Some permanence results of properties of Banach spaces

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Abstract. Using some known lifting theorems we present three-space property type and permanence results; some of them seem to be new, whereas other are improvements of known facts.

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This short note is devoted to the proof of certain results showing that some isomorphic properties of Banach spaces or of operators between Banach spaces have some nice permanence property. We use lifting results for sequences due to Lohman [17] and Gonzalez and Onieva [10] and a lifting result for vector measures obtained by the author and Rao in [7]. Some of our theorems are improvements of known ones, whereas others of them seem to be completely new.

We start recalling the old Lohman result of lifting

Theorem 1 ([17]). *Let E be a Banach space and F be a subspace of E not containing copies of ℓ^1. Let $Q : E \to E/F$ denote the quotient map. Every weakly Cauchy sequence $(\hat{x}_n) \subset E/F$ admits a subsequence that is the image under the quotient map of a weakly Cauchy sequence in E.*

It will be applied to get a permanence result for the Dunford-Pettis Property and a variation, introduced in [8], of this famous notion

Definition 1 (see, for instance, [1], [8]). A Banach space E is said to possess the *Dunford-Pettis Property* (the *Alternative Dunford-Pettis Property*, resp.) if for any weakly null sequences $(x_n) \subset E$ (any $(x_n) \subset E$ weakly converging to some $x \in E$ with $\|x_n\| = \|x\| = 1$, resp.) and any weak Cauchy sequence $(x_n^*) \subset E^*$ one has $\lim_n x_n^*(x_n) = 0$ ($\lim_n x_n^*(x_n - x) = 0$, resp.).

We observe that this is not the original definition of Dunford-Pettis Property (of Alternative Dunford-Pettis Property, resp.), but an equivalent formulation (see, for instance, [1] and [8]) more useful than the original definition for our purpose. It is well known that $C[0, 1]$ enjoys Dunford-Pettis Property, whereas infinite dimensional reflexive Banach spaces cannot possess it; furthermore, since any

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separable Banach space is isomorphic to a subspace of $C[0,1]$ ([19, Theorem 3.12 on p. 142]), we may conclude that such a property is not usually inherited by subspaces. However

Theorem 2. *Let E be a Banach space with the Dunford-Pettis Property (the Alternative Dunford-Pettis Property, resp.) and F be a w*-closed subspace of E^* not containing copies of ℓ^1. Thus ${}^{\perp}F \subset E$ has the Dunford-Pettis Property (the Alternative Dunford-Pettis Property, resp.), too.*

Proof: We present the proof of the result relative to the Dunford-Pettis Property, since the "Alternative" case is similar at all. Let $(x_n) \subset {}^{\perp}F$ be a weakly null sequence and $(\hat{x}^*_n) \subset ({}^{\perp}F)^* = E^*/F$ be a weak Cauchy sequence with $\inf_n |\hat{x}^*_n(x_n)| > 0$. Thanks to Theorem 1 we may find a weak Cauchy subsequence $(x^*_{k_n}) \subset E^*$ with $Q(x^*_{k_n}) = \hat{x}^*_{k_n}$, where $Q : E^* \to E^*/F$ is the quotient map. If $i : {}^{\perp}F \to E$ denotes the isomorphic embedding, we have ([18, Theorem 1.10.16])

$$\langle x_{k_n}, Q(x^*_{k_n})\rangle = \langle i(x_{k_n}), x^*_{k_n}\rangle \qquad \forall\, n \in \mathbb{N}.$$

Hence, thanks to the Dunford-Pettis Property of E, we deduce that

$$\langle x_{k_n}, \hat{x}_{k_n}\rangle = \langle x_{k_n}, Q(x^*_{k_n})\rangle = \langle i(x_{k_n}), x^*_{k_n}\rangle \to 0$$

and this clearly contradicts our assumption. □

Pelczynski, in [20], introduced the following property

Definition 2. A Banach space E has *Property* (u) if for any weak Cauchy sequence (x_n) there is a weakly unconditionally converging series $\sum_{n=1}^{\infty} y_n$ such that $x_n - \sum_{i=1}^{n} y_i \xrightarrow{w} \theta$.

