FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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1. INTRODUCTION

The study in fixed point theory has generally developed in three main directions: generalization of conditions which ensure existence, and, if possible, uniqueness, of fixed points; investigation of the character of the sequence of iterates \( \{T^n x\}_{n=1}^{\infty} \), where \( T: X \to X \), \( X \) a complete metric space, is the map under consideration; study of the topological properties of the set of fixed points, whenever \( T \) has more than one fixed point. This note treats only some aspects of the first and second question, along a line followed by many other authors. We mention, in particular, De Blasi [3], Kannan [4], Opial [7], Reich [8–10].

More precisely we consider maps \( T: X \to X \), which satisfy conditions of the type

\[
d(Tx, Ty) \leq \varphi(d(x, y)) + \psi(d(Tx, x)) + \zeta(d(Ty, y))
\]

for each \( x, y \in X \), and for these mappings we prove, under suitable hypotheses, existence and uniqueness of fixed points.

The paper consists of four sections. In Section 1 we prove a fundamental lemma. In Section 2 we use this in order to establish the main fixed point theorems. In Section 3 we obtain some well-known results as corollaries of our theorems. Further results are presented in Section 4.

1. Through all the paper, \( X \) denotes a complete metric space and \( T: X \to X \), an asymptotically regular mapping (see [2]); i.e., a function satisfying \( \lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0 \) for each \( x \in X \).

Furthermore, we suppose that there exist three functions \( \varphi, \psi, \chi \), from \([0, +\infty[\) into \([0, +\infty[\), which satisfy the assumptions:

\[
(I_1) \quad \varphi(r) < r \quad \text{if} \quad r > 0,
\]

\[
(I_2) \quad \text{there exists} \lim_{r \to \infty} \varphi(r) < \varphi(\bar{r}) \quad \text{for each} \quad \bar{r} \in [0, +\infty[.
\]

\[
(I_3) \quad \psi(0) = \chi(0) = 0.
\]

Moreover, we suppose that \( T, \varphi, \psi, \chi \) satisfy the inequality

\[
(I_4) \quad d(Tx, Ty) \leq \varphi(d(x, y)) + \psi(d(Tx, x)) + \chi(d(Ty, y)) \quad \text{for each} \quad x, y \in X.
\]

**Lemma.** Under the above assumptions on \( X \) and \( T \) and if, in addition, \( \psi \) and \( \chi \) are continuous at \( r = 0 \), then, for each \( x \in X \), there exists \( z \in X \) such that \( \{T^n x\}_{n=0}^{\infty} \) converges to \( z \).
Proof: Suppose that there exists \( x \in X \) such that the sequence of iterates is not a Cauchy sequence. Then, following [9] there exist \( \varepsilon > 0 \), \( \{m(j)\}_{j=0}^{\infty} \), \( \{n(j)\}_{j=0}^{\infty} \) which satisfy the conditions

\[
m(j) > n(j) \quad \text{for each } j \in \mathbb{N} \quad (1.1)
\]
\[
\lim_{j \to \infty} n(j) = +\infty \quad (1.2)
\]
\[
d(T^{m(j)}x, T^{n(j)}x) \geq \varepsilon \quad (1.3)
\]
\[
d(T^{m(j)-1}x, T^{n(j)}x) < \varepsilon. \quad (1.4)
\]

Then, we have

\[
\varepsilon \leq d(T^{m(j)}x, T^{n(j)}x) \leq d(T^{m(j)}x, T^{m(j)-1}x) + d(T^{m(j)-1}x, T^{n(j)}x) < \varepsilon + d(T^{m(j)}x, T^{m(j)-1}x)
\]

which implies

\[
\lim_{j \to \infty} d(T^{m(j)}x, T^{n(j)}x) = \varepsilon. \quad (1.5)
\]

