Sequential w-right continuity and summing operators

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We continue the study of the w-right and strong∗ topologies in general Banach spaces started in [36, 37] and [35]. We show that in $L_1(\mu)$-spaces the w-right convergence of sequences admits a simpler control. Some considerations about these topologies will be contemplated in the particular cases of C*-algebras and JB*-triples in connection with summing operators. We also study (sequential) w-right-norm and strong*-norm continuity for holomorphic mappings.

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1 Preliminaries

I. Villanueva, J. D. M. Wright, K. Ylinen and the second author of the present note introduced in [36] two interesting topologies: the strong∗ and the w-right topology in the following way: let $X$ and $Y$ be two Banach spaces, for every bounded linear operator $T : X \to Y$, we can consider a seminorm on $X$ defined by $\|x\|_T := \|T(x)\|$. The strong∗-topology is the topology generated by the family of seminorms $\|\cdot\|_T$, where $T : X \to H$ is a bounded linear operator from $X$ to some Hilbert space $H$ (such a topology is denoted by $S^*(X,X^*)$). Similarly, the w-right-topology is the topology generated by the family of seminorms $\|\cdot\|_T$ where $T$ runs in the set of all bounded linear operators from $X$ to a reflexive space [36]. In Section 2, we establish new methods for controlling w-right convergent sequences in $L_1(\mu)$ spaces. Section 3 is devoted to a more detailed study of strong*-norm continuous operators between Banach spaces. In the particular cases of operators whose domain is a C*-algebra or a JB*-triple, we explore the connections with $p$-C*-summing and $p$-JB*-triple-summing operators. We prove an extension property for 2-C*-summing and 2-JB*-triple-summing operators (see Theorems 3.6 and 3.9). In this section we shall also introduce and develop $p$-JB*-triple-summing operators on JB*-triples as suitable generalizations of $p$-C*-summing operators on C*-algebras in the sense introduced by Pisier in [39].

The last section of the paper is devoted to the study of those holomorphic mappings of bounded type which are sequentially w-right-norm continuous. The main result in [35] establishes that a bounded linear operator $T : X \to Y$ is weakly compact if and only if $T$ is w-right-norm continuous. We shall provide examples showing that none of these implications holds for continuous polynomials in general Banach spaces. In the linear case, $T$ is weakly compact if and only if $T^{**}$ is $Y$-valued. In the setting of multilinear operators, this equivalence has been recently studied in [37]. One of the main results in the just quoted paper proves that when $X_1, \ldots, X_k$ are non zero sequentially right Banach spaces and $T : X_1 \times \cdots \times X_k \to X$ is a multilinear operator, then $T$ is RQCC (i.e., $T$ is jointly sequentially w-right-norm continuous) if and only if all of the Aron-Berner extensions of $T$ are $X$-valued if and only if $T$ has an $X$-valued Aron-Berner extension. We shall consider here holomorphic mappings of bounded type $f$ between two Banach spaces $X$ and $Y$ with $X$ being a sequentially right Banach
space. We shall prove that such a mapping $f$ is sequentially $w$-right-norm continuous if and only if its Aron-Berner extension, $AB(f) : X^{**} \rightarrow Y^{**}$, is $Y$-valued.

### 1.1 Notation

Except otherwise stated, all the Banach spaces considered in this paper will be complex. Given a Banach space $X$, $S(X)$ and $B(X)$ denote, respectively, the unit sphere and the closed unit ball of $X$. For any pair of Banach spaces $X$, $Y$, $L(X, Y)$ will stand for the space of all bounded linear operators between $X$ and $Y$, while $X \otimes Y$ and $X \otimes \overline{Y}$ will denote the injective and projective tensor product of $X$ and $Y$, respectively.

### 2 When the $w$-right-topology and the weak-topology coincide sequentially

In [36, Proposition 2.7] the authors remarked the following.

**Proposition 2.1** Let $X$ be a Banach space. If the $w$-right-topology coincides with the weak topology on $X$, then $X$ is finite dimensional.

Given a set $X$ with two topologies $\tau_1$ and $\tau_2$, we say that $\tau_1$ and $\tau_2$ coincide sequentially if both topologies define the same convergent sequences on $X$, that is, a sequence $(x_n)_n$ in $X$ is $\tau_1$-convergent to $x \in X$ if and only if $(x_n)_n$ converges to the same $x$ in the $\tau_2$-topology.

