Minimum embedding of Steiner triple systems into \((K_4 - e)\)-designs

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Abstract

A \((K_4 - e)\)-design on \(v + w\) points embeds a Steiner triple system if there is a subset of \(v\) points on which the graphs of the design induce the blocks of a Steiner triple system. It is established that \(w \geq v/3\), and that when equality is met that such a minimum embedding of an STS\((v)\) exists, except when \(v = 15\).

1 Introduction and Definitions

Let \(\mathcal{G}\) be a set of graphs. A \(\mathcal{G}\)-design of order \(v\) and index \(\lambda\) is a partition of the edges of the \(\lambda\)-fold complete graph \(\lambda K_n\) into subgraphs, each isomorphic to a graph in \(\mathcal{G}\). When \(\mathcal{G}\) contains a single graph \(G\), the design is a \(G\)-design. A Steiner triple system (STS) is a \(K_3\)-design of index one.

A natural class of questions arises. Given two graphs \(G\) and \(H\), \(H\) being an induced subgraph of \(G\), when does there exist a \(G\)-design of order \(v + w\)
and index λ on the point set \( V \cup W \), with \(|V| = v\) and \(|W| = w\), so that restricting the graphs of the \( G\)-design to the points in \( V \), we obtain the graphs of an \( H\)-design of order \( v \) and index \( \lambda \). This is an embedding of the \( H\)-design into the \( G\)-design. In this vein, substantial research has been done when \( H = P_3 \), the simple path with 2 edges, [5, 6, 7, 8].

In this note, we consider the minimum embedding of a Steiner triple system into a \( G\)-design where \( G \) has four points. By ‘minimum’, we mean that \( w \) is the smallest that it can be in relation to \( v \).

There are only three graphs on four vertices containing a triangle. They are

- \( G_1 \): triangle and a pendant edge;
- \( G_2 \): \( K_4 - e \); and
- \( G_3 \): \( K_4 \).

Suppose we are to embed the STS(\( v \)) into a \( G_i\)-design on \( v + w \) points; \( w \) is the number of new points. Each block in the STS(\( v \)) is attached to \( i \) edges from the \( v \) points to the \( w \) points. By counting edges from the original \( v \) points in the STS to the \( w \) new points, we obtain that \( vw \geq \frac{iv(v-1)}{6} \) where \( \frac{iv(v-1)}{6} \) is the number of blocks in the STS(\( v \)). For both \( G_1 \) and \( G_3 \), we show that the bound \( w \geq \frac{iv(v-1)}{6} \) can hold with equality. However, when \( i = 2 \), we prove that \( w \neq \frac{v-1}{3} \). Therefore, we are interested in the situation when \( w \) is the next best possible, namely \( w = \frac{v}{3} \). Before we proceed to the case when \( G = G_2 = K_4 - e \), we discuss the simpler cases when \( G \) is a triangle and a pendant edge and when \( G = K_4 \).

When \( G = G_1 \), we have \( w \geq \frac{v-1}{6} \). Equality can only occur when \( v \equiv 1 \pmod{6} \). Furthermore, we need \( v + w \equiv 0, 1 \pmod{8} \) since we construct a \( G\)-design on \( v + w \) points. If \( v = 6t + 1 \), we must have \( 7t + 1 \equiv 0, 1 \pmod{8} \) and hence \( t \equiv 0, 1 \pmod{8} \). Therefore, equality only holds when \( v \equiv 1, 7 \pmod{48} \). If \( v \equiv 1, 7 \pmod{48} \), then such an embedding exists: Take a cyclic STS of order \( v \); as \( v \equiv 1 \pmod{6} \), it has no short orbit, and has exactly \( \frac{v-1}{6} \) orbits of length \( v \). In the \( v \) blocks generated by a single orbit of the cyclic STS, we can pick a point from each of the \( v \) blocks so that these \( v \) points are distinct (we can simply take the “first” point of the block and apply the cyclic group of order \( v \) to obtain the \( v \) points, one from each block.) For the \( v \) blocks from each orbit in turn, we associate them to a new point by joining the new point to the distinguished point in each triple. In this
way, we form $v$ copies of $G_1$ and the new point appears in a copy of $G_1$ with each of the $v$ original points. Finally, since $t = \frac{v-1}{6} \equiv 0, 1 \pmod{8}$, we can put a $G_1$-design on the $t$ new points to obtain the required design.