On the other hand

\[
d(T^{m(j)}x, T^{n(j)}x) \leq d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x)
\]
\[
T^{m(j)+1}x \leq d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x) + \varphi(d(T^{m(j)}x, T^{n(j)}x))
\]
\[
+ \psi(d(T^{m(j)+1}x, T^{m(j)}x)) + \chi(d(T^{n(j)+1}x, T^{m(j)}x))
\]

that is

\[
d(T^{m(j)}x, T^{n(j)}x) - \varphi(d(T^{m(j)}x, T^{n(j)}x)) \leq d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x) + \psi(d(T^{m(j)+1}x, T^{m(j)}x)) + \chi(d(T^{n(j)+1}x, T^{m(j)}x))
\]

and, letting \( j \to +\infty \)

\[
\varepsilon - \lim_{j \to \infty} \varphi(d(T^{m(j)}x, T^{n(j)}x)) \leq 0.
\]

Hence, by (1.3) and (1.5) it follows that \( \varepsilon = 0 \), a contradiction. Since \( X \) is a complete metric space, the proof is complete.

2. In this section we shall prove two fixed point theorems: the first for noncontinuous, the second for continuous mappings.

**Theorem 1.** Let \( X, T, \varphi, \psi, \chi \) be as in the Lemma. Furthermore, we suppose that \( \chi(r) < r \) if \( r > 0 \). Then \( T \) has a unique fixed point.

**Proof.** Uniqueness is obvious by virtue of hypotheses \((I_1), (I_2)\) and \((I_4)\). So let us show existence. From the Lemma there is a \( z \in X \) such that \( T^n x \to z \), as \( n \to +\infty \), for each \( x \in X \). Since

\[
d(z, Tz) \leq d(z, T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, Tz) \leq d(z, T^n x) + d(T^n x, T^{n+1}x)
\]
\[
+ \varphi(d(T^n x, z)) + \chi(d(Tz, z)) + \psi(d(T^n x, T^{n+1}x))
\]
we have
\[ d(z, Tx) - \chi(d(z, Tz)) \leq 2d(z, T^n x) + d(T^n x, T^{n+1} x) + \psi(d(T^n x, T^{n+1} x)) \]
and, letting \( n \to +\infty \), we have
\[ 0 \leq d(z, Tz) - \chi(d(z, Tz)) \leq 0; \]
thus, \( z = Tz \) follows.

**Remark 1.** Obviously, the role of the functions \( \psi \) and \( \chi \) can be reversed.

For continuous functions, we can prove the following result.

**Theorem 2.** Let \( X, T, \phi, \psi, \chi \) be as in the Lemma. If, in addition, \( T \) is continuous, then it has a unique fixed point.

**Proof.** Uniqueness follows as in Theorem 1. Let \( z \in X \) be such that \( T^n x \to z \), as \( n \to +\infty \). From continuity of \( T \) we obtain
\[ \lim_n T^{n+1} x = Tz \]
and, since \( T \) is asymptotically regular, we have \( z = Tz \).

**Remark 2.** If \( X \) is a closed, non-void, subset of a Banach space \( B \), Theorem 2 is still true under the assumption: "\( T \) is a strongly–weakly continuous mapping from \( X \) into \( X \)."

**Remark 3.** From the Lemma, Theorem 1 and Theorem 2, there follows that the sequence of iterates \( \{ T^n x \}_{n=0}^\infty \) converges to the unique fixed point of \( T \), for each \( x \in X \).

**Remark 4.** In [3] the author puts the following question: "Let \( T \) be an asymptotically regular, continuous mapping and let \( S \) be a non-void, weakly closed subset of a Hilbert space \( X \). If \( T, T:S \to S \), satisfies the condition
\[ \| Tx - Ty \| \leq p\| x - y \| + q(\| Tx - x \| + \| Ty - y \|), \quad p^2 + q^2 = 1, \quad p, q \neq 0, \]
for each \( x, y \in S \), does the sequence \( \{ T^n x \}_{n=0}^\infty \) converge to the unique fixed point of \( T \), if it exists?"

By using Theorem 2, with \( \phi(r) = pr, \psi(r) = \chi(r) = qr \) for each \( r \in [0, +\infty[ \), we answer, positively, to this question. Furthermore, if we use Theorem 1, we can dispense with the continuity of \( T \).

