Answer to a Question by
M. Feder About $K(X, Y)$

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ABSTRACT. We show that a Banach space $E$ construed by Bourgain-Delbaen in 1980 answers a question put by Feder in 1982 about spaces of compact operators.

Let $X, Y$ be two Banach spaces. By $K(X, Y), W(X, Y), L(X, Y)$ we denote the Banach spaces of all compact, weakly compact and bounded linear operators from $X$ into $Y$, respectively. In the paper [4] Feder put a question that in light of recent results in [3] can be reformulated as it follows

**Question.** Do Banach spaces $X$ and $Y$ exist so that $K(X, Y) \neq L(X, Y)$ and however $K(X, Y)$ does not contain a copy of $c_0$?

Fedor's question is related to the following problem: let us assume $X, Y$ are such that $L(X, Y) \neq K(X, Y)$; is $K(X, Y)$ uncomplemented in $L(X, Y)$?

The results in [3] and [4] show that the best result known is the following one: if $c_0$ embeds into $K(X, Y)$, then $K(X, Y)$ is uncomplemented in $L(X, Y)$; so it remains to study the case of $K(X, Y)$ containing no copy

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of $c_0$, if such two spaces exist (i.e. if Feder's question has a positive answer).

Up to now, no answer to Feder's question has been given as far as we know.

In this short note we want to show that a Banach space constructed by Bourgain and Delbaen in [1] (before Feder's paper appeared) answers positively to the above question. The space $E$ constructed by Bourgain and Delbaen is a $\mathcal{L}_\infty$-space with the Radon-Nikodym property such that $E^*$ is isomorphic to $\ell^1$. If we take $X = Y = E$ we clearly get $K(X, Y) = W(X, Y) \neq L(X, Y)$. Now, let us assume that $c_0$ lives inside $K(X, Y)$. We recall that $K(X, Y) = K_w(X^{**}, Y)$ and $W(X, Y) = W_w(X^{**}, Y)$ where $K_w(X^{**}, Y)$, $W_w(X^{**}, Y)$ denote the spaces of all $w* - w$ continuous compact, bounded operators from $X^{**}$ into $Y$, respectively. So we can act in $K_w(X^{**}, Y)$. Let $(T_n)$ be a copy of the unit vector basis of $c_0$ in $K_w(X^{**}, Y)$. For $x^{**} \in X^{**}$, the series $\Sigma T_n(x^{**})$ is unconditionally converging in $Y$ and so, for any $\xi_{\epsilon} \in \ell^\infty$, the series $\Sigma \xi_{\epsilon} T_n(x^{**})$ is also unconditionally converging. It is not difficult to see that the map $\xi_{\epsilon} \rightarrow \Sigma \xi_{\epsilon} T_n$ defines a bounded, linear operator from $\ell^\infty$ into $L(X^{**}, Y)$. We shall prove that, actually, $\Sigma \xi_{\epsilon} T_n \in \mathcal{L}_w(X^{**}, Y)$. Let $(x^{**}_\alpha)$ be a $w*$-null net in $B_{x^{**}}$ and $y^* \in Y^*$. If we denote by $\varphi_{\alpha}$ the operator $\Sigma \xi_{\alpha} T_n$, we have to show that

$$\lim_{\alpha} |\varphi_{\alpha}(x^{**}_\alpha)(y^*)| = 0$$

Since $\Sigma \xi_{\alpha} T_n(y^*)$ is unconditionally converging, we have

$$\lim_{\alpha} \sup_{p=1}^{\infty} \left| \xi_{p} T_p(y^*) (x^{**}) \right| = 0.$$ 

So, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\sum_{\alpha=1}^{n_0} |\xi_{p} T_p(y^*)(x^{**})| < \epsilon / 2$ for all $\alpha$; since $x^{**}_\alpha \xrightarrow{w^*} \theta$, it is obvious that

$$\lim_{\alpha} \sum_{\rho=1}^{n_0-1} |\xi_{p} T_p(y^*)(x^{**})| = 0,$$

and so for $\alpha$ sufficiently large we have

$$\sum_{\rho=1}^{n_0-1} |\xi_{p} T_p(y^*)(x^{**})| < \epsilon / 2.$$
Hence, for $\alpha$ sufficiently large, we get

$$\sum_{\rho=1}^{\alpha} |\xi_{\rho} T_{\rho}^*(y^*)/(x^*_{\alpha})| < \varepsilon,$$

which means that

$$\lim_{\alpha} |\varphi_{\xi}(x^*_{\alpha})(y^*)| = 0.$$

Hence, $\sum \xi_{\alpha} T_{\alpha} \in L_{w^*}(X^{**}, Y)$. In this way, we have defined a bounded, linear operator from $\ell_\infty$ into $L_{w^*}(X^{**}, Y) = W(X, Y)$ such that the unit vector basis of $c_0$ is mapped onto a not relatively compact sequence. A result due to Rosenthal ([5]) implies that $\ell_\infty$ must live inside $W(X, Y)$. Since $W(X, Y) = K(X, Y)$, $\ell_\infty$ embeds into $K(X, Y)$, too. But this contradicts a corollary of the main result of [2].

We also observe that in the paper [1] another class of Banach spaces $F$ has been introduced; as remarked in the NOTES ADDED to our paper [3] if $X = Y = F$ we get a second example of Banach spaces answering positively Feder's question; even in that case $W(X, Y) = K(X, Y)$. So we can conclude this note with the following questions

**Question A.** Do Banach spaces $X$, $Y$ exist so that $K(X, Y) \neq W(X, Y)$ and $c_0$ does not embed into $K(X, Y)$?

**Question B.** Let $X = Y = E$ (or $F$) be the Bourgain-Delbaen space. Is $K(X, Y)$ uncomplemented in $L(X, Y)$?

**References**


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