variety X. This is not so; a counterexample is provided by the nonsingular plane cubic curve, which has a subgroup $\mathrm{Cl}^0 X \subset \mathrm{Cl}\, X$ with $\mathrm{Cl}\, X/\mathrm{Cl}^0 X \cong \mathbb{Z}$, and the elements of $\mathrm{Cl}^0 X$ are in one-to-one correspondence with points of X. Hence, for example, if $k = \mathbb{C}$ then Cl^0 is not even countable.

This big subgroup $\mathrm{Cl}^0 X$ has, however, no effect on intersection numbers, since $\deg D = 0$ for $D \in \mathrm{Cl}^0 X$. The same thing also holds for an arbitrary nonsingular projective variety. Namely, one can prove[14] that if a divisor D is algebraically equivalent to 0 (see Chap. III, 4.4 for the definition), then $D_1 \cdots D_{n-1} D = 0$ for any divisors $D_1, \dots, D_{n-1}$. Thus intersection numbers depend only on the classes of divisors in $\mathrm{Div}\, X/\mathrm{Div}^a X$ (the Néron–Severi group $\mathrm{NS}\, X$). Chap. III, 4.4, Theorem D asserts that this group is finitely generated. Obviously, if $E_1, \dots, E_r$ are generators of this group, then in order to know any intersection numbers of divisors on X, it is enough to know the finitely many numbers $E_1^{i_1} \cdots E_r^{i_r}$ with $i_1 + \cdots i_r = \dim X$, by analogy with what we saw in Examples 1–2. In other words, an analogue of Bézout's theorem holds for X.

2.2. Varieties over the Reals

The different versions of Bézout's theorem proved in 2.1 have some pretty applications to algebraic geometry over $\mathbb{R}$.

We return to 2.1, Example 1, and suppose that the equations $F_i = 0$ for $i = 1, \dots, n$ have real coefficients, and that we are interested in real solutions. If $\deg F_i = m_i$ and the divisors D_i are in general position then $D_1 \cdots D_n = m_1 \cdots m_n$, as proved in 2.1, Example 1. By definition, $D_1 \cdots D_n = \sum (D_1 \cdots D_n)_x$, where the sum runs over solutions x of the system of equations $F_1 = \cdots = F_n = 0$. In this we must of course consider both real and complex solutions x. However, since the F_i have real coefficients, whenever x is a solution then so is the complex conjugate $\overline{x}$. By definition of the intersection number it follows at once that $(D_1 \cdots D_n)_x = (D_1 \cdots D_n)_{\overline{x}}$, and hence $D_1 \cdots D_n \equiv \sum (D_1 \cdots D_n)_y \bmod 2$, where now the sum takes place only over real solutions. In particular if $D_1 \cdots D_n$ is odd (which holds if and only if all the degrees $\deg F_i = m_i$ are odd), then we deduce that there exists at least one real solution. This assertion is proved under the assumption that the D_i are in general position. But the following simple argument allows us to get rid of this restriction.

The point is that in our case the theorem on moving the support of a divisor can be proved very simply and in a more explicit form. Namely, we can choose a linear form l nonzero at all the points $x_1, \dots, x_r$ we want to move the support of the divisor away from. If D is defined by a form F of degree m then the divisor D' defined by the form $F_\varepsilon = F + \varepsilon l^m$ will satisfy all the conditions in the conclusion of the theorem if $F(x_i) + \varepsilon l(x_j)^m \neq 0$ for

[14] See Fulton [27], Chap. 10 for a proof (in much more advanced terms).

$j = 1, \ldots, r$. These conditions can be satisfied for arbitrarily small values of ε.

We now show how to get rid of the general position restriction in the assertion we proved above on the existence of a real solution of a system of equations of odd degrees. Let

$$F_1 = \cdots = F_n = 0 \tag{1}$$

be any such system. By what we have said above we can find linear forms l_i and arbitrarily small values of ε such that the divisors defined by the forms $F_{i,\varepsilon} = F_i + \varepsilon l_i^{m_i}$ are in general position. Now we proved above that the system $F_{1,\varepsilon} = \cdots = F_{n,\varepsilon} = 0$ has a real solution x_ε. Because projective space is compact, we can find a sequence of numbers $\varepsilon_m \to 0$ such that the sequence x_{ε_m} converges to a point $x \in \mathbb{P}^n$. Now $F_{j,\varepsilon_m} \to F_j$ as $\varepsilon \to 0$, so that x is a solution of the system (1).

We state the result we have just proved.

Theorem 1. *A system of n homogeneous real equations in $n+1$ variables has a nonzero real solution if the degree of each equation is odd.* □

Entirely analogous arguments apply to the variety $\mathbb{P}^n \times \mathbb{P}^m$ (see 2.1, Example 2). We get the following result.

Theorem 2. *A system of real equations*

$$F_i(x_0 : \cdots : x_n; y_0 : \cdots : y_m) = 0 \qquad \text{for } i = 1, \ldots, n+m$$

has a nonzero real solution if the number $\sum k_{i_1} \cdots k_{i_n} l_{j_1} \cdots l_{j_m}$ is odd. Here k_i and l_i are the degrees of homogeneity of F_i in the two sets of variables, and we consider a solution to be zero if either $x_0 = \cdots = x_n = 0$ or $y_0 = \cdots = y_m = 0$. □

Theorem 2 has interesting applications to algebra. One of these is concerned with the question of division algebras over $\mathbb{R}$. If an algebra over $\mathbb{R}$ has rank n then it has a basis $e_1, \ldots, e_n$, and the algebra structure is determined by a multiplication table

$$e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k \qquad \text{for } i, j = 1, \ldots, n. \tag{2}$$