3. In this section we obtain some well-known results, as corollaries of our previous theorems.

**Corollary 1 [6].** Let \( T \) be a function from \( X \) into \( X \). We suppose that there exists a map \( f, f: [0, +\infty[ \to [0, +\infty[ \), continuous from the right for each \( r \in [0, +\infty[ \), such that
\[ d(Tx, Ty) \leq f(d(x, y)), \quad \text{for each } x, y \in X. \]
If \( f(r) < r \), for \( r > 0 \), then the sequence \( \{ T^n x \}_{n=0}^\infty \) converges to the unique fixed point of \( T \).
**Proof.** We use Theorem 2. In this theorem let \( \varphi(r) = f(r) = 0 \), \( \psi(r) = \chi(r) = 0 \) be, for each \( r \in [0, +\infty] \). Moreover, if \( \nu \in \mathbb{N} \) exists such that \( T^{n+1}x = T^nx \), for each \( x \in X \), \( T \) is a asymptotically regular mapping. On the contrary, one has, for each \( x \in X \) such that this \( \nu \) doesn't exist,

\[
d(T^{n+1}x, T^nx) \leq f(d(T^nx, T^{n-1}x)) < d(T^nx, T^{n-1}x) \quad \text{for each } n \in \mathbb{N}.
\]

Put \( d = \lim d(T^{n+1}x, T^nx) \), we have \( d \leq f(d) \leq d \) and so \( d = 0 \).

**COROLLARY 2 [1].** Let \( X, T \) be as in Corollary 1. Suppose that \( f \), from \( [0, +\infty] \) into \( [0, +\infty] \) is a non-decreasing function, continuous from the right such that

\[
d(Tx, Ty) \leq f(d(x, y)), \quad \text{for each } x, y \in X.
\]

If \( f(r) < r \) \( (r > 0) \) and if \( X \) is bounded, then \( T \) has a unique fixed point.

**COROLLARY 3 [3].** Let \( T \) be an asymptotically regular and continuous function, such that \( T \) maps \( S \) into \( S \), with \( S \) a non-void weakly closed subset of a Hilbert space \( H \). Also, we suppose that \( T \) satisfies

\[
\|Tx - Ty\| \leq \|Tx - x\| + \|Ty - y\|, \quad \text{for each } x, y \in X.
\]

Then, for each \( x \in S \), the sequence \( \{T^n x\}_{n=0}^{\infty} \) converges to the unique fixed point of \( T \).

**Proof.** In Theorem 2, we take \( \varphi(r) = 0 \), \( \psi(r) = \chi(r) = r \) for each \( r \geq 0 \).

**COROLLARY 4 [3].** Let \( T \) be, \( T : S \to S \), \( S \) a non-void weakly closed subset of a Hilbert space \( H \), an asymptotically regular and continuous mapping, such that

\[
\|Tx - Ty\| \leq p\|x - y\| + q(\|Tx - x\| + \|Ty - y\|), \quad \text{for each } x, y \in X,
\]

with \( p^2 + q^2 < 1 \). Then, the thesis of Corollary 3 is true.

**Proof.** In Theorem 1, we put \( \varphi(r) = pr \), \( \psi(r) = \chi(r) = qr \), for each \( r \geq 0 \). We observe that, by using Theorem 1, we can dispense with the continuity of \( T \).

**COROLLARY 5 [8].** Let \( (X, d) \) be a complete metric space and let \( a, b, c \) be nonnegative numbers, with \( a + b + c < 1 \). Furthermore, suppose that \( T : X \to X \) satisfies

\[
d(Tx, Ty) \leq ad(x, y) + bd(Tx, x) + cd(Ty, y), \quad \text{for each } x, y \in X.
\]

Then, \( T \) has a unique fixed point.

**Proof.** We can take, in Theorem 1, \( \varphi(r) = ar \), \( \psi(r) = br \), \( \chi(r) = cr \), for each \( r \in [0, +\infty] \). Since, for each \( n \in \mathbb{N} \), we have

\[
d(T^{n+1}x, T^nx) \leq ad(T^nx, T^{n-1}x) + bd(T^{n+1}x, T^nx) + cd(T^nx, T^{n-1}x), \quad x \in X,
\]

if follows that

\[
d(T^{n+1}x, T^nx) \leq \left( \frac{a + c}{1 - b} \right)^n d(Tx, x), \quad x \in X.
\]
This fact implies that $T$ is asymptotically regular, being $a + b + c < 1$.