We recall that a bounded linear operator $T : X \rightarrow Y$ is called completely continuous if it maps weakly convergent sequences to norm convergent sequences. A Banach space $X$ has the Dunford-Pettis property (DPP) if, for every Banach space $Y$, every weakly compact operator from $X$ to $Y$ is completely continuous. $X$ satisfies the (weaker) alternative Dunford-Pettis property (DP1) if every weakly compact operator $T : X \rightarrow Y$ is a DP1 operator, that is, $T(x_n)$ converges in norm to $T(x)$ whenever $x_n \rightarrow x$ weakly in $X$ and $\|x_n\| = \|x\| = 1$.

The following result was established in [36, Remark 4.5].

**Proposition 2.2** Let $X$ be a Banach space.

(a) The $w$-right and the weak topologies coincide sequentially if and only if $X$ has the DPP.

(b) The $w$-right and the weak topologies coincide sequentially on the unit sphere of $X$ if and only if $X$ has the DP1.

To formulate the next result we first recall a deep result due to Rieffel (see [27] for more details).

**Theorem 2.3** (Rieffel) Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ be a Banach space. A vector measure $F : \Sigma \rightarrow X$ is Bochner-representable with respect to $\mu$ (i.e., $F(B) = \text{Bochner} - \int_B f \, d\mu$ for all $B \in \Sigma$ and some $f \in L_1(\mu, X)$) if and only if $F$ is $\mu$-continuous, $F$ is of bounded variation, and for each $\epsilon > 0$ there exists $B_\epsilon \in \Sigma$ with $\mu(\Omega \setminus B_\epsilon) < \epsilon$ such that

$$\left\{ \frac{F(B)}{\mu(B)} : B \subseteq B_\epsilon, B \in \Sigma, \mu(B) > 0 \right\}$$

is relatively weakly compact.

**Remark 2.4** Note that we can reformulate Rieffel’s theorem in terms of operators as follows: a bounded linear operator $T : L_1(\mu) \rightarrow X$ is Bochner-representable (i.e., there is a $g \in L_\infty(\mu, X)$ so that $T(f) = \text{Bochner} - \int_\Omega f \cdot g \, d\mu$ for every $f \in L_1(\mu)$) if and only if for each $\epsilon > 0$ there exists $\Omega_\epsilon \in \Sigma$ with $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ so that $T : L_1(\Omega_\epsilon, \Sigma_{\Omega_\epsilon}, \mu|_{\Sigma_{\Omega_\epsilon}}) \rightarrow X$ is weakly compact, where $\Sigma_{\Omega_\epsilon}$ denotes the $\sigma$-field $\Sigma_{\Omega_\epsilon} = \{ F \in \Sigma : F \subseteq \Omega_\epsilon \}$.

**Proposition 2.5** Let $(\Omega, \Sigma, \mu)$ be a finite measure space, and let $(f_n)_n$ be a sequence in $L_1(\mu)$. Then the following are equivalent:

1. $(f_n)_n$ is weakly null.
2. $(f_n)_n$ is $w$-right null.
3. For every Banach space $X$, and every weakly compact operator $T : L_1(\mu) \rightarrow X$, the sequence $(T(f_n))_n$ is norm null.
4. For every Banach space $X$, and every representable operator $T : L_1(\mu) \rightarrow X$, the sequence $(T(f_n))_n$ is norm null.
5. For every Banach space $X$ and every $g \in L_\infty(\mu, X)$, the sequence $(f_n \cdot g)_n$ is weakly null in $L_1(\mu, X)$.  

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Proof. Since $L_1(\mu)$ has the DPP, then (0) and (0′) are equivalent by Proposition 2.2. The equivalence of (0′) and (1) follows directly from the definition of the w-right topology.

(1) $\Rightarrow$ (2) Let $X$ be a Banach space and let $T : L_1(\mu) \longrightarrow X$ be a representator operator. Fixing $\epsilon > 0$, by the previous remark, there exists $\Omega_\epsilon \in \Sigma$ with $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ so that

$$T : L_1(\Omega, \Sigma, \mu|_{\Sigma_\epsilon}) \longrightarrow X$$

is weakly compact.

Moreover, since $(f_n)_n$ is weakly null in $L_1(\Omega, \Sigma, \mu)$, by a classical result of Dunford and Pettis (see [15] Chapter IV.2) the sequence $(f_n)_n$ is uniformly integrable. We can then assume that $\|f_n \chi_{\Omega_n}\|_{L_1(\Omega, \Sigma, \mu)} < \frac{\epsilon}{\sqrt{n}}$ uniformly in $n \in \mathbb{N}$.