We do not assume that the algebra is associative, so that the structure constants c_{ij}^k can be arbitrary. The algebra is called a *division algebra* if the equation

$$ax = b \tag{3}$$

has a solution for every $a \neq 0$ and every b. It is easy to see that this is equivalent to the nonexistence of zerodivisors in the algebra. For this it is enough to consider the linear map φ given by $\varphi(x) = ax$ in the real vector space formed by elements of the algebra. The condition that (3) has a solution means that the image of φ is the whole space, and this is equivalent to $\ker \varphi = 0$, as is well known. This condition means just that the algebra has no zerodivisors, that is $xy = 0$ implies either $x = 0$ or $y = 0$. If $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{j=1}^{n} y_j e_j$, then (2) gives

$$xy = \sum_{k=1}^{n} z_k e_k, \quad \text{where} \quad z_k = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^k x_i y_j \quad \text{for } k = 1, \dots, n.$$

Thus the algebra is a division algebra if the system of equations

$$F_k(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^k x_i y_j = 0 \quad \text{for } k = 1, \dots, n \tag{4}$$

has no real solutions with $(x_1, \dots, x_n), (y_1, \dots, y_n) \neq (0, \dots, 0)$. These equations very nearly satisfy the conditions of Theorem 2. The difference is that the F_k define divisors in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, the number n of which is not equal to the dimension $2n - 2$ of the variety. We therefore choose any integer r with $1 \leq r \leq n - 1$ and set $x_{r+2} = \cdots = x_n = 0$ and $y_{n-r+2} = \cdots = y_n = 0$. The equations $F_k\big((x_1, \dots, x_{r+1}, 0, \dots, 0), (y_1, \dots, y_{n-r+1}, 0, \dots, 0)\big) = 0$ for $k = 1, \dots, n$ are now defined in $\mathbb{P}^r \times \mathbb{P}^{n-r}$, and a fortiori have no nonzero real roots. According to Theorem 2 this is only possible if the sum

$$\sum k_{i_1} \cdots k_{i_r} l_{j_1} \cdots l_{j_{n-r}} \tag{5}$$

is even, and this must moreover hold for all $r = 1, \dots, n - 1$. In our case the forms F_k are bilinear, so that $k_i = l_i = 1$, and the sum (5) equals the number of summands, which is $\binom{r}{n}$. We see that if (4) has no nonzero real solutions then all the integers $\binom{r}{n}$ are even for $r = 1, \dots, n - 1$. This is only possible if $n = 2^k$. Indeed, our condition on $\binom{r}{n}$ can be expressed as follows: over the field with 2 elements $\mathbb{F}_2$ we have $(T + 1)^n = T^n + 1$. If $n = 2^l m$ with m odd and $m > 1$ then over $\mathbb{F}_2$,

$$(T + 1)^{2^l m} = (T^{2^l} + 1)^m = T^{2^l m} + mT^{2^l (m-1)} + \cdots + 1 \neq T^{2^l m} + 1.$$

We have proved the following result:

Theorem 3. *The rank of a division algebra over* $\mathbb{R}$ *is a power of* 2. □

It can be proved that a division algebra over $\mathbb{R}$ exists only for $n = 1, 2, 4$ and 8. The proof of this fact uses rather delicate topological arguments.

Applying analogous arguments, one can investigate for which values of m and n the system of equations

$$\sum_{k=1}^{n}\sum_{j=1}^{m} c_{ij}^{k} x_k y_j = 0 \qquad \text{for } i = 1, \dots, n, \tag{6}$$

does not have nonzero real solutions. Based on the interpretation of the tangent space to $\mathbb{P}(V)$ given in Chap. II, 1.3, one can easily show that under the stated assumption, (6) defines $(m-1)$ linearly independent tangent vectors at each point of $\mathbb{P}^{n-1}$, that is, $(m-1)$ everywhere linearly independent vector fields on $\mathbb{P}^{n-1}$. In this form, the question of the possible values of m and n has been completely answered by topological methods. The question is interesting because it is equivalent to that of knowing whether the system of partial differential equations

$$\sum_{k=1}^{n}\sum_{j=1}^{m} c_{ij}^{k} \frac{\partial u_j}{\partial x_k} = 0 \qquad \text{for } i = 1, \dots, m$$

is elliptic.

2.3. The Genus of a Nonsingular Curve on a Surface

The following formula plays an enormous role in the geometry on a nonsingular projective surface X. It is usually called the *adjunction formula* or the *genus formula*, and expresses the genus of a nonsingular curve $C \subset X$ in terms of certain intersection numbers:

$$g_C = \frac{1}{2}C(C+K) + 1; \tag{1}$$

here g_C is the genus of C and K the canonical class of X.

This formula can be proved using only the methods we already know. However, a clearer and more transparent geometric proof follows from the elementary properties of vector bundles. This is given in Chap. VI, 1.4, Theorem 4. Here we only discuss a number of applications.

Example 1. The projective plane. If $X = \mathbb{P}^2$ then $\operatorname{Cl} X = \mathbb{Z}$, with generator L, the class containing all the lines of $\mathbb{P}^2$. If $C \subset \mathbb{P}^2$ has degree n then $C \sim nL$. In view of $K = -3L$ and $L^2 = 1$, in this case (1) gives

$$g = \frac{n(n-3)}{2} + 1 = \frac{(n-1)(n-2)}{2}.$$

We obtained the same result in Chap. III, 6.4 by a different method.

Example 2. The nonsingular quadric surface. Let $X \subset \mathbb{P}^3$ be a nonsingular quadric surface in $\mathbb{P}^3$. Let's see how to classify nonsingular curves on X in terms of their geometric properties.