**Remark 5.** Under the assumptions of Corollary 5, we have, by virtue of Remark 3, that the sequence $\{T^nx\}_{n=0}^\infty$ converges to the unique fixed point of $T$.

**Corollary 6** [5]. Let $T: X \to X$, be a mapping satisfying the condition

$$d(Tx, Ty) \leq (x, y) - \Delta(d(x, y)),$$

for each $x, y \in X$ where $\Delta, \Delta: [0, + \infty[ \to [0, + \infty[$, is a continuous function such that $\Delta(r) > 0$, if $r > 0$. Then, for each $x \in X$, the sequence $\{T^nx\}_{n=0}^\infty$ converges to the unique fixed point of $T$.

**Proof.** Asymptotical regularity follows as in Corollary 1. By putting, in Theorem 1, $\phi(r) = r - \Delta(r), \psi(t) = \chi(r) = 0$ for each $r \in [0, + \infty[$ one obtains the thesis. We observe that we can dispense with the continuity of $\Delta$. In fact it suffices that $\Delta$ is a right continuous or a monotonically nondecreasing function.

4. In this section, we consider mappings $T, T: X \to X$, satisfying all assumptions of n. 1, with the exception of $(I_4)$, which is replaced with

$$(I_4') \quad d(Tx, Ty) \leq \phi(d(x, y)) + p(d(x, y)) \psi(d(Tx, x)) + q(d(x, y)) \chi(d(Ty, y))$$

for each $x, y \in X$. Here $p, p: [0, + \infty[ \to [0, + \infty[$, is a function such that

$$(I_3) \quad \lim_{s \to s^+} p(s) < + \infty, \text{ for each } s \in [0, + \infty[;$$

and $q, q: [0, + \infty[ \to [0, + \infty[$, is a function such that

$$(I_4) \quad \lim_{s \to 0^+} q(s) < 1 \text{ and } \lim_{s \to s^+} q(s) < + \infty, \text{ for each } s \in [0, + \infty[.$$

Before proving the announced fixed point theorems, we observe that in this case the Lemma is still true if we suppose that $\psi, \chi$ are continuous functions as $r \to 0^+$. Then, with standard arguments, we prove

**Theorem 3.** If $X, T, \phi, \psi, \chi, p, q$ satisfy all previous assumptions and, in addition, $\chi(r) < r$ for each $r \in [0, + \infty[$, then $T$ has a unique fixed point. Moreover, the sequence $\{T^nx\}_{n=0}^\infty$ converges to the unique fixed point of $T$.

This last Theorem 3 generalizes the following result of [10].

"Let $(X, d)$ be a complete metric space. If $T, T: X \to X$, satisfies

$$d(Tx, Ty) \leq a(d(x, y))d(x, y) + b(x, y)d(Tx, x) + c(d(x, y))d(Ty, y),$$

where $x, y \in X, x \neq y$, and $a, b, c$ are monotonically decreasing functions from $[0, + \infty[$ into $[0, 1]$ such that $a(s) + b(s) + c(s) < 1$, the $T$ has a unique fixed point".

To this end, we observe, first of all, that $T$ is an asymptotically regular mapping (see [10]). Moreover, we can take

$$\phi(r) = \begin{cases} 0 & r = 0; \\ 0 & r \neq 0 \end{cases}, \quad \psi(r) = \chi(r) = r \text{ for each } r \in [0, + \infty[; \quad p(s) = 1$$
for each $s \in [0, +\infty[$; since we may assume $c(s) < \frac{1}{2}$ (see [9])
\[
q(s) = \begin{cases} 
\frac{1}{2} & s = 0 \\
c(s) & s \neq 0
\end{cases}
\]

By using Theorem 3 we obtain that $T$ has a unique fixed point in $X$ and, furthermore, that the sequence of iterates converges to this unique fixed point.

Finally, if $T$ is a continuous mapping satisfying assumptions as in the Lemma we may prove a result as Theorem 2, without assumptions $\lim_{s \to 0^+} p(s) < +\infty$ and $\lim_{s \to 0^+} q(s) < 1$.

REFERENCES