

Randomly pick a point \(x_0 \notin R\)

For \(i = 1\) to \(i < k\)

Randomly pick \(x_i\) such that Either the pair \(x_{i-1}x_i \notin B\) OR \(x_i \notin R\)

If placing the block \(x_0x_1 \ldots x_k\) conflicts with up to one other
block

\[ B \leftarrow B \cup x_0 x_1 \ldots x_k, \text{ deleting the conflicting block if any} \]
\[ R \leftarrow R \cup x_0 x_1 \ldots x_k \]

If we do not find the design within a certain number of tries we start again. This algorithm works reasonably well for small \( k \), but the running time increases dramatically as \( k \) increases, as there will often be more than one conflicting block.

**Divide and Conquer** For a given \( k \) the running time increases as the number of blocks to be found increases, if this is too large it may be possible to reduce this number by dividing the problem into two problems on about half the number of blocks and piecing together the results. This can of course be done recursively and could be cleverly worked into a single a pseudo hill climbing algorithm. Though we did not need to do this for the cases we considered, our approach to the case \( P_4, v = 34 \) illustrates the idea:

1. Hill climb to find a \( P_4 \)-decomposition of \( K_{17} \) missing one edge,
2. Hill climb again to find a \( P_4 \)-decomposition of \( K_{17,17} \) missing one edge,
3. Apply the RGDD construction, ensuring that the missing edges are adjacent, so this path becomes the leftover block. See the Appendix for details.

### 2.2 Chromatic index tables

We give two tables showing the parameters associated with \( P_k \) decompositions, \( k = 3, 4 \), we include the smallest value the chromatic index could be.

With these values we may use the \( P_k \)-GDD’s and \( P_k \)-frames from Theorem 7 and 8 in the constructions above to obtain the entire spectrum. It remains to find the ingredient designs for \( P_3 \) and \( P_4 \), which we do in the next section.

<table>
<thead>
<tr>
<th>( v )</th>
<th>blocks per class</th>
<th>( # ) near ( \parallel ) classes</th>
<th>( # ) of blocks in leftover class</th>
<th>expected ( \chi' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12t+</td>
<td>4t+</td>
<td>9t+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>( t )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( 3t )</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>4</td>
<td>( 2t + 1 )</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>6</td>
<td>( 3t + 2 )</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>
Table 2: Values for $P_4$

<table>
<thead>
<tr>
<th>v=</th>
<th>blocks per class</th>
<th># near $\parallel$ classes</th>
<th># of blocks in leftover class</th>
<th>expected $\chi'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12t+</td>
<td>3t+</td>
<td>8t+</td>
<td>leftover class</td>
<td>leftover class</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2t</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>$t+1$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>4</td>
<td>2t+1</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>7</td>
<td>$t+1$</td>
<td>8</td>
</tr>
</tbody>
</table>

2.3 Ingredients for $P_3$

When denoting a path we shall omit the commas and brackets, and we enclose the paths of partial parallel classes between $\parallel$, for example

$\parallel 012,345,678,9ab \parallel$

- $v \equiv 0 \pmod{12}$.

We use the RGDD Construction, filling in the groups of a $P_3$–RGDD of type $12^t$ with a nearly resolvable $P_3$–design on 12 points with one block in the leftover class. The resulting $t$ disjoint blocks form the leftover partial parallel class of the result.

Resolution classes of a nearly resolvable $P_3$–design on 12 points:

$\parallel 314,526,709,ba8 \parallel$, $\parallel 516,24a,3b0,798 \parallel$, $\parallel 173,28b,546,0a9 \parallel$

$\parallel 183,275,499,06a \parallel$, $\parallel 193,204,658,a7b \parallel$, $\parallel 103,294,a5b,768 \parallel$

$\parallel 1a2,435,96b,780 \parallel$, $\parallel 1b2,63a,748,950 \parallel$

Leftover = \{123\}

- $v \equiv 1 \pmod{12}$.

We use the Frame Construction, filling in a $P_3$–frame of type $12^t$ using the following nearly resolvable $P_3$–design on 13 points with a hole of size one. $V = \{0,1,\ldots,9,a,b,\infty\}$, $\infty$ is the hole and the leftover class contains three blocks, none of which contain $\infty$. We pair up these nine classes with the 9 frame parallel classes that miss the particular group. The 3t disjoint blocks form the leftover partial parallel class of the result.

Resolution classes of the required nearly resolvable $P_3$–design:

$\parallel 143,526,09\infty,ba8 \parallel$, $\parallel 516,2\infty4,603,798 \parallel$, $\parallel 173,28b,546,0a\infty \parallel$

$\parallel 183,275,499,\infty6a \parallel$, $\parallel 931,204,5\infty8,a7b \parallel$, $\parallel \infty10,294,a5b,768 \parallel$

$\parallel a21,\infty35,96b,780 \parallel$, $\parallel \infty b2,63a,748,950 \parallel$, $\parallel 3b2,658,7\infty0,a19 \parallel$

Leftover = \{4a9,607,3b1\}
• $v \equiv 4 \pmod{12}$.

We start with a $P_3$-frame of type $12^f$ and apply the frame construction to create a nearly resolvable $P_3$-design on $12t + 4$ points. We require a nearly resolvable $P_3$-design on 16 points with a subdesign of size 4, given below, the point set is $V \cup W$, where $V = \{0, 1, \ldots, 9, a, b\}$ and $W = \{\infty_0, \infty_1, \infty_2, \infty_3\}$. The first three classes contain the subdesign, the first block from each. We obtain 3 classes by taking the union of these classes across the groups. For each group of the frame, the remaining 9 classes are paired with the 9 classes from the frame missing that group. The result is a nearly resolvable $P_3$-design on $12t + 4$ points with $9t + 3$ classes.