For this property we have the following result

Theorem 3. *Let E be a Banach space with Property (u) and F be a subspace not containing copies of ℓ^1. Thus E/F has Property (u).*

Proof: Let $(\hat{x}_n) \subset E/F$ be a weak Cauchy sequence. Using Theorem 1 we may found a sequence $(x_{k_n}) \subset E$ that is weak Cauchy and such that $x_{k_n} \in \hat{x}_{k_n}$, for all $n \in \mathbb{N}$. By our assumption there is a weakly unconditionally converging series $\sum_{n=1}^{\infty} y_n$ such that $x_{k_n} - \sum_{i=1}^{n} y_i \xrightarrow{w} \theta$. It is now very easy to show that $\hat{x}_n - \sum_{i=1}^{n} \hat{y}_i \xrightarrow{w} \theta$, with $\sum_{n=1}^{\infty} \hat{y}_n$ weakly unconditionally converging. □

The next group of results utilizes the following Gonzalez-Onieva result.

Theorem 4 ([10]). *Let E be a Banach space and F be a reflexive subspace of E. Let $Q : E \to E/F$ denote the quotient map. Let $(x_n) \subset E$ be a bounded sequence such that $(Q(x_n))$ converges weakly to some $Q(x) \in E/F$. Thus (x_n) is relatively weakly compact.*

We now introduce certain well known isomorphic properties of Banach spaces.

Definition 3 ([11], [20]). Let E be a Banach space. We say that E has *Property (V) of Pelczynski* (resp. *Dieudonné Property* or *Reciprocal Dunford-Pettis Property*) if any unconditionally converging (resp. weakly completely continuous or Dunford-Pettis) operator defined on E is weakly compact.

For such properties we have the following permanence property by "small" perturbation.

Theorem 5. *Let E be a Banach space and F be a reflexive subspace of E^*. Also assume that ${}^{\perp}F$ has Property (V) of Pelczynski (resp. Dieudonné Property or Reciprocal Dunford-Pettis Property). Then E has Property (V) (resp. Dieudonné Property or Reciprocal Dunford-Pettis Property).*

Proof: The proof is the same in all of the cases, so that we perform just the one about Property (V) of Pelczynski. Let $Q : E^* \to E^*/F$ be the quotient map and $i : E^*/F \to ({}^{\perp}F)^*$ be the existing surjective isomorphism ([18, Theorem 1.10.16]); it is well known ([18, Theorem 1.10.16]) that $i \circ Q : E^* \to ({}^{\perp}F)^*$ is w*-w* continuous, since $i \circ Q(x^*)$ is just the restriction of x^* to ${}^{\perp}F$; so there is $S : {}^{\perp}F \to E$ with $S^* = i \circ Q$. Hence, for any unconditionally converging operator $T : E \to Y$ the operator $T \circ S : {}^{\perp}F \to Y$ is an unconditionally converging operator, that must so be weakly compact; hence, also $S^* \circ T^* = i \circ Q \circ T^*$ must be weakly compact; this in turn gives that $Q \circ T^*$ must be weakly compact, since i is a surjective isomorphism. We may so assume that, for a bounded sequence $(y_n^*) \subset Y^*$, the sequence $(Q \circ T^*)(y_n^*))$ is weakly converging to some $Q(x^*)$; Theorem 4 gives that $(T^*(y_n^*))$ is relatively weakly compact. The arbitrariness of $(y_n^*) \subset Y^*$ gives that T^* (and so T) is weakly compact. □

We remark that Theorem 5 improves a result (about Property (V)) due to Godefroy and P. Saab obtained in [9] under the assumption *"E is separable"*; the part relative to the other two properties seems to be new.

In order to present our second block of results which use Theorem 4 we need to introduce the following

Definition 4 ([4], [5], [15]). Let E be a Banach space and M a bounded subset of E. Then M is called

(i) *limited* if any operator $T : E \to c_0$ maps M onto a relatively compact subset of c_0,
(ii) *Dunford-Pettis* if any weakly compact operator $T : E \to c_0$ maps M onto a relatively compact subset of c_0,

(iii) *Grothendieck* if any operator $T : E \to c_0$ maps M onto a relatively weakly compact subset of c_0.

We shall say that E has the

(j) (BD) *Property* if any limited set in E is relatively weakly compact,

(jj) (RDP^*) *Property* if any Dunford-Pettis set in E is relatively weakly compact,

(jjj) (GPw) *Property* if any Grothendieck set in E is relatively weakly compact.

