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ON COMPLEMENTED COPIES OF c_0 AND ℓ_∞

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Abstract. We furnish examples of pairs of Banach spaces X, Y so that both c_0 and l_∞ do not live inside X^* and Y, but they embed complementably into the space $DP(X,Y)$ of the Dunford–Pettis operators from X into Y.

1. Introduction

Many papers have been devoted to the question of when a space of operators contains complemented copies of c_0 or (complemented) copies of ℓ_∞, since the appearance of the paper [21] (see for instance [2], [3], [4], [5], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [22], [23], [24], [26]). But such a long list is far from being exhaustive, and to the consequences of such containments on the geometry of spaces of operators.

In our paper [10] we proved that certain spaces of operators between two Banach spaces X and Y, like the space $W(X,Y)$ of weakly compact operators or the space $UC(X,Y)$ of unconditionally converging operators or the space $L(X,Y)$ of bounded operators (all endowed with the usual sup norm), contain a complemented copy of c_0 *only if* c_0 lives inside either X^* or Y, leaving open the question of whether the same happens for the space $DP(X,Y)$ of Dunford–Pettis operators (i.e. operators mapping weakly null sequences into norm null ones), when such a space is not equal to the space of compact operators $K(X,Y)$. (Indeed it is well known that, if $DP(X,Y) = K(X,Y)$, then c_0 can embed complementably into $DP(X,Y) = K(X,Y)$, even if X^*, Y do not contain copies of c_0; see [3], [11], [13], [14], [21]). These facts motivated the following

QUESTION 1. *Does c_0 live complementably inside $DP(X,Y)$ only if it lives inside either X^* or Y?*

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Here, we answer in the negative Question 1, by showing that there exist Banach spaces X, Y for which

(i) $K(X,Y) \neq DP(X,Y)$,

(ii) c_0 does not embed into X^* and Y, and

(iii) $DP(X,Y)$ contains a complemented copy of c_0.

So, we may conclude that, from this point of view, the space $DP(X,Y)$ behaves as the space $K(X,Y)$. On the other hand, it is also well known that ℓ_∞ embeds inside $K(X,Y)$ only if it embeds into either X^* or Y ([4], [21]), so that, in view of the quoted similarity between $DP(X,Y)$ and $K(X,Y)$, it appears natural to put the following other question, too.

QUESTION 2. *Does ℓ_∞ live inside $DP(X,Y)$ only if it lives inside either X^* or Y?*

Again, we answer here in the negative, so deducing that, this time, the behaviour of $DP(X,Y)$ is really different from that of $K(X,Y)$, but is similar to that of other spaces of operators, like the space of weakly compact operators (or the superspace of unconditionally converging operators and the one of bounded operators), that are known to contain ℓ_∞ even if neither X^* nor Y do (see [2], [12]). Of course, in our counterexample, we avoid that $W(X,Y)$ is contained inside $DP(X,Y)$, otherwise the result would be well known.

After this introduction explaining the reasons of this paper, we may pass to describe our examples answering the previous questions.

ANSWER TO QUESTION 1. We consider $X = Y = l_2 \oplus_2 Z$, where Z is an infinite dimensional Banach space with Schur property such that Z^* is weakly sequentially complete (an example of such a space can be found in [1]), so that Y is weakly sequentially complete and it is not allowed to contain copies of c_0. Since $X^* = l_2 \oplus_2 Z^*$, X^* is weakly sequentially complete, too; we deduce that property (ii) above is easily verified. On the other hand, if $\phi : l_2 \to l_2$ is a compact operator, then the operator

$$T(l,z) = \big(\phi(l), z\big) : l_2 \oplus_2 Z \to l_2 \oplus_2 Z \qquad \forall\, (l,z) \in l_2 \oplus_2 Z$$

is easily seen to be an element of $DP(X,Y) \setminus K(X,Y)$; hence, (i) is satisfied. It remains to prove just (iii). It is well known ([25], see also [7]) that it is enough to show that there are $(T_n) \subset DP(X,Y)$, equivalent to the unit vector basis of c_0, and $(D_n^*) \subset \big(DP(X,Y)\big)^*$ such that (D_n^*) is weak* null and $\inf_n \big|D_n^*(T_n)\big| > 0$. To this purpose, we define

$$i(l) = (l, \theta_Z) : l_2 \to X = Y, \quad j(l) = (l, \theta_{Z^*}) : l_2 \to X^* = Y^*.$$