Similarly, to embed an STS($v$) into a $K_4$-design on $v + w$ points, simple counting reveals that $w \geq \frac{v-1}{2}$. Equality only holds when $v \equiv 3, 9 \pmod{24}$.

We refer the reader to [4] for more details.

2 Minimum embedding in $(K_4 - e)$-designs

In the remainder of the note, we deal with $G_2 = K_4 - e$.

If we embed an STS($v$) into a $(K_4 - e)$-design on $v + w$ points, we must have $vw \geq \frac{v(v-1)}{3}$. Therefore $w \geq \frac{v-1}{3}$. We next analyze the situation when $w = \frac{v-1}{3}$. Suppose that we can embed an STS($v$) into a $(K_4 - e)$-design on $v + \frac{v-1}{3}$ points. Considering $K_4 - e$’s containing each of the $\frac{v-1}{3}$ new points, the blocks of the STS($v$) must be partitioned into $\frac{v-1}{3}$ classes, each class containing exactly $\frac{v}{2}$ blocks. But for an STS($v$) to exist, $v$ must be odd. Therefore, it is impossible to embed an STS($v$) into a $(K_4 - e)$-design on $v + \frac{v-1}{3}$ points.

We therefore consider as the minimum the next possible embedding, when $w = \frac{v}{3}$. Equality can only occur when $v \equiv 15, 27 \pmod{30}$. The goal that we address next is to establish that, whenever $v \equiv 15, 27 \pmod{30}$, there exists a $(K_4 - e)$-design on $\frac{4v}{3}$ points such that an STS($v$) is embedded in it.

In order to prove the result, we introduce a variant of “resolvable” designs. An STS($6t + 3$) on $V = \{x\} \cup I_{6t+2}$ is $\frac{2}{3}$-resolvable if we can select two distinguished elements in each triple, and then we can partition the blocks into $2t+1$ classes, each containing $3t+1$ blocks such the distinguished points in each block of each class cover all $6t + 3$ points except the point $x$.

Lemma 2.1 If there exists a $\frac{2}{3}$-resolvable STS($6t + 3$), $t \equiv 2, 4 \pmod{5}$, and $t \neq 2$, then there exists an STS($6t + 3$) that can be embedded into a $(K_4 - e)$-design on $8t + 4$ points.

Proof: Since there exists a $\frac{2}{3}$-resolvable STS($6t + 3$), we let $V = \{x\} \cup I_{6t+2}$ be the point set of the STS. Treat each of the $2t + 1$ classes of blocks: For class $i$, attach a new point $\infty_i$ and form copies of $K_4 - e$ by adding the edges from $\infty_i$ to the two distinguished points in the block. In this way, every pair of the form $\infty_i$ and $j$ for $j \in I_{6t+2}$ is in exactly one $K_4 - e$.  

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To obtain the $(K_4 - e)$-design on $8t + 4$ points, we put a $(K_4 - e)$-design on $\{\infty_i : 1 \leq i \leq 2t + 1\} \cup \{x\}$. Since $2t + 2 \equiv 0, 1 \pmod{5}$, such a $(K_4 - e)$-design exists [3].

The construction of the $(K_4 - e)$-design on $8t + 4$ points requires a $(K_4 - e)$-design on $2t + 2$ points. When $t = 2$, no such $(K_4 - e)$-design exists. Hence, the construction fails to construct an STS(15) which is embeddable into a $(K_4 - e)$-design on twenty points. It is indeed impossible to have such design exists, as we see next.

**Lemma 2.2** No STS(15) can be embedded into a $(K_4 - e)$-design on 20 points.

**Proof:** Suppose that such an STS(15) exists. Each of the 5 additional points determines a set of blocks of the STS(15) to which the additional point has been adjoined, and also determines two distinguished point in each block. Since the sets of distinguished points must be disjoint, one of the five additional points can be associated with at most seven blocks of the STS(15). Now the STS has 35 blocks, and hence each additional point is associated with exactly seven of its blocks, and defines all but one element as the distinguished elements in the blocks of the class. Then deleting all copies of $K_4 - e$ which induce a block on the 15 points, and collapsing the 15 points into one, we would produce a $(K_4 - e)$-design on 6 points, which does not exist [3].