\[
\begin{array}{c}
|| \infty_0 \infty_2 \infty_3, 204, 715, ab8, 936 || & || \infty_0 \infty_1 \infty_3, 653, 09a, 814, b72 || \\
|| \infty_2 \infty_0 \infty_3, 705, 486, 962, 3a1 || & || 0 \infty_0 \infty_1, 38, 24 \infty_2, 75a, 896 || \\
|| 82 \infty_0, \infty_1 54, \infty_7 76, a \infty_3 9, 01b || & || 8 \infty_0 5, a \infty_1 4, \infty_2 92, 7 \infty_3 3, 6a0 || \\
|| 4 \infty_c 7, 8 \infty_9 9, a \infty_c 2b, 0 \infty_c 1, 325 || & || 0 \infty_9 7, a \infty_1 1, 3 \infty_c 8, 4 \infty_c 3b, 260 || \\
|| 2 a \infty_0, \infty_2 0 \infty_1, 837, 6b4, \infty_c 59 || & || 2 \infty_2 5, \infty_1 7a, 08 \infty_3, \infty_9 61, 943 || \\
|| \infty_c b0, \infty_2 64, 78a, 913, \infty_c 2 \infty_3 || & || \infty_0 30, \infty_2 12, \infty_3 6 \infty_1, a 47, b 58 || \\
\end{array}
\]

• $v \equiv 5 \pmod{12}$.

We start with a $P_3$-frame of type $12^f$ and apply the frame construction to create a nearly resolvable $P_3$-design with $12t + 5$ points. We require a nearly resolvable $P_3$-design on 17 points with a subdesign of size 5, given below. The point set is $V \cup W$, where $V = \{0, 1, \ldots, 9, a, b\}$ and $W = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. The first four classes contain the subdesign, the first block from each. We obtain 4 classes by taking the union of these classes across the groups. For each group of the frame, the remaining 9 classes are paired with the 9 classes from the frame missing that group. The leftover class comes from the union of the leftover classes across the groups. The result is a nearly resolvable $P_3$-design on $12t + 5$ points with $9t + 4$ classes and a leftover class of size $2t + 1$.

\[
\begin{array}{c}
|| \infty_1 \infty_2 \infty_0, 314, 526, 709, ba8 || & || \infty_2 \infty_3 \infty_1, 516, 24a, 360, 798 || \\
|| \infty_3 \infty_4 \infty_2, 173, 28b, 546, 0a9 || & || \infty_4 \infty_0 \infty_3, 183, 275, 469, 062 || \\
|| b \infty_0 4, 5 \infty_0 1, 03 \infty_2, 8 \infty_3 1, 7 \infty_4 6 || & || \infty_0 2a, 4 \infty_1 1, 6 \infty_2 7, 9 \infty_3 3, \infty_4 5b || \\
|| \infty_0 80, 5 \infty_0 1 6, 4 \infty_0 2 9, b7 \infty_3, a \infty_4 1 || & || 5 \infty_0 3, 2 \infty_1 7, 1 \infty_0 2 0, b \infty_3 6, 4 \infty_4 8 || \\
|| t \infty_0 6, 0 \infty_1 8, \infty_2 2b, a \infty_4 1 4, \infty_4 93 || & || 19 \infty_1 1, 204, 658, a 7 \infty_0 3, 3 \infty_4 b || \\
|| 10 \infty_1 3, 294, a \infty_2 5, 768, 3 \infty_1 b || & || 1a \infty_0 0, 435, 96b, 78 \infty_2 2, 2 \infty_4 4 0 || \\
|| 1 b \infty_2, 63a, 748, 9 \infty_0 0, 5 \infty_0 2 || & \text{Leftover} = \{\infty_0 \infty_1 \infty_4, 950, 123\}
\end{array}
\]

• $v \equiv 8 \pmod{12}$.

Similar to the case $v \equiv 4 \pmod{12}$. We add points $\infty_i$, $i = 0, 1, \ldots, 7$ and use the following nearly resolvable $P_3$-design on 20 points having a nearly resolvable subsystem on $\{\infty_i \mid i = 0, 1, \ldots, 7\}$. The first six classes contain the subdesign, the first two blocks from each. The result is
a nearly resolvable $P_3$–design on $12t + 8$ points with $9t + 6$ classes and a leftover class of size $3t + 2$. $V = \{0, 1, \ldots, 9, a, b\} \cup \{\infty_i \mid i = 0, 1, \ldots, 7\}$

\[
\begin{align*}
| & \infty_0 \infty_1 \infty_2, \infty_3 \infty_4 \infty_5, 314, 526, 709, ba8 | \\
| & \infty_4 \infty_0 \infty_7, \infty_1 \infty_5 \infty_3, 516, 24a, 360, 798 | \\
| & \infty_3 \infty_1 \infty_4, \infty_0 \infty_2 \infty_6, 173, 28b, 546, 0a9 | \\
| & \infty_4 \infty_7 \infty_2, \infty_3 \infty_6 \infty_4, 193, 204, 658, a7b | \\
| & \infty_5 \infty_7 \infty_6, \infty_3 \infty_0 \infty_4, 1a2, 435, 96b, 780 | \\
| & \infty_0 \infty_5 \infty_2, \infty_3 \infty_7 \infty_4, 1b2, 63a, 748, 950 | \\
| & 183, 275, \infty_0 \infty_4 \infty_1, \infty_2 \infty_9 \infty_3, \infty_4 \infty_6 \infty_5, \infty_6 \infty_4 \infty_7 | \\
| & 103, 294, \infty_0 \infty_5 \infty_7, \infty_2 \infty_8 \infty_3, \infty_4 \infty_7 \infty_5, \infty_6 \infty_0 \infty_7 | \\
| & \infty_9 \infty_8 \infty_6, \infty_0 \infty_2 \infty_3, \infty_2 \infty_9 \infty_3, \infty_4 \infty_3 \infty_5, \infty_5 \infty_7 \infty_7 | \\
| & \infty_0 \infty_4 \infty_6, \infty_1 \infty_2 \infty_3, 1\infty_34, 6\infty_30, 3\infty_36, 9\infty_76 | \\
| & \infty_0 \infty_6 \infty_5, \infty_2 \infty_3 \infty_7, \infty_3 \infty_0 \infty_6, 9\infty_18, 2\infty_24, 3\infty_35 | \\
| & \infty_0 \infty_5 \infty_1, \infty_2 \infty_3 \infty_4, \infty_3 \infty_2 \infty_4, 0\infty_6b, 7\infty_78 | \\
| & \infty_4 \infty_9 \infty_5, \infty_0 \infty_2 \infty_7, 7\infty_30, 0\infty_16, a\infty_23, 1\infty_34 | \\
| & \infty_0 \infty_4 \infty_1, \infty_2 \infty_3 \infty_5, 5\infty_46, 2\infty_51, 9\infty_68, 0\infty_73 | \\
| & \infty_4 \infty_8 \infty_5, \infty_6 \infty_7, 9\infty_30, 7\infty_1b, 1\infty_25, 2\infty_3a |
\end{align*}
\]