Now, since $(f_n \chi_{\Omega_n})_n$ is weakly null in $L_1(\Omega, \Sigma, \mu|_{\Sigma_\epsilon})$, then there is an $n_0 \in \mathbb{N}$ so that $\|T(f_n \chi_{\Omega_n})\| < \epsilon$, for all $n > n_0$. Therefore, for $n > n_0$, we have

$$\|T(f_n)\| \leq \|T(f_n \chi_{\Omega_n})\| + \|T(f_n \chi_{\Omega_\epsilon}\setminus\Omega_n)\| < \epsilon + \|T\|\|f_n \chi_{\Omega_\epsilon}\|_{L_1(\Omega, \Sigma, \mu)} < 2\epsilon.$$

(2) $\Rightarrow$ (0) Fix $g \in L_\infty(\mu)$; it’s enough to choose $T : L_1(\mu) \longrightarrow \mathbb{C}$ defined by $T(f) = \int_{\Omega} f \, g \, d\mu$. $T$ is trivially representable and so $(f_n, g) \xrightarrow{n \to \infty} 0$, that is, $(f_n)_n$ is weakly null in $L_1(\mu)$.

(0) $\Rightarrow$ (3) Fix a Banach space $X$ and an element $g \in L_\infty(\mu, X)$. Let $(f_n)_n$ be a weakly-null sequence in $L_1(\mu)$. Since $(\Omega, \Sigma, \mu)$ is a finite measure space we can apply the Diestel-Ruess-Schachermayer Theorem (see [14]) to the weakly relatively compact set $(f_n)_n \subseteq L_1(\mu)$. Then for each subsequence $(f_{n_k})_k$ of $(f_n)_n$, there exists a sequence $(g_{n_k})_k$ with $g_{n_k} \in X_{\Sigma \setminus \Sigma_{n_k}}$ such that $(g_{n_k}(\omega))_n$ is a null sequence of scalars for $a.e. \omega \in \Omega$. But the sequence $(g_n \cdot g)_k$ is such that $g_n \cdot g \in co\{f_{n_k} \cdot g : n_k \geq n\}$ and so $(g_n \cdot g(\omega))_k$ is norm null in $X$ for $a.e. \omega \in \Omega$. By the Diestel-Ruess-Schachermayer’s Theorem the sequence $(f_n \cdot g)_n \subseteq L_1(\mu, X)$ is weakly null in $L_1(\mu, X)$.

(3) $\Rightarrow$ (0) It’s enough to choose $X = \mathbb{K}$ and $g(\omega) = 1$ for each $\omega \in \Omega$. \qed

Corollary 2.6 (0), (0′), (1), (2) of the proposition above are equivalent for any measure space $(\Omega, \Sigma, \mu)$.

Proof. Note that (0), (0′) and (1) are trivially equivalent because $L_1(\mu)$ has the DPP.

(0) $\Rightarrow$ (2) Let $(f_n)_n$ be a weakly null sequence in $L_1(\mu)$. It is well known that (see [18], III.8.5) there exists a set $\Omega_1 \subseteq \Sigma$, a sub $\sigma$-field of $\Sigma$ such that the restriction $\mu_1$ of $\mu$ to $\Sigma_1$ has the properties

(i) the measure space $(\Omega_1, \Sigma_1, \mu_1)$ is $\sigma$-finite;
(ii) $\cap_{n \geq 1} \{f_n : n \geq 1\} \subseteq L_1(\Omega_1, \Sigma_1, \mu_1)$.

Since $L_1(\Omega_1, \Sigma_1, \mu_1)$ is a closed subspace of $L_1(\Omega, \Sigma, \mu)$, we can assume, without loss of generality that $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space. Thus there exists a sequence $(A_n)$ in $\Sigma$ of pairwise disjoint sets with finite and positive $\mu$-measure such that $\Omega = \bigcup_{n \in \mathbb{N}} A_n$.

Now, we define $\mu_0 : \Sigma \longrightarrow [0, +\infty)$ as

$$\mu_0(E) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \mu(A_n)$$

for every $E \in \Sigma$.