**Lemma 2.3** There exists a $\frac{2}{3}$-resolvable STS(9), and hence a $(K_4 - e)$-GDD of type $1^8 4^1$.

**Proof:** An STS(9) on $\{x\} \cup I_8$ is displayed below. The four blocks of each class are listed in each row, and the two marked points of every block are underlined.

\[
\begin{align*}
\{x,0,4\}, \{x,1,5\}, \{x,2,6\}, \{x,3,7\} \\
\{0,1,3\}, \{4,5,7\}, \{1,2,4\}, \{5,6,0\} \\
\{2,3,5\}, \{6,7,1\}, \{3,4,6\}, \{7,0,2\}
\end{align*}
\]
Lemma 2.4 If there exists a cyclic STS(6t + 1) such that each base block contains a difference which is relatively prime to 6t + 1, then there exists a a $\frac{2}{3}$-resolvable STS(12t + 3).

Proof: Form a cyclic STS(6t + 1). Choose a difference $d_x$ in each starter block which forms a single cycle of edges mod 6t + 1. Represent starter blocks as $\{ -\frac{d_x}{2}, \frac{d_x}{2}, a_x \}, \{ -\frac{d_x}{2}, \frac{d_x}{2}, a_x \}, \ldots, \{ -\frac{d_x}{2}, \frac{d_x}{2}, a_x \}$. Now, we form the $\frac{2}{3}$-resolvable STS(12t + 3) on $\{ \infty \} \cup \left( \mathbb{Z}_{6t+1} \times \{0, 1\} \right)$. For each starter block $\{ -\frac{d_x}{2}, \frac{d_x}{2}, a_x \}$ form

1. one class containing all translates of $\{ \left( -\frac{d_x}{2} \right)_0, \left( \frac{d_x}{2} \right)_1, (a_x)_1 \}$,
2. one class containing all translates of $\{ \left( \frac{d_x}{2} \right)_1, \left( \frac{d_x}{2} \right)_0, (a_x)_1 \}$,
3. form $\{ \left( -\frac{d_x}{2} \right)_0, (\frac{d_x}{2})_0, (a_x)_0 \} , \{ \left( -\frac{d_x}{2} \right)_1, (\frac{d_x}{2})_1, (a_x)_0 \}$ and
   A. add $2i(d_x)$ to each for $i = 1, 2, \ldots, 3t$ and adjoin $\{ \left( \frac{d_x}{2} \right)_0, \left( \frac{d_x}{2} \right)_1, \infty \}$ to form one class
   B. add $(2i - 1)d_x$ to each for $i = 1, 2, \ldots, 3t$ and adjoin $\{ \left( -\frac{d_x}{2} \right)_0, \left( -\frac{d_x}{2} \right)_1, \infty \}$ to form other class.

Now form the final class as $\{ \left( -\frac{d_x}{2} \right)_0, \left( \frac{d_x}{2} \right)_0, (a_x)_0 \} , \{ \left( -\frac{d_x}{2} \right)_1, (\frac{d_x}{2})_1, (a_x)_0 \}$ for $x = 1, 2, \ldots, t$ and $\{ d_0, d_1, \infty \}$ for $d \in \mathbb{Z}_{6t+1} \setminus \{ \frac{d_x}{2}, \frac{-d_x}{2}, x = 1, 2, \ldots, t \}$.

A $\frac{2}{3}$ 3-frame of type $(3n)^k$ is a 3-GDD of type $(3n)^k$ such that the blocks can be partitioned into $nk$ classes, so that

1. Each class consists of $\frac{3n(k-1)}{2}$ blocks of size 3 in which it is possible to mark 2 points, and these $3n(k-1)$ marked points cover all $3n(k-1)$ points in all but one group.
2. Each group is missed in this way by exactly $n$ class of blocks.