Leftover = \{\infty_1 \infty_6 \infty_5, \infty_3 \infty_2 \infty_4, 123, a5b, 768\}.

- $v \equiv 9 \pmod{12}$.

No construction needed, just apply Lemma 6 to a resolvable $K_3$–design ([14]).

### 2.4 Ingredients for $P_4$

- $v \equiv 0 \pmod{12}$.

Use the GDD construction with the following $P_4$–design on 12 points.

$V = \{0, 1, \ldots, 9, a, b\}$

\[
\begin{align*}
| & 0489, 156a, 237b | \\
| & 0579, 138a, 246b | \\
| & 0125, 4367, 9ab8 | \\
| & 0957, 1586, 82a3 | \\
| & 6265, 70a4, 8193 |
\end{align*}
\]

Leftover = \{4587\}.

- $v \equiv 1 \pmod{12}$.

We use the frame construction using the following nearly resolvable $P_4$–design on $V = \{0, 1, \ldots, 9, a, b, c\}$ as ingredient.

\[
\begin{align*}
| & c123, a45b, 0789 | \\
| & 9364, 8ab2, 1057 | \\
| & 8ab0, 9ca71, 2435 | \\
| & 69e2, 0665, 8473 | \\
| & b7c6, 5830, a192 |
\end{align*}
\]

Leftover = \{80b1, 652a\}.

- $v \equiv 3 \pmod{12}$.
We use the frame construction, however in this case it is not possible to have a $P_4$-design on 3 points, and so we must find a nearly resolvable $P_4$-design and a nearly resolvable $P_4$-design with the hole of size 3 separately. We take $V = \{0, 1, \ldots, 9, a, b, \infty_1, \infty_2, \infty_3\}$.

1. A nearly resolvable $P_4$-design

\[
\begin{array}{ll}
& \| \infty_1 \infty_2 \infty_3 0, 8412, 567b \| & \| a \infty_1 \infty_3 b, 0951, 3682 \| \\
& 0 \infty_1 1a, 72 \infty_2 3, 4 \infty_3 58 \| & \| 2 \infty_1 31, 4 \infty_2 50, 6 \infty_3 78 \| \\
& b \infty_1 5, 06 \infty_2 7, 8 \infty_3 9a \| & \| 6 \infty_1 75, 8 \infty_2 92, a \infty_3 10 \| \\
& 8 \infty_1 94, a \infty_2 b5, 62 \infty_3 3 \| & \| \infty_1 b2a, 40 \infty_2 1, 7389 \| \\
& 25a0, b397, 4618 \| & \| 2068, 3a69, 1745 \| \\
& 6b19, 07a8, 2435 \| & \text{Leftover} = \{4ab9, 8032\} \\
\end{array}
\]

2. A nearly resolvable $P_4$-design with the hole of size 3 - $\{\infty_1, \infty_2, \infty_3\}$.

\[
\begin{array}{ll}
& \| 8 \infty_1 94, a \infty_2 b5, 2 \infty_3 36 \| & \| 6 \infty_1 75, 8 \infty_2 92, a \infty_3 10 \| \\
& b \infty_1 5, 06 \infty_2 7, 8 \infty_3 9a \| & \| a \infty_2 3, 4 \infty_2 50, 6 \infty_3 78 \| \\
& 0 \infty_1 1a, 62 \infty_2 3, 4 \infty_3 58 \| & \| a \infty_1 b2, 40 \infty_2 1, 7389 \| \\
& 0 \infty_3 b9, a841, 5672 \| & \| 0951, 6823, 4ab7 \| \\
& 6b19, 807a, 2435 \| & \| 2068, 3a69, 1745 \| \\
& 25a0, b397, 4618 \| & \text{Leftover} = \{0312\} \\
\end{array}
\]

- $v \equiv 4 \pmod{12}$.
  Just apply lemma 6 to a resolvable $K_4$-design on $12k + 4$ points ([14]).

- $v \equiv 6 \pmod{12}$.
  We use the frame construction with ingredient a nearly resolvable $P_4$-design on 18 points with a subdesign of size 6, given below. The first four classes contain the subdesign, the first block from each.

$V = \{0, 1, \ldots, 9, a, b, \infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$:

\[
\begin{array}{ll}
& \| \infty_0 \infty_1 \infty_3 \infty_5, 0489, 156a, 237b \| & \| \infty_1 \infty_2 \infty_4 \infty_5, 0579, 138a, 246b \| \\
& \| \infty_2 \infty_3 \infty_0 \infty_5, 2035, 147a, 6968 \| & \| \infty_2 \infty_4 \infty_1 \infty_5, 0125, 4367, 9ab8 \| \\
& 61a5, \infty_0 7 \infty_1 9, \infty_2 0 \infty_3 8, \infty_5 2 \infty_4 b \| & \| 7294, \infty_1 6 \infty_0 a, \infty_2 1 \infty_3 5, \infty_4 8 \infty_5 3 \| \\
& \| 8063, \infty_1 5 \infty_0 9, \infty_3 2 \infty_2 6, \infty_5 4 \infty_0 7 \| & \| 62b5, \infty_0 4 \infty_3 8, \infty_3 3 \infty_2 7, \infty_4 0 \infty_5 9 \| \\
& \| 70a4, \infty_1 3 \infty_0 8, \infty_4 9 \infty_3 6, \infty_2 5 \infty_5 b \| & \| 8193, \infty_0 2 \infty_0 b, \infty_3 4 \infty_2 7, \infty_5 6 \infty_4 5 \| \\
& 4587, \infty_0 0 \infty_1 b, \infty_3 a \infty_2 9, \infty_5 3 \infty_4 3 \| & \| 6095, \infty_0 1 \infty_1 a, \infty_3 b \infty_2 8, \infty_4 4 \infty_5 7 \| \\
\end{array}
\]