It is easy to see that $\mu_0$ is a finite measure (it is a probability measure) and, if we consider the function $h = \sum_{n \in \mathbb{N}} 2^n \mu(A_n) \chi_{A_n}$, the map

$$T : L_1(\mu) \longrightarrow L_1(\mu_0)$$

$$f \longmapsto h \cdot f$$

is a surjective isometry. Since $\mu$ and $\mu_0$ have the same null sets, for each Banach space $X$ the identity

$$id : L_\infty(\mu, X) \longrightarrow L_\infty(\mu_0, X)$$

is a surjective isometry.
Fix a Banach space $X$ and a representable operator $S : L_1(\mu) \to X$. Then there is a $g \in L_\infty(\mu, X)$ so that $S(f) = \int_\Omega f \cdot g \, d\mu$. By the last paragraph above, we can consider $g$ as element of $L_\infty(\mu_0, X)$. Then the operator

$$
\tilde{S} : L_1(\mu_0) \to X
$$

$$
\tilde{S}(f) = \int_\Omega f \cdot g \, d\mu_0 \quad \text{for each} \quad f \in L_1(\mu_0)
$$

is trivially representable. Since $(T(f_n))_n$ is w-right null in $L_1(\mu_0)$ (actually, every bounded linear operator maps w-right null sequences into w-right null sequences), by the previous proposition we have

$$
\|S(f_n)\| = \|\tilde{S}(f_n \cdot h)\| \to 0 \quad \text{as} \quad n \to \infty
$$

because for each $f \in L_1(\mu, X)$ we have $\int_\Omega f \, d\mu = \int_\Omega f \cdot h \, d\mu_0$, which gives (2).

The implication $(2) \Rightarrow (0)$ follows similarly. \hfill \Box

### 3 Strong*-norm continuous operators

We recall a result established in [35]. We should note here that after the publication of the just quoted paper, we were told about the signifcative papers [40] and [42], which are directly connected with the results obtained in [35]. In fact, the main result in [35] follows as a consequence of [40, Proposition 2.6 and Theorems 3.1 and 3.2], proved there in a more general setting. The equivalence of (i) and (iii) in [35, Corollary 5] can be also obtained from [42, Lemmas 2.1 and 3.2].

**Theorem 3.1** Let $X$ and $Y$ be two Banach spaces, and let $T : X \to Y$ be a bounded linear operator. Then the following are equivalent

(a) $T$ is w-right-norm continuous.

(b) $T$ is w-right-norm continuous on the closed unit ball of $X$.

(c) $T$ is weakly compact.

Similarly we have:

**Theorem 3.2** Let $X$ and $Y$ be two Banach spaces, and let $T : X \to Y$ be a bounded linear operator. Then $T$ is strong*-norm continuous if and only if $T$ factors through a Hilbert space.

**Proof.** Let $T : X \to Y$ be a strong*-norm linear operator. The set

$$
U := \{x \in X : \|T(x)\| \leq 1\}
$$

is a strong*-neighborhood of zero in $X$. Then there exist Hilbert spaces $H_1, \ldots, H_n$, and operators $G_i : X \to H_i$, $i = 1, \ldots, n$, satisfying that $\bigcap_{i=1}^n \{x \in X : \|G_i(x)\| \leq 1\} \subseteq U$. Consider $H := (\bigoplus_{i=1}^n H_i)_{\ell_2}$, and $G : X \to H$ defined by $G(x) = (G_i(x))_{i=1}^n$. The inclusion

$$
\{x \in X : \|G(x)\| \leq 1\} \subseteq \bigcap_{i=1}^n \{x \in X : \|G_i(x)\| \leq 1\}
$$

implies that

$$
\|T(x)\| \leq \|G(x)\| \quad \text{for each} \quad x \in X. \tag{3.1}
$$

The kernel of $G$ is a closed subspace of $X$ and the mapping $x + \ker(G) \to |||x|||_G = \|G(x)\|$ is a prehilbertian norm on the quotient $X/\ker(G)$. The inequality (3.1) guarantees that the law

$$
R : X/\ker(G) \to Y
$$

$$
x + \ker(G) \mapsto T(x)
$$

is a well-defined continuous operator on $X/\ker(G)$ with $\|R\| \leq \|T\|$. If $H_G$ denotes the completion of $X/\ker(G)$, then $H_G$ is a Hilbert space and $R$ admits an extension $\hat{R} : H_G \to Y$. If $\pi$ denotes the canonical projection of
X onto \(X/\ker(G)\) and \(j_G\) the inclusion of \(X/\ker(G)\) into \(H_G\), then we have \(T = \hat{R} j_G \pi\), which shows that \(T\) factors through a Hilbert space (i.e., \(T \in \Gamma_2(X, Y)\)). Clearly every operator in \(\Gamma_2(X, Y)\) is strong*-norm continuous.

**Corollary 3.3** Every strong*-norm continuous operator between two Banach spaces is uniformly convexifying in the sense of Beaudzamy [7].