For these properties we have the following three spaces property result.

Theorem 6. *Let E be a Banach space and F be a reflexive subspace of E. If E/F has the (BD) Property or the (RDP*) Property or the (GPw) Property, then E has the same property.*

Proof: The proofs are the same in all of the cases, so we just perform the one relative to (BD) Property. Let (x_n) be a limited set in E. If $Q : E \to E/F$ denotes the quotient map, also $(Q(x_n))$ is a limited set in E/F and, by our assumption, it must be relatively weakly compact; hence some subsequence $(Q(x_{n_k}))$ has to converge weakly to some element of E/F, say $Q(x)$. Thanks to Theorem 4 a further subsequence of (x_{n_k}) has to converge weakly in E. □

Now we present an application of Theorem 4 to a property of Banach spaces that is not invariant under general isomorphisms, but just under isometries (like the Alternative Dunford-Pettis Property already quoted).

Definition 5 (see, for instance, [8])**.** We say that a Banach Space E has the *Kadec-Klee Property* if each sequence $(x_n) \subset E$ weakly converging to some $x \in E$, actually converges strongly to x, provided that $\|x_n\| \to \|x\|$.

Theorem 7. *Let E be a Banach space with the Kadec-Klee Property and F be a reflexive subspace of E. Then the space E/F satisfies the Kadec-Klee Property.*

Proof: By contradiction suppose there are $(\hat{x}_n), \hat{x}$ in E/F with

$$\hat{x}_n \xrightarrow{w} \hat{x},\ \|\hat{x}_n\| \to \|\hat{x}\|,\ \lim_n \|\hat{x}_n - \hat{x}\| = \eta > 0.$$

Theorem 4 gives that there are a sequence $(x_n) \subset E$, $x_n \in \hat{x}_n$, for all $n \in \mathbb{N}$, and a subsequence (x_{n_k}) weakly converging to some $y \in E$; clearly $y \in \hat{x}$. Choose $h_n \in \hat{x}_n, n \in \mathbb{N}$, so that

$$\|\hat{x}_n\| \le \|h_n\| \le \|\hat{x}_n\| + \frac{1}{n} \qquad \forall\, n \in \mathbb{N}.$$

Since $(h_{n_k} - x_{n_k}) \subset F$, a reflexive space, there is a further subsequence $(h_{n_{k_p}} - x_{n_{k_p}})$ weakly converging to some $z \in F$; it follows that $h_{n_{k_p}} \xrightarrow{w} y + z \in \hat{x}$. Now, we have

$$\lim_p \|\hat{x}_{n_{k_p}}\| + \frac{1}{n_{k_p}} \ge \lim_p \|h_{n_{k_p}}\| \ge \|y + z\| \ge \|\hat{x}\|.$$

Hence $\|h_{n_{k_p}}\| \longrightarrow \|y+z\|$. It follows that $h_{n_{k_p}} \longrightarrow y+z$ strongly since E has the Kadec-Klee Property, from which it follows that $(\hat{x}_{n_{k_p}})$ converges strongly to $\hat{x}$, a contradiction that concludes our proof. □

The last applications of Theorem 4 we want to present are to weakly compact operators between Banach spaces; by $L(X,Y)$ ($W(X,Y)$, resp.) we shall denote the space of all linear and bounded (linear and weakly compact, resp.) operators from X into Y.

Theorem 8. *Let X, E be two Banach spaces and F be a reflexive subspace of E. Then $L(X,E/F)=W(X,E/F)$ implies that $L(X,E)=W(X,E)$.*

Proof: Let $T \in L(X,E)$ be; hence $Q \circ T : X \to E/F$ is weakly compact. Let (x_n) be a bounded sequence in X; we have that $(Q \circ T(x_n)) = (Q\,[T(x_n)])$ is relatively weakly compact, which means that a suitable subsequence $(Q \circ T(x_{n_k}))$ must converge weakly to some element, say $Q(y)$, in E/F. Theorem 4 gives now that $(T(x_{n_k}))$ must have a further subsequence that converges in the weak topology of E, so that T is weakly compact. □

Another result about operators is

Theorem 9. *Let Y be a $\mathcal{L}_1-$space. Let E be a Banach space and F be a reflexive subspace of E. Let $Q : E \to E/F$ denote the quotient map. If $T : Y \to E/F$ is a weakly compact operator, then there is a weakly compact operator $\tilde{T} : Y \to E$ such that $Q \circ \tilde{T} = T$.*

Proof: The existence of $\tilde{T}$ under our assumptions is guaranteed by a result due to Kalton and Pelczynski ([13]). We have just to show that $\tilde{T}$ is weakly compact. This part follows as in the proof of Theorem 8, so that we are done. □

Of course, results of the same nature as Theorem 9 are also possible each time we have a lifting result for operators (see, for instance, [12], [16]).