- $v \equiv 7 \pmod{12}$.
  We use the frame construction with ingredient a nearly resolvable $P_4$-design on 19 points with a subdesign of size 7, given below. The first six classes
contain the subdesign, the first block from each.
\[ V = \{0, 1, \ldots, 9, a, b, \infty_0, \infty_1, \ldots, \infty_6\}: \]

\[
\begin{array}{c}
\| \infty_0 \infty_1 \infty_6 \infty_2, 0579, 138a, 246b \| \\
\| \infty_1 \infty_2 \infty_0 \infty_3, 2035, 147a, b968 \| \\
\| \infty_2 \infty_3 \infty_1 \infty_4, 0125, 4367, 9ab8 \| \\
\| \infty_3 \infty_4 \infty_2 \infty_5, 6095, 71b4, 82a3 \| \\
\| \infty_4 \infty_5 \infty_3 \infty_6, 61a5, 7294, 80b3 \| \\
\| \infty_5 \infty_6 \infty_4 \infty_0, 6265, 70a4, 8193 \| \\
\| \infty_6 \infty_0 \infty_1 \infty_7, \infty_2 \infty_3 \infty_2, \infty_4 \infty_5 \infty_3 \| \\
\| \infty_0 \infty_1 \infty_8, \infty_3 \infty_2 \infty_3, \infty_5 \infty_4 \infty_7 \| \\
\| \infty_2 \infty_3 \infty_5, \infty_4 \infty_6 \infty_0, \infty_1 \infty_7 \infty_8 \| \\
\| \infty_5 \infty_6 \infty_4, \infty_2 \infty_7 \infty_3, \infty_3 \infty_4 \infty_0 \| \\
\| \infty_7 \infty_8 \infty_5, \infty_2 \infty_3 \infty_3, \infty_4 \infty_5 \infty_6 \| \\
\| \infty_8 \infty_6, \infty_1 \infty_3 \infty_5, \infty_3 \infty_1 \infty_6, \infty_5 \infty_4 \| \\
\| \infty_6 \infty_0 \infty_5 \infty_1 \| \\
\end{array}
\]

Leftover = \[\{\infty_0 \infty_0 \infty_5 \infty_1\}\]

- \(v \equiv 9 \pmod{12}\).

We use the frame construction with ingredient \(a\) nearly resolvable \(P_4\)-design on 21 points with a subdesign of size 9, given below. The first six classes contain the subdesign, the first two blocks from each.

\[ V = \{0, 1, \ldots, 9, a, b, \infty_0, \infty_1, \ldots, \infty_8\}: \]

\[
\begin{array}{c}
\| \infty_2 \infty_4 \infty_5 \infty_3, \infty_0 \infty_8 \infty_5 \infty_7, 0579, 138a, 246b \| \\
\| \infty_3 \infty_0 \infty_2 \infty_1, \infty_5 \infty_3 \infty_6 \infty_7, 2035, 147a, b968 \| \\
\| \infty_4 \infty_6 \infty_0 \infty_5, \infty_7 \infty_8 \infty_2 \infty_6, 0125, 4367, 9ab8 \| \\
\| \infty_5 \infty_0 \infty_6 \infty_4, \infty_7 \infty_1 \infty_3 \infty_8, 6095, 71b4, 82a3 \| \\
\| \infty_6 \infty_4 \infty_7 \infty_0, \infty_3 \infty_2 \infty_6, 61a5, 7294, 80b3 \| \\
\| \infty_7 \infty_6 \infty_4 \infty_2, \infty_5 \infty_1 \infty_8, 6265, 70a4, 8193 \| \\
\| \infty_8 \infty_0 \infty_1 \infty_2, \infty_3 \infty_5 \infty_3, \infty_6 \infty_5 \infty_7 \| \\
\| \infty_0 \infty_4 \infty_6 \infty_3, \infty_1 \infty_5 \infty_9 \infty_0, \infty_8 \infty_3 \infty_3, \infty_7 \infty_2 \infty_0 \| \\
\| \infty_4 \infty_0 \infty_1 \infty_6, \infty_0 \infty_0 \infty_5, \infty_2 \infty_1 \infty_4, \infty_9 \infty_8 \infty_2 \| \\
\| \infty_5 \infty_4 \infty_8 \infty_6, \infty_2 \infty_3 \infty_7, \infty_0 \infty_9 \infty_3, \infty_7 \infty_1 \infty_0 \| \\
\| \infty_6 \infty_5 \infty_1 \infty_0, \infty_2 \infty_4 \infty_6, \infty_7 \infty_8 \infty_4, \infty_2 \infty_0 \infty_3, \infty_7 \infty_0 \infty_9 \| \\
\| \infty_7 \infty_8 \infty_6, \infty_4 \infty_3 \infty_5, \infty_5 \infty_7 \infty_1, \infty_6 \infty_2 \infty_4, \infty_3 \infty_3 \infty_9 \| \\
\| \infty_8 \infty_2, \infty_1 \infty_0 \infty_3, \infty_2 \infty_6 \infty_0, \infty_3 \infty_4 \infty_7, \infty_4 \infty_5 \infty_1 \| \\
\end{array}
\]

- \(v \equiv 10 \pmod{12}\).