We call each class a $\frac{2}{3}$ frame parallel class.

Theorem 2.5 If $n \notin \{2, 6, 8\}$, then there exists a $\frac{2}{3}$ 3-frame of type $6^n$.  

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Proof: We first construct a $\frac{2}{3}$-3-frame of type $3^p$ for all odd integers $p \geq 3$. Recall the Bose construction (see, for example, [2]) of a 3-GDD of type $3^p$ as follows: $V = \mathbb{Z}_p \times \mathbb{Z}_3$ and the blocks are $\{(-i + k)_j, (i + k)_j, k_{j+1}\}$ for $i = 1, 2, \ldots, \frac{p-1}{3}$, $j = 0, 1, 2$ and $k \in \mathbb{Z}_p$ with groups $\{i_0, i_1, i_2\}$ for $i \in \mathbb{Z}_p$. By marking the first two points in the blocks $\{-i_j, i_j, 0_{j+1}\}$ and taking $i = 1, 2, \ldots, \frac{p-1}{2}$ we obtain a $\frac{2}{3}$ frame parallel class for the groups $\{0, 0, 0\}$. Applying the action of $\mathbb{Z}_p$ on the blocks gives the $\frac{2}{3}$ frame. Next, we can inflate the $\frac{2}{3}$-3-frame of type $3^p$ using weight two to obtain a $\frac{2}{3}$ frame of type $6^p$, since any TD(3,2) is “resolvable” on any two groups. Therefore, we obtain a $\frac{2}{3}$-3-frame of type $6^3$ and $6^5$. Finally, we construct a $\frac{2}{3}$-3-frame of type $6^4$. Let $V = \mathbb{Z}_24$ and let the groups be $\{i, i + 4, i + 8, i + 12, i + 16, i + 20\}$ for $i = 0, 1, 2, 3$. The base blocks are $\{1, 2, 12\}, \{11, 5, 2\}, \{0, 23, 1\}$. The six marked points are distinct (mod 8) and are not multiples of 4. We can therefore add 8 and 16 to these three blocks to produce altogether nine blocks which form a $\frac{2}{3}$ frame parallel class for the first group. Then adding 4 to these nine blocks generates a second $\frac{2}{3}$ frame parallel class for this group. Finally, we can add $i = 1, 2, 3$ to obtain two $\frac{2}{3}$ frame parallel classes for the $i$th group. Applying a simple result on PBD($\{3, 4, 5\}$) [1] and PBD closure of $\frac{2}{3}$-3-frames, we conclude that if $n \notin \{2, 6, 8\}$, then there exists a $\frac{2}{3}$ 3-frame of type $6^n$. ❑

Lemma 2.6 There exists a $\frac{2}{3}$-resolvable STS($6t + 3$) for all $t \geq 1$.

Proof: Such an STS exists for $t = 1$ by Lemma 2.3. If there exists a $\frac{2}{3}$ 3-frame of type $6^n$, we can adjoin three new points. Then, for each group together with the three new points, we place a $\frac{2}{3}$-resolvable STS(9) with a block on the three new points. We pair up two $\frac{2}{3}$ frame parallel classes to two of the three “parallel classes” in the STS(9), using those two classes not containing the block on the three new points. We produce the final class by merging the classes in the STS(9)s containing the block in the three new points. When $n = 2, 6, 8$, since $6n + 3 = 2 \ast p + 1$ for $p = 7, 19, 25$, we apply Lemma 2.4. Any cyclic STS of order 7 and 19 would be sufficient. For $p = 25$, we take a STS(25) with base blocks $\{0, 1, 12\}, \{0, 2, 9\}, \{0, 3, 8\}, \{0, 4, 10\}$. The difference formed by the first two elements of each block is relatively prime to 25; hence Lemma 2.4 applies. ❑

Combining Lemmas 2.1, 2.6 and 2.2, we have the following theorem.
Theorem 2.7 If \( v \equiv 15,27 \pmod{30} \) and \( v \neq 15 \), then there exists a \( \text{STS}(v) \) that can be embedded into a \( (K_4-e) \)-design on \( \frac{4v}{3} \) points.

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