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ON COMPLEMENTED COPIES OF c_0 IN L_X^p , $1 \leq p < \infty$

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Let (S, Σ, μ) be a not purely atomic measure space and X be a Banach space. In this note we want to show that if X contains a copy of c_0 , then the usual Banach space of the Lebesgue-Bochner integrable functions L_X^p , $1 \leq p < \infty$, contains a complemented copy of c_0 . Our result is similar in spirit to one obtained in [1] by Cembranos concerning the Banach space $C_K(K)$; in passing we observe that the Cembranos result has been extended in [3] to the case of ϵ -tensor products and then in [2] to the case of the Banach space of compact weak*-weak continuous operators.

In order to prove our theorem we need the definition of limited sets. A (bounded) subset M of a Banach space X is said to be limited if for each weak* null sequence $(x_n^*) \subset X^*$ we have $\lim_n \sup_{x \in M} |x_n^*(x)| = 0$. Further we use the following result obtained in [2]

Lemma. *If X contains an unlimited sequence (x_n) that is equivalent to the unit basis of c_0 , then X contains a complemented copy of c_0 .*

Now, we are ready to show our theorem.

Theorem. *Assume X contains a copy of c_0 . Then L_X^p , $1 \leq p < \infty$, contains a complemented copy of c_0 .*

Proof. We shall construct a sequence of functions in L_X^p which is equivalent to the unit basis of c_0 and is not limited in L_X^p , so by virtue of the Lemma we will be done. Let (x_n) be a sequence in X equivalent to the unit basis of c_0 and (x_n^*) be a bounded sequence in X^* such that $x_m^*(x_n) = \delta_{mn}$. It suffices to consider the case of $[0,1]$ with Lebesgue measure. We

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consider Rademacher functions r_n and define a sequence (f_n) in L_X^p by putting $f_n = r_n x_n$ and a sequence in $(L_X^p)^*$ by putting $f_n^* = r_n x_n^*$. First of all we show that (f_n) is a sequence equivalent to the unit basis of c_0 . Since (x_n) is a copy of the unit basis of c_0 , there are $h_1, h_2 \in \mathbf{R}^+$ such that, for all finite sequences $(a_i)_{i=1}^s$ of real numbers, we have

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i x_i \right\|_X \leq h_2 \max_{1 \leq i \leq s} |a_i|.$$

Since $|r_i(t)| = 1$ on $[0,1]$ for all $i \in \mathbf{N}$, we have

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i r_i(t) x_i \right\|_X \leq h_2 \max_{1 \leq i \leq s} |a_i| \quad t \in [0, 1].$$

This easily gives that

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i f_i \right\|_{L_X^p} \leq h_2 \max_{1 \leq i \leq s} |a_i|,$$

i.e. (f_n) is equivalent to the unit basis of c_0 . Now, we observe that

$$f_n^*(f_n) = \int_{[0,1]} x_n^*(x_n) r_n^2(t) dm = 1 \quad \text{for all } n \in N.$$

So it remains only to prove that $f_n^* \xrightarrow{w^*} \theta$. To this purpose take $h \in L_X^p$ and observe that $|f_n^*(h)| = \left| \int_{[0,1]} x_n^*(h(t)) r_n(t) dm \right| \leq \|x_n^*\| \left\| \int_{[0,1]} h(t) r_n(t) dm \right\|$ for all $n \in N$. Since (x_n^*) is bounded and moreover $\lim_n \left\| \int_{[0,1]} h(t) r_n(t) dm \right\| = 0$, we get $\lim_n f_n^*(h) = 0$. Arbitrariness of h in L_X^p gives that $f_n^* \xrightarrow{w^*} \theta$. Our proof is complete.

The above Theorem has the following Corollary

Corollary. *Let X contain a copy of c_0 . Then $L_X^p, 1 \leq p < \infty$, is neither a Grothendieck space nor a dual space.*

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