We use the frame construction with ingredient \(a\) nearly resolvable \(P_4\)-design on 22 points with a subdesign of size 10, given below. The first seven classes contain the subdesign, the first two blocks from each.
\[ V = \{0, 1, \ldots, 9, a, b, \infty_0, \infty_1, \ldots, \infty_9\} : \]

\[ \| \infty_2 \infty_4 \infty_9 \infty_5, \infty_4 \infty_2 \infty_9 \infty_3 \infty_7, 0489, 156a, 237b \| \]
\[ \| \infty_8 \infty_1 \infty_7 \infty_0, \infty_8 \infty_2 \infty_9 \infty_4, 0579, 138a, 246b \| \]
\[ \| \infty_9 \infty_1 \infty_5 \infty_2, \infty_9 \infty_8 \infty_4 \infty_0, 2035, 147a, b688 \| \]
\[ \| \infty_7 \infty_2 \infty_9 \infty_9, \infty_6 \infty_1 \infty_3 \infty_4, 0125, 4367, 9ab8 \| \]
\[ \| \infty_0 \infty_1 \infty_9 \infty_3, \infty_4 \infty_5 \infty_6 \infty_7, 6095, 71b4, 82a3 \| \]
\[ \| \infty_2 \infty_0 \infty_3 \infty_8, \infty_6 \infty_9 \infty_7 \infty_5, 61a5, 7924, 8063 \| \]
\[ \| \infty_1 \infty_4 \infty_5 \infty_8, \infty_5 \infty_9 \infty_0 \infty_6, 62b5, 70a4, 8193 \| \]
\[ \| \infty_0 \infty_0 \infty_8, \infty_2 \infty_3 \infty_1, \infty_4 \infty_5 \infty_2, \infty_6 \infty_6 \infty_7, \infty_8 \infty_4 \infty_9 \infty_6 \| \]
\[ \| \infty_1 \infty_0 \infty_9, \infty_3 \infty_2 \infty_0, \infty_5 \infty_4 \infty_0, \infty_7 \infty_4 \infty_7, \infty_9 \infty_2 \infty_5 \| \]
\[ \| \infty_0 \infty_2 \infty_1 \infty_0, \infty_2 \infty_9 \infty_3, \infty_4 \infty_4 \infty_0, \infty_6 \infty_5 \infty_7, \infty_9 \infty_1 \infty_8 \| \]
\[ \| \infty_0 \infty_3 \infty_0 \infty_4, \infty_3 \infty_8 \infty_2 \infty_2, \infty_5 \infty_5 \infty_4, \infty_7 \infty_6 \infty_0, \infty_9 \infty_3 \infty_8 \| \]
\[ \| \infty_0 \infty_4 \infty_0 \infty_9, \infty_2 \infty_7 \infty_3 \infty_0, \infty_4 \infty_6 \infty_5, \infty_6 \infty_8 \infty_2, \infty_8 \infty_6 \infty_5 \| \]
\[ \| \infty_1 \infty_5 \infty_0 \infty_8, \infty_3 \infty_6 \infty_2 \infty_1, \infty_5 \infty_7 \infty_4, \infty_7 \infty_3 \infty_0, \infty_9 \infty_3 \infty_5 \| \]
\[ \| \infty_0 \infty_6 \infty_1 \infty_1, \infty_2 \infty_5 \infty_3, \infty_4 \infty_8 \infty_5, \infty_6 \infty_0 \infty_7, \infty_8 \infty_7 \infty_9 \| \]
\[ \| \infty_1 \infty_7 \infty_0 \infty_0, \infty_3 \infty_4 \infty_2 \infty_3, \infty_5 \infty_9 \infty_4, \infty_7 \infty_1 \infty_6, \infty_9 \infty_8 \infty_5 \| \]

\[ \text{Leftover} = \{ \infty_8 \infty_5 \infty_4 \infty_9, 4587 \} \]

### 3 Conclusions

We have thus shown that the chromatic values in the tables given in Section 2.2 are attained for every value of \( v \), except for some small cases where the required frames do not exist, \( v = 25, 28, 29, 32 \) for \( P_5 \) and \( v = 25, 27, 30, 31, 33, 34 \) for \( P_4 \). These cases are covered in the appendix.

The use of \( \mathcal{G} \)–frames and resolvable is a powerful framework in the solution of the Minimal Chromatic Index Problem. The necessary GDD’s and frames can often be found by applications of Wilson’s construction [14]. We believe that this type of methodology will be useful for solving a large class of these kinds of problems.

If \( \mathcal{G} = \bigcup K_i \), we have a context for looking at “proportional” class-uniform like structures. In this case the number of copies of each \( K_i \) (\( i \neq 1 \)) in the leftover class is in the same proportion to those in the nearly resolvable classes. This is allows the number uncovered points (\( K_1 \)'s) in the leftover class to vary, which is why we exclude isolated points (\( K_1 \)). The case where \( \mathcal{G} = K_2 \bigcup K_3 \) is particularly interesting and generalises the work of Danziger and Stevens [3].

The case of the Walke problem where one assumes that a fixed number of people are gone on a hike and thus are not present for the meal and further that many have left for home before the last meal is served, needs to be settled.

When \( \mathcal{G} \) has fewer than four edges and no isolated points the only outstanding cases are two or three disjoint edges for which the solution is trivial, and the claw \( K_{1,3} \).

The idea of subdividing hill climbing problems as was done to solve \( v = 34 \) in the appendix is a new a very useful technique for graph decomposition problems which should be exploited further.
Appendix: Small Cases

In this appendix we give those cases which are missed by the recursive constructions.

- \( P_3, v = 25 \).