**Corollary 3.4** Every strong*-norm continuous operator between two Banach spaces is a Banach-Saks operator [8].

**Corollary 3.5** The class of all strong*-norm continuous operators between Banach spaces is an injective (closed) operator ideal in the Pietsch sense [38].

When the domain space is a \(C^*\)-algebra (respectively, a \(JB^*\)-triple) then strong*-norm continuous operators coincide with 2-\(C^*\)-summing (respectively, 2-\(JB^*\)-triple-summing) operators. \(p\)-\(C^*\)-summing operators on \(C^*\)-algebras were introduced by Pisier in [39]. We recall that an operator \(T\) from a \(C^*\)-algebra \(A\) to a Banach space \(Y\) is said to be \(p\)-\(C^*\)-summing \((p > 0)\) if there exists a constant \(C\) such that for every finite sequence of \(a_1, \ldots, a_n\) in \(A\) the next inequality holds

\[
\left(\sum_{i=1}^{n} \|T(a_i)\|^p\right)^{\frac{1}{p}} \leq C \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}},
\]

where, for each \(x \in A\), we denote \(|x| = \sqrt{\frac{xx^* + x^*x}{2}}\). The smallest constant \(C\) verifying the above inequality is denoted by \(C_p(T)\).

The following Pietsch’s factorization theorem for \(p\)-\(C^*\)-summing operators was established by Pisier in [39]: if \(T : A \to Y\) is a bounded linear operator from a \(C^*\)-algebra to a complex Banach space, then \(T\) is a \(p\)-\(C^*\)-summing operator if and only if there is a norm-one positive linear functional \(\varphi\) in \(A^*\) and a positive constant \(K_p(T)\) such that

\[
\|T(x)\| \leq K_p(T)(\varphi(|x|^p))^{\frac{1}{p}}
\]

for every \(x \in A\). Every \(p\)-summing operator from a \(C^*\)-algebra to a Banach space is \(p\)-\(C^*\)-summing but the converse is false in general (compare [39, Remark 1.2]). It follows from the little Grothendieck’s inequality for \(C^*\)-algebras (see [23, 39]) that an operator \(T : A \to Y\) is 2-\(C^*\)-summing if and only if it is strong*-norm continuous.

The following result shows that 2-\(C^*\)-summing operators enjoy an extension property which is the appropriate version of [13, Theorem 4.15].

**Theorem 3.6** Let \(A\) and \(B\) be two \(C^*\)-algebras with \(B\) a \(C^*\)-subalgebra of \(A\) and let \(Y\) be a Banach space. Then every 2-\(C^*\)-summing operator \(T : B \to Y\) admits a norm preserving 2-\(C^*\)-summing extension \(\hat{T} : A \to Y\).

**Proof.** Let \(T : B \to Y\) be a 2-\(C^*\)-summing operator. According to the Pietsch factorisation theorem, there is a norm-one positive linear functional \(\varphi\) in \(B^*\) and a positive constant \(K_2(T)\) such that

\[
\|T(x)\| \leq K_2(T)(\varphi(|x|^2))^{\frac{1}{2}}
\]

for every \(x \in B\). Proposition 3.1.6 in [29] implies the existence of a positive functional \(\phi \in A^*\) satisfying that \(\|\phi\| = \|\varphi\|\) and \(\phi|_B = \varphi\). The set \(N_\phi = \{x \in A : \phi(xx^* + x^*x) = 0\}\) is a closed subspace of \(A\) and the sesquilinear form

\[
(x + N_\phi, y + N_\phi) \mapsto \frac{1}{2} \phi(xy^* + y^*x)
\]

defines a pre-inner product on the preHilbert space \(A/N_\phi\). The completion of the latter space is a Hilbert space that will be denoted by \(H_\phi\). Let \(j_\phi : A \to H_\phi\) denote the composition of the canonical projection and inclusion.
The norm-closure of \( j_\phi(B) = B/(N_\phi \cap B) \) is a closed subspace of \( H_\phi \) which is denoted by \( K \). Let \( \pi \) be the orthogonal projection of \( H_\phi \) onto \( K \).

The operator \( B/(N_\phi \cap B) \rightarrow Y, x + (N_\phi \cap B) \rightarrow T(x) \) is well-defined by (3.3). Therefore there exists a unique operator \( R : K \rightarrow Y \) satisfying \( R(x + (N_\phi \cap B)) = T(x) \). Finally, \( \tilde{T} : A \rightarrow Y, \tilde{T} = R \circ \pi \circ j_\phi \) is a norm preserving 2-C*-summing extension of \( T \).