If we denote by $\left(e_n^2\right)$ the canonical basis of l_2, we observe that, obviously,

$$T_n = j\left(e_n^2\right) \otimes_\varepsilon i\left(e_n^2\right) \in DP(X,Y),$$

$$D_n^* = i\left(e_n^2\right) \otimes_\pi j\left(e_n^2\right) \in \left(DP(X,Y)\right)^* \qquad \forall n \in \mathbb{N}$$

and also

$$\left[i\left(e_n^2\right) \otimes_\pi j\left(e_n^2\right)\right]\left[j\left(e_n^2\right) \otimes_\varepsilon i\left(e_n^2\right)\right] = 1 \qquad \forall n \in \mathbb{N}.$$

On the other hand, Theorem 3 in [11] gives that $\left(j\left(e_n^2\right) \otimes_\varepsilon i\left(e_n^2\right)\right)$ is a copy of the unit vector basis of c_0; furthermore $i\left(e_n^2\right) \otimes_\pi j\left(e_n^2\right) \xrightarrow{w^*} \theta_{(DP(X,Y))^*}$ since, for any $T \in DP(X,Y)$, one has

$$\left|\left[i\left(e_n^2\right) \otimes_\pi j\left(e_n^2\right)\right](T)\right| \leqq \left|T\left(i\left(e_n^2\right)\right)\right| \to 0$$

where we used the facts that $T \in DP(X,Y)$ and $i\left(e_n^2\right) \xrightarrow{w} \theta_X$. $\square$

In passing we observe that our spaces X^*, Y are both weakly sequentially complete, but $DP(X,Y)$ is not.

Answer to Question 2. Now we wish to introduce a pair of Banach spaces X, Y such that

(j) $W(X,Y) \nsubseteqq DP(X,Y)$

(jj) ℓ_∞ does not embed into X^* and Y, and

(jjj) $DP(X,Y)$ contains a copy of ℓ_∞.

We consider $X = Y = C[0,1] \oplus_2 \ell_2$ and we immediately note that ℓ_∞ does not embed into Y, since this last space is separable, and into $X^* = \left(C[0,1]\right)^* \oplus_2 \ell_2$ since it is weakly sequentially complete; hence (jj) is true. Moreover, if $\phi : C[0,1] \to C[0,1]$ is a weakly compact operator, then the operator

$$T(x,y) = \left(\phi(x), y\right) : X \to Y$$

is clearly weakly compact, but it is not a Dunford–Pettis operator, otherwise in ℓ_2 weak and strong convergence would be the same; hence (j) is verified. Now, we are going to show (jjj). Since any Dunford–Pettis operator defined on $C[0,1]$ is weakly compact (see [20]) and ℓ_2 is reflexive, it is very simple to show that any Dunford–Pettis operator defined on X is weakly compact, a fact giving that any bounded (L)-set M in X^*, i.e. a subset $M \subset X^*$ such that $\lim_n \sup_M \left|x_n(x^*)\right| = 0$ for each w-null sequence $(x_n) \subset X$, is relatively weakly compact (see [8]). On the other hand, X contains copies of ℓ_1, and this gives that there is an (L)-set inside X^* that is not relatively compact (see [6]). Hence there is a sequence $(x_n^*) \subset X^*$ that is an L-subset, converges weakly to θ, as we may suppose after a translation, but not strongly. Now, let (y_n) be a copy of the unit vector basis of c_0 inside Y. As usual after [21], for $\xi = (\xi_n) \in \ell_\infty$ and $x \in X$, we consider the series $\sum_{n=1}^{\infty} \xi_n (x_n^* \otimes y_n)(x)$ that is unconditionally converging in Y and then we define

$$\psi(\xi) = \sum_{n=1}^{\infty} \xi_n (x_n^* \otimes y_n) : \ell_\infty \to L(X, Y)$$

that is a well-defined linear operator (see [21]). Moreover, let (x_h) be a weak null sequence in X. Thanks to the nature of (y_n), for each $\xi = (\xi_n) \in \ell_\infty$, there is $C_\xi > 0$ such that

$$\Big\| \sum_{n=1}^{\infty} \xi_n x_n^*(x_h) y_n \Big\|_Y \leqq C_\xi \sup_n \big| \xi_n x_n^*(x_h) \big| \leqq C_\xi \|\xi\|_{\ell_\infty} \sup_n \big| x_n^*(x_h) \big| \quad \forall\, h \in \mathbb{N}.$$

Recall now that (x_n^*) is an L-subset, so that $\sup_n \big| x_n^*(x_h) \big| \xrightarrow{h} 0$ which implies that $\psi(\xi) \in DP(X, Y)$. To conclude the proof it is enough to observe that $\psi(e_n) \not\to \theta$ and then apply a famous result from [24]. $\square$

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