A \( P_3 \) - design on 25 points with 18 classes, 6 blocks in leftover class, \( V = \{a, \ldots, y\} \).

\[
\begin{align*}
\parallel vcg, bhs, pzx, yol, kae, mfj, rtu, nid \parallel, \\
\parallel iqo, vwe, klx, prj, dst, yng, ncb, uba \parallel, \\
\parallel duo, veg, cyf, txj, yrb, nsw, hql, kpa \parallel, \\
\parallel smw, hjc, yug, wdp, itu, brv, ofe, aqk \parallel, \\
\parallel zmf, skb, iel, bne, htw, ybn, xgp, mwv \parallel, \\
\parallel rux, kcg, chl, ybn, xgp, mwv, tdf, oao \parallel, \\
\parallel ecf, ort, arn, ped, ghn, isq, tkb, jlu \parallel, \\
\parallel mwd, lan, ceh, tys, rkv, bfp, lpu, goy \parallel, \\
\parallel wug, fil, rse, jyh, gha, rdk, mnx, ptc \parallel, \\
\parallel mbh, res, oie, fuj, wza, pqk, phr, dno \parallel.
\end{align*}
\]

Leftover = \{yel, qvt, jix, ohk, fpm, sgw\}

- \( P_3, v = 28 \).

A \( P_3 \) - design on 28 points with 21 classes, \( V = \{a, \ldots, z\} \cup \{A, B\} \).

\[
\begin{align*}
\parallel yrb, Bzm, loc, ine, qxo, wpk, hju, vad, Afj \parallel, \\
\parallel Biz, luq, bkt, xjx, pum, arc, qhd, Afe, ncy \parallel, \\
\parallel Bre, bdo, pwk, Agk, taz, hmn, wfp, tlc, zly \parallel, \\
\parallel smn, kir, qeo, ncz, djx, tvw, Bbu, yhA, fgl \parallel, \\
\parallel jyn, lbr, qve, ygb, inc, hoa, dza, xfk, upB \parallel, \\
\parallel mxo, pdc, het, ual, yjr, nbw, Bgv, fsi, kqA \parallel, \\
\parallel rub, cle, zog, fmt, hax, vkd, sny, piq, wAB \parallel, \\
\parallel jqd, pem, uac, jos, jkx, pde, hwr, dno \parallel.
\end{align*}
\]

Leftover = \{yne, qtx, jiz, ohk, fpm, sqw\}

- \( P_3, v = 29 \).

A \( P_3 \) - design on 29 points with 22 classes, \( V = \{a, \ldots, z\} \cup \{A, B, C\} \).

\[
\begin{align*}
\parallel yrz, Bzm, loc, ine, qxo, wpk, hju, vad, Afj \parallel, \\
\parallel Biz, luq, bkt, xjx, pum, arc, qhd, Afe, ncy \parallel, \\
\parallel Bre, bdo, pwk, Agk, taz, hmn, wfp, tlc, zly \parallel, \\
\parallel smn, kir, qeo, ncz, djx, tvw, Bbu, yhA, fgl \parallel, \\
\parallel jyn, lbr, qve, ygb, inc, hoa, dza, xfk, upB \parallel, \\
\parallel mxo, pdc, het, ual, yjr, nbw, Bgv, fsi, kqA \parallel, \\
\parallel rub, cle, zog, fmt, hax, vkd, sny, piq, wAB \parallel, \\
\parallel jqd, pem, uac, jos, jkx, pde, hwr, dno \parallel.
\end{align*}
\]

Leftover = \{yne, qtx, jiz, ohk, fpm, sqw\}

- \( P_3, v = 25 \).

A \( P_3 \) - design on 25 points with 18 classes, 6 blocks in leftover class, \( V = \{a, \ldots, y\} \).

\[
\begin{align*}
\parallel vcg, bhs, pzx, yol, kae, mfj, rtu, nid \parallel, \\
\parallel iqo, vwe, klx, prj, dst, yng, ncb, uba \parallel, \\
\parallel duo, veg, cyf, txj, yrb, nsw, hql, kpa \parallel, \\
\parallel smw, hjc, yug, wdp, itu, brv, ofe, aqk \parallel, \\
\parallel zmf, skb, iel, bne, htw, ybn, xgp, mwv \parallel, \\
\parallel rux, kcg, chl, ybn, xgp, mwv, tdf, oao \parallel, \\
\parallel ecf, ort, arn, ped, ghn, isq, tkb, jlu \parallel, \\
\parallel mwd, lan, ceh, tys, rkv, bfp, lpu, goy \parallel, \\
\parallel wug, fil, rse, jyh, gha, rdk, mnx, ptc \parallel, \\
\parallel mbh, res, oie, fuj, wza, pqk, phr, dno \parallel.
\end{align*}
\]

Leftover = \{yel, qtx, jiz, ohk, fpm, sqw\}
\[ P_3, v = 32. \]
A \( P_3 \)-design on 32 points with 24 classes, \( V = \{ a, \ldots, z \} \cup \{ A, \ldots, F \} \)

\[
\begin{align*}
\{A, yf, uD, CrE, mpk, syi, bqF, teh, zm\} & \cup \{wz, ghB, xCl, nAf, wkq, dba, cot, Fslm, jyE, iDe \} \\
\{bJ, huc, gpe, cis, qls, rDd, kFC, xao, yAE, mnf \} & \cup \{Eij, bhe, uzF, wan, gBl, sdv, Cof, yDk, Acm, vro \} \\
\{gjd, opt, uyg, nzz, jsf, Fai, DbE, khr, vAC, mqB \} & \cup \{moD, pmw, lEv, ytc, Agx, scJ, zkd, urf, FBB, qhi \}
\end{align*}
\]

\[
\parallel \{Alo, cjw, uDf, CrE, mpk, syi, bqF, teh, zm\} \| \parallel \{wz, ghB, xCl, nAf, wkq, dba, cot, Fslm, jyE, iDe \} \\
\parallel \{bJ, huc, gpe, cis, qls, rDd, kFC, xao, yAE, mnf \} \| \parallel \{Eij, bhe, uzF, wan, gBl, sdv, Cof, yDk, Acm, vro \} \\
\parallel \{gjd, opt, uyg, nzz, jsf, Fai, DbE, khr, vAC, mqB \} \| \parallel \{moD, pmw, lEv, ytc, Agx, scJ, zkd, urf, FBB, qhi \}
\]