**Proposition 3.7** Let \( X \) be a Banach space. Suppose that the w-right topology and the \( S^*(X, X^*) \)-topology coincide on bounded subsets of \( X \). Then the following statements holds:

(a) \( X \) satisfies the DPP if and only if every strong*-norm continuous operator from \( X \) to a Banach space is completely continuous.

(b) \( X \) satisfies the DP1 if and only if every strong*-norm continuous operator from \( X \) to a Banach space is a DP1 operator.

**Proof.** (a) We prove only the if-implication, because the other implication follows easily. Suppose that every strong*-norm continuous operator from \( X \) to a Banach space is completely continuous. Let \( (x_n) \) be a weakly-null sequence in \( X \) and let \( T : X \rightarrow Y \) be a weakly compact operator. Since the w-right topology and the \( S^*(X, X^*) \)-topology coincide on bounded subsets of \( X \), then there exist a bounded linear operator \( G \) from \( X \) to a Hilbert space and a mapping \( N : (0, \infty) \rightarrow (0, \infty) \) satisfying

\[
\|T(x)\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|,
\]

for all \( x \in X \) and \( \varepsilon > 0 \) (compare [36, Proposition 5.1]).

Let us fix \( \delta > 0 \). Since \( (x_n) \) is bounded, we can find an appropriate \( \varepsilon_0 > 0 \) satisfying that \( \varepsilon_0 \|x_n\| < \frac{\delta}{2} \), for every natural \( n \). By hypothesis, \( G(x_n) \rightarrow 0 \) in norm. So there exists a natural \( m \) satisfying that

\[
N(\varepsilon_0)\|G(x_n)\| < \frac{\delta}{2}, \quad \text{for all} \quad n \geq m,
\]

which gives that \( \|T(x_n)\| < \delta \), for all \( n \geq m \).

The proof of statement (b) follows similarly. □

For each \( C^* \)-algebra \( A \), the w-right and the strong* topologies coincide on bounded sets of \( A \) (compare [2, Theorem II.7]). The extension property of 2-C*-summing operators proved in Theorem 3.6 together with the above Proposition 3.7 give an alternative proof to [11, Corollary 2] and [19, Corollary 3.2].

**Corollary 3.8** Every \( C^* \)-subalgebra of a \( C^* \)-algebra satisfying the DPP (respectively, the DP1) also satisfies the same property.

Let \( u \) be a norm-one element in a Banach space \( X \). The set of states of \( X \) relative to \( u \), \( D(X, u) \), is defined as the non empty, convex, and weak*-compact subset of \( X^* \) given by

\[
D(X, u) := \{ \Phi \in X^* : \Phi(u) = 1 = \|\Phi\| \}.
\]

For \( x \in X \), the numerical range of \( x \) relative to \( u \), \( V(X, u, x) \), is given by \( V(X, u, x) := \{ \Phi(x) : \Phi \in D(X, u) \} \). The numerical radius of \( x \) relative to \( u \), \( v(X, u, x) \), is given by

\[
v(X, u, x) := \max\{|\lambda| : \lambda \in V(X, u, x)\}.
\]

It is well-known that a bounded linear operator \( T \) on a complex Banach space \( X \) is hermitian if and only if \( V(L(X), I_X, T) \subseteq \mathbb{R} \) (compare [9, Section 5, Lemma 2]). If \( T \) is a bounded linear operator on \( X \), then we have \( V(L(X), I_X, T) = \sigma T \) where

\[
W(T) = \{ x^*(T(x)) : (x, x^*) \in \Gamma \},
\]

and \( \Gamma \subseteq \{(x, x^*) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\} \) verifies that its projection onto the first coordinate is norm dense in the unit sphere of \( X \) [9, Section 9]. Moreover, the numerical radius of \( T \) can be calculated as follows

\[
v(L(X), I_X, T) = \sup\{|x^*(T(x))| : (x, x^*) \in \Gamma \}.
\]
In particular if \( X = Y^* \), then by the Bishop-Phelps-Bollobás theorem, it follows that
\[
v(\mathcal{L}(X), L_X, T) = \sup \{|x^*(T(x))| : x \in S_X, \ x^* \in S_Y, \ x^*(x) = 1\}.
\]

Originally introduced by Kaup in [26], the class of complex Banach spaces called JB*-triples includes all C*-algebras, Hilbert spaces, spin factors and operators between complex Hilbert spaces. A JB*-triple is a complex Banach space \( E \) with a continuous triple product \( \{\ldots\} : E \times E \times E \longrightarrow E \) which is bilinear and symmetric in the outer variables and conjugate linear in the middle one, and satisfies:

- (JB1) (Jordan Identity) \( L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} - \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\} \), for all \( a, b, c, x, y, z \) in \( E \), where \( L(a, b)x := \{a, b, x\} \);
- (JB2) The map \( L(a, a) : E \rightarrow E \) is an hermitian operator with non negative spectrum for all \( a \) in \( E \);
- (JB3) \( \|\{a, a, a\}\| = \|a\|^3 \), for all \( a \) in \( E \).