Now, we pass to another result of permanence of a different isomorphic property of Banach spaces, in the same spirit of the results we got in [6].

Definition 6 ([14])**.** A bounded linear operator $T : L^1[0,1] \to X$ is called *Nearly Representable* if for each Dunford-Pettis operator $D : L^1[0,1] \to L^1$ the composition $T \circ D : L^1[0,1] \to X$ is Bochner representable. A Banach space X has the (*NRNP*) if every nearly representable operator $T : L^1[0,1] \to X$ is Bochner representable.

We have to use the following lifting result for vector measures obtained by ourselves and Rao in [7].

Theorem 10 ([7])**.** *Let Ω be a set and Σ be a $\sigma-$algebra of subsets of Ω. Let E be a Banach space and F be a subspace of E that is w*-closed in some dual space G in turn containing E. Then, for any countably additive vector measure* *of bounded variation $\nu : \Sigma \to E/F$ there is countably additive vector measure of bounded variation $\tilde{\nu} : \Sigma \to E$ such that*

$$Q(\tilde{\nu}(A)) = \nu(A) \qquad \forall\, A \in \Sigma$$

where Q is the quotient map from E onto E/F.

Our result is an immediate consequence of Theorem 1 in [6], so we state it without proof.

Theorem 11. *Assume that E is a Banach space with the (NRNP), F is a closed subspace of E with the Radon Nikodym Property that is also w*-closed in some dual space G in turn containing E. Then E/F has the (NRNP).*

This result improves Corollaries 2 and 3 obtained in [6], under the assumption *"E is a dual space"* and *"F is reflexive"*, respectively. We now present an occurrence in which our new result is applicable, but not the cited corollaries from [6]. We choose $E = K(c_0, T) = L(c_0, T)$ where T is the weakly sequentially complete Banach lattice constructed by Talagrand in [23]; from results in [4] and [21] it follows that E enjoys the (NRNP); also choose $G = L(c_0, T^{**})$ a dual space and $F = K(c_0, R) = L(c_0, R)$, with R any reflexive subspace of T; T^{**} is a Banach lattice containing copies of c_0, otherwise by results in [22] $\mathrm{cabv}(\Sigma, T)$ would be weakly sequentially complete, that is not the case ([23]); this implies that G does not enjoy the (NRNP) (see [14]). Furthermore, it is not difficult to see that F is a w*-closed subspace in the dual space G; indeed, let $g \in G$ so that some net $(h_\alpha) \subset F$ converges weak* to g; it follows that

$$\lim_{\alpha} h_\alpha(x \otimes t^*) = g(x \otimes t^*) \qquad \forall\, x \in c_0,\ t^* \in T^*\,;$$

we deduce that $(h_\alpha(x))$, a net in R, converges weakly, inside R, to $g(x) \in T$, so that $g(x) \in R$, too, and $g \in F$. Also, we observe that such an F has the Radon-Nikodym Property thanks to an old result due to Diestel-Morrison ([3]). Hence all of the hypotheses of Theorem 11 are verified and we may so conclude that E/F enjoys the (NRNP).

Another occurrence in which Theorem 11 is applicable is the following. Let T be the unit circle and let $\wedge$ be a Riesz subset of $\mathbb{Z}$ that is not nicely placed. Then $L^1_\wedge$ is a w^*-closed subspace of $C(T)^*$ having the Radon-Nikodym property. Therefore $\tilde{Q} : \mathrm{cabv}(\mu, L^1) \to \mathrm{cabv}(\mu, L^1/L^1_\wedge)$, defined by $\tilde{Q}(\nu)(A) = Q[\nu(A)]$, $A \in \Sigma$, is a quotient map and so $L^1/L^1_\wedge$ has the (NRNP).

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