\[
\text{Leftover} = \{BFt, bwv, cDE, pDv, dmk, zlk, fmj, yeq\}
\]

\[ P_4, v = 25. \]
A \( P_4 \)-design on 25 points with 16 classes, \( V = \{ a, \ldots, y \} \)

\[
\begin{align*}
\{mhit, nps, kfc, gvy, azd, jow \} & \cup \{vfj, umeo, bcpn, taod, pzxk, yriq \} \\
\{zhb, cvujy, mqs, raxf, ftdn, ikpo \} & \cup \{jlq, equf, ackd, stvo, thnb, xrgp \} \\
\{heq, lnuw, tycx, akro, vdfb, imnsj \} & \cup \{coy, ajhp, ldgt, kznm, irou, qsvb \} \\
\{nybq, wkq, tmjd, afxj, clue, hosl \} & \cup \{slyp, igmt, ofec, bhdk, ruxz, aqgg \} \\
\{mxgo, hesh, qtwy, ams, dfrj, eknf \} & \cup \{wzyl, shbl, qoca, prmf, kjtn, vwnq \} \\
\{lomp, jvkn, geda, stuh, qfxw, cibr \} & \cup \{lmvn, dpfi, ajbe, hrsw, ykte, xqoq \} \\
\{fsyq, upai, hbbk, nced, eltr, jqmU \} & \cup \{ilw, manr, ztej, knuf, hqyd, hpcv \}
\end{align*}
\]

\[
\text{Leftover} = \{vezo, kgc, pmj, wmrq\}
\]
• $P_4, v = 27$ A $P_4$ - design on 27 points with 19 classes, $V = \{a, \ldots, z\} \cup \{A\} \\
\left\| uqxj, lhtk, mAzv, fgri, waeb, sdny \right\|, \left\| vcle, hAwr, kbin, qszm, yfta, xpoj \right\|, \left\| kjsl, bdp, czu, zmwe, yrAq, iaej \right\|, \left\| tqeh, nkDA, wprz, yyjf, ljun, Aboc \right\|, \left\| azfn, isAp, vktro, tgyx, blqy, wudc \right\|, \left\| ysuC, bgpj, olvA, rfjn, hqdz, wixz \right\|, \left\| cvra, gyev, gstm, qvex, zrnA, hikf \right\|, \left\| nxzd, Afca, jlpq, yhog, varl, baok \right\|, \left\| zkny, wosA, zdhf, c7gA, zBsn, urcA \right\|, \left\| rlkp, vncq, swzh, xmpg, tjaA, byio \right\|, \left\| pfaj, ogdr, nzbA, khec, zltw, vmeA \right\|, \left\| hzfA, dcej, Aeql, innr, bsksw, fglf \right\|, \left\| iAnt, bhjc, eynq, rfxs, kauA, oqzA \right\|, \left\| ozxe, hubw, fdnj, psew, etyk, iqew \right\|, \left\| yAuk, gypA, clfd, ztnA, nofA, exrh \right\|, \left\| cuwj, edyu, ghqi, anrA, kZpb, onlz \right\|, \left\| ipsm, zybs, agjr, rxAk, towA, lnhw \right\|

Leftover = $\{Abjd, mcei, nzoy\}$

• $P_4, v = 30$. 

A $P_4$ - design on 30 points with 20 classes, $V = \{a, \ldots, z\} \cup \{A, \ldots, D\} \\
\left\| qxuD, ltod, kave, jmAn, szgw, fBhy, rpBc \right\|, \left\| xajn, gbwr, epcz, Chvt, squm, dyio, AflD \right\|, \left\| ocni, vzlw, djDe, Bygr, xCQf, mhAk, tgpu \right\|, \left\| vgyj, zkce, f7hl, ipdq, anth, oqzB, CDsA \right\|, \left\| mGsw, hzyA, goky, kwPA, hzA, zrQe, eCui \right\|, \left\| ftDy, iApA, smpQ, qGkl, nAhr, hzA, zrQe, eCui \right\|, \left\| sjoy, wosA, hzyA, zrQe, eCui \right\|, \left\| kjsl, bdtp, cxuz, hmwe, yrAq, iavf \right\|, \left\| tqeh, nkDA, wprz, yyjf, ljun, Aboc \right\|, \left\| azfn, isAp, vktro, tgyx, blqy, wudc \right\|, \left\| ysuC, bgpj, olvA, rfjn, hqdz, wixz \right\|, \left\| cvra, gyev, gstm, qvex, zrnA, hikf \right\|, \left\| nxzd, Afca, jlpq, yhog, varl, baok \right\|, \left\| zkny, wosA, zdhf, c7gA, zBsn, urcA \right\|, \left\| rlkp, vncq, swzh, xmpg, tjaA, byio \right\|, \left\| pfaj, ogdr, nzbA, khec, zltw, vmeA \right\|, \left\| hzfA, dcej, Aeql, innr, bsksw, fglf \right\|, \left\| iAnt, bhjc, eynq, rfxs, kauA, oqzA \right\|, \left\| ozxe, hubw, fdnj, psew, etyk, iqew \right\|, \left\| yAuk, gypA, clfd, ztnA, nofA, exrh \right\|, \left\| cuwj, edyu, ghqi, anrA, kZpb, onlz \right\|, \left\| ipsm, zybs, agjr, rxAk, towA, lnhw \right\|

Leftover = $\{Abjd, mcei, nzoy\}$

• $P_4, v = 31$. 