For each element \( x \) in a JB*-triple \( E \), we shall denote \( x^{[1]} := x, x^{[3]} := \{x, x, x\} \), and \( x^{[2n+1]} := \{x, x, x^{[2n-1]}\}, (n \in \mathbb{N}) \). Given a subset \( F \subset E \), the symbol \( F \sqcap F \) will denote the set
\[
\{L(x, y) : x, y \in F \} \subset \mathcal{L}(E).
\]

Examples: every C*-algebra is a JB*-triple with respect to the product \( \{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a) \). The above product remains valid for the space \( \mathcal{L}(H, K) \) of all continuous operators between two complex Hilbert spaces \( H, K \).

For each JB*-triple \( E \) and every state \( \Phi \in D(\mathcal{L}(E), I_E) \), the assignment \( x \longmapsto \|x\|_\Phi := \Phi(L(x, x))^{\frac{1}{2}} \) defines a prehilbertian seminorm on \( E \). Further, whenever \( \varphi \) is a norm-one element in \( E^* \) and \( z \in S_{E^*} \) with \( \varphi(z) = 1 \), the mapping \( x \longmapsto \|z\|_\varphi = \varphi\{x, x, z\}^{\frac{1}{2}} \) does not depend on the point of support \( z \), and defines also a prehilbertian seminorm on \( E \) (compare [34, Section 1]).

In the more general setting of JB*-triples, only the notion of 2-C*-summing operator has been generalized in [32]. An operator \( T \) from a JB*-triple \( E \) to a Banach space \( Y \) is said to be 2-JB*-triple-summing if there exists a positive constant \( C \) such that for every finite sequence \( (x_1, \ldots, x_n) \) of elements in \( E \) we have
\[
\sum_{i=1}^n \|T(x_i)\|^2 \leq C \sum_{i=1}^n L(x_i, x_i).
\]

The smallest constant \( C \) for which (3.4) holds is denoted by \( C_2(T) \).

The corresponding Pietsch factorization theorem for 2-JB*-triple-summing operators was established in [32, Theorem 3.6]. Indeed: if \( T : E \rightarrow Y \) is a 2-JB*-triple-summing operator then there are norm-one functionals \( \varphi_1, \varphi_2 \) in \( E^* \) and a positive constant \( C(T) \) such that
\[
\|T(x)\| \leq C(T) \|x\|_{\varphi_1, \varphi_2}
\]
for all \( x \in E \). This result together with the little Grothendieck inequality for JB*-triples and the Hahn-Banach theorem allow us to prove the following result with a verbatim adaptation of the proof of Theorem 3.6.

**Theorem 3.9** Let \( E \) and \( F \) be two JB*-triples with \( F \) a JB*-subtriple of \( E \) and let \( Y \) be a Banach space. Then every 2-JB*-triple-summing operator \( T : F \rightarrow Y \) admits a norm preserving 2-JB*-triple-summing extension \( \hat{T} : E \rightarrow Y \).

Having in mind that for every JB*-triple \( E \), the w-right and strong* topologies coincide on bounded subsets of \( E \) (compare [33, p. 621]), the results [12, Corollary 6] and [1, Corollary 1] follow now as a direct consequence.

**Corollary 3.10** Every JB*-subtriple of aJB*-triple satisfying the DPP (respectively, the DP1) also satisfies the same property.

Our next goal is to introduce a suitable variation of \( p \)-summing operators in JB*-triples.

Let \( x \) be an element in a (general) JB*-triple \( E \) and let \( E_x \) denote the JB*-subspace generated by \( x \). It is known that \( E_x \) is a commutative JB*-triple. Therefore, the closed linear span of \( E_x \sqcap E_x \subset L(E_x) \) is an abelian C*-algebra (compare [26, Proposition 1.5]). This structure allows to define \( L(x, x)^\frac{1}{2} \) as an element in \( L(E_x) \).
However, this definition, based on “local theory,” does not satisfy our needs because $L(x, x)^\sharp$ should be an element in $L(E)$.