A $P_4$ - design on 31 points with 22 classes, $V = \{a, \ldots, z\} \cup \{A, \ldots, E\} \\
\left\| qxuD, ltod, kave, jmAn, szgw, fBhy, rpBc \right\|, \left\| xajn, gbwr, epcz, Chvt, squm, dyio, AflD \right\|, \left\| ocni, vzlw, djDe, Bygr, xCQf, mhAk, tgpu \right\|, \left\| vgyj, zkce, f7hl, ipdq, anth, oqzB, CDsA \right\|, \left\| mGsw, hzyA, goky, kwPA, hzA, zrQe, eCui \right\|, \left\| ftDy, iApA, smpQ, qGkl, nAhr, hzA, zrQe, eCui \right\|, \left\| sjoy, wosA, hzyA, zrQe, eCui \right\|, \left\| kjsl, bdtp, cxuz, hmwe, yrAq, iavf \right\|, \left\| tqeh, nkDA, wprz, yyjf, ljun, Aboc \right\|, \left\| azfn, isAp, vktro, tgyx, blqy, wudc \right\|, \left\| ysuC, bgpj, olvA, rfjn, hqdz, wixz \right\|, \left\| cvra, gyev, gstm, qvex, zrnA, hikf \right\|, \left\| nxzd, Afca, jlpq, yhog, varl, baok \right\|, \left\| zkny, wosA, zdhf, c7gA, zBsn, urcA \right\|, \left\| rlkp, vncq, swzh, xmpg, tjaA, byio \right\|, \left\| pfaj, ogdr, nzbA, khec, zltw, vmeA \right\|, \left\| hzfA, dcej, Aeql, innr, bsksw, fglf \right\|, \left\| iAnt, bhjc, eynq, rfxs, kauA, oqzA \right\|, \left\| ozxe, hubw, fdnj, psew, etyk, iqew \right\|, \left\| yAuk, gypA, clfd, ztnA, nofA, exrh \right\|, \left\| cuwj, edyu, ghqi, anrA, kZpb, onlz \right\|, \left\| ipsm, zybs, agjr, rxAk, towA, lnhw \right\|

Leftover = $\{cmBr, atwx, eCyp, fDzn, ubdQ\}$

• $P_4, v = 33$. 

A $P_4$ - design on 33 points may be found [7].
\[ P_4, v = 34 \]

In this case the ordinary hill climbing methods failed. We subdivided the problem into two tractable problems (the latter taking two days running time on a PC). We give a \( P_4 \)-design on 17 points, missing an edge, i.e. a \( P_4 \)-decomposition of \( K_{17} \setminus e \) with point set \( \{a, \ldots, p\} \cup \infty \), the edge \( \infty a \) is left uncovered.

\[
\{ ghnp, ciam, djbl, \infty fek, \} \cup \{ chej, ltdn, bniwcc, ogka \} \[ \]
\[
\{ mibh, nyck, \infty da, fhjp \} \cup \{ mecf, ikpb, denj, \infty cloa \} \]
\[
\{ gpmm, ea fj, tdse, iokk \} \cup \{ phag, elfk, dino, mjcb \} \]
\[
\{ egno, fplj, nkhi, dboc \} \cup \{ fbal, spem, kdah, gocrc \} \]
\[
\{ jqgd, pockm, ambh, ecof \} \cup \{ hocmjk, obei, fndp, acly \} \]
\[
\{ kind, \infty omh, paci, ojgb \} \]. Leftover = \{epee\}

We now give a \( P_4 \)-GDD of type 17² missing an edge, with point set \( V = (\{a, \ldots, p\} \cup \infty) \times \mathbb{Z}_2 \), the edge \( \infty a \infty 1 \) is left uncovered.

\[
\{ i1oa, c1oa, h0h0, k0c1, mali1na1, b0a1c0k1, e1i0p1n1, f1g0a0, j1d0a0, l0d1p0b1 \} \]
\[
\{ g1k00, c1, o0e0a0, p0f10b0c1, k0d1m0a0, a1i0a1ma, f001, l0b0a1, c01i0a1 \} \]
\[
\{ l0d1h0h0, c1, o0m0p1j0, h0a1k0f1, c1, o0c0e1p0, o0k0e0j1, d1c0a0h0, b0n1f0a0, l0h1g0b0a0 \} \]
\[
\{ g0e1b0p1, n1a0f1c1, j1a1l0a1m1, b1o1g1e0, f001a0k1, c1, o1a0d0, n01p0a1, h0j0k0l1 \} \]
\[
\{ o0o1a0h1, p0j0j0d1, l1b0b0ma0, a1d0c1g0, k0p0f11e1, i0o1j0h0, i0o1m0c1, i1c0a0l0 \} \]
\[
\{ c1, o0i0b0a0, c01a0k0a1, a1n0a1j0, g1p0p0a0, f1f0h1c1, j1c0c1a0, o0b0e0a1, l1j0a1e0 \} \]
\[
\{ h0a1j0g1, f0m0a0c1, n1, k0c1a0, g0j0c0c1, a1, b0d01a0, k110f0, j1a0f0, a0l1p0d0, p0a1e0b1 \} \]
\[
\{ k0i0a0p0, f1i0b1h0, l0j0a1m0, c1, o0d0e1c0, a0c1j0a0, p0d0e0a0, g1i0n0c0, g0o1d0k0, o0o1f0h1 \} \]
\[
\{ n1p0c01h0, a0l0h0a1, e1i0m0j1f0, d1k0k1j0, c0f1k0c0, h0c0o0d0, o0g1b0m1, b1g0p0e0 \} \]
\[
\{ j1o0d1g0, b0h1e0, n0a1c0g1, p1h0f0m0, d0a1f0k1, c0i0f1c0, c0, n01a1d0, i0m1p0b1 \} \]
\[
\{ c0p0a1d0, e1h0g1f0, l0n0a1m0, b0k10a1, j1, c0a0m1, c0m0g0c1, a1, a0a1a1, d0b0j0f1 \} \]
\[
\{ m01a0c0, j1b0e0c1, d1e0f0d0, k0e0d0h1, g0p0l0b1, c0l1f0a1, a0i0a0m1, h0c0p0k1 \} \]

We now apply the GDD construction, ensuring that the \( \infty \)'s match up. Add the block \( \{20a0\infty 0\infty 1\{a1\} \} \) to the leftover class. This gives the required \( P_4 \)-design on 34 points.

References