We shall see how local theory, wisely applied, can help us to avoid this obstacle. Let $x$ be an element in a JB$^*$-triple $E$. It is known that $E_x$ is JB$^*$-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space $\Omega$ contained in $(0, \|x\|)$, such that $\Omega \cup \{0\}$ is compact and $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism $\Psi$ from $E_x$ onto $C_0(\Omega)$, $\Psi(x)(t) = t (t \in \Omega)$ (cf. [25, 4.8], [26, 1.15] and [20]). The set $\Omega = \text{Sp}(x)$ is called the triple spectrum of $x$. We should note that $C_0(\text{Sp}(x)) = C(\text{Sp}(x))$, whenever $0 \notin \text{Sp}(x)$.

Local theory in JB$^*$-triples gave rise to the so-called triple functional calculus. To avoid possible confusion with the classical continuous functional calculus in C$^*$-algebras, given a function $f \in C_0(\text{Sp}(x))$, $f(x)$ shall have its usual meaning when $E_x$ is regarded as an abelian C$^*$-algebra and $f_1(x)$ shall denote the same element of $E_x$ when the latter is regarded as a JB$^*$-subtriple of $E$. Thus, for any odd polynomial, $P(\lambda) = \sum_{k=0}^n \alpha_k \lambda^{2k+1}$, we have $P_1(x) = \sum_{k=0}^n \alpha_k x^{2k+1}$. The symbol $x[\sharp]$ will stand for $f_1(x)$, where $f(\lambda) := \lambda^p$, $(\lambda \in \text{Sp}(x))$.

The general lack of order in JB$^*$-triples of the same kind that exists for C$^*$-algebras prevents us to affirm any property on a finite sum of the form $\sum_j x_j^{[\sharp]}$, where $x_1, \ldots, x_k$ are arbitrary elements in a JB$^*$-triple $E$. In order to have a common order, not depending on the local structure, we make use of the space $L(E)$. The following definition does not require the existence of an order.

**Definition 3.11** Let $E$ be a JB$^*$-triple, let $Y$ be a Banach space and let $p > 0$. An operator $T : E \to Y$ is said to be $p$-JB$^*$-triple-summing if there exists a positive constant $C$ such that for every finite sequence $(x_1, \ldots, x_n)$ of elements in $E$ we have

$$\sum_{i=1}^n \|T(x_i)\|^p \leq C \sum_{i=1}^n L \left(x_i^{[\sharp]}, x_i^{[\sharp]}\right).$$

The smallest constant $C$ for which (3.5) holds is denoted by $C_p(T)$.

Let $A$ be a C$^*$-algebra. We recall that two elements $a$ and $b$ in $A$ are said to be orthogonal if $ab^* = b^*a = 0$, equivalently, $L(a, b) = 0$. When $a$ and $b$ belong to a JB$^*$-triple $E$, we say that $a$ and $b$ are orthogonal whenever $L(a, b) = 0$. When a C$^*$-algebra is regarded as a JB$^*$-triple, these two notions of orthogonality coincide on $A$.

We refer to [10, Lemma 1] for several reformulations of orthogonality in C$^*$-algebras and JB$^*$-triples.

C$^*$-algebras have a dual structure as JB$^*$-triples and C$^*$-algebras. Our next result shows that, in the setting of C$^*$-algebras, $p$-C$^*$-summing operators and $p$-JB$^*$-triple-summing operators coincide.

**Lemma 3.12** Let $(x_1, \ldots, x_n)$ be a finite sequence of elements in the C$^*$-algebra $A$ and let $X$ be a Banach space. The following statements hold:

(a) $\left\| \sum_{i=1}^n \|x_i\|^2 \right\| \leq \left\| \sum_{i=1}^n L(x_i, x_i) \right\| \leq 2 \left\| \sum_{i=1}^n \|x_i\|^2 \right\|$.

(b) When $x_1, \ldots, x_n$ are assumed to be hermitian we have

$$\left\| \sum_{i=1}^n L \left(x_i^{[\sharp]}, x_i^{[\sharp]}\right) \right\| = \left\| \sum_{i=1}^n \|x_i\|^p \right\|,$$

for every $p > 0$.

(c) If $T \in L(A, X)$, then $T$ is $p$-C$^*$-summing whenever it is $p$-JB$^*$-triple-summing. Moreover, if $T$ is $p$-C$^*$-summing then there exists $C > 0$ satisfying

$$\sum_{i=1}^n \|T(x_i)\|^p \leq C \sum_{i=1}^n L \left(x_i^{[\sharp]}, x_i^{[\sharp]}\right),$$

for every finite sequence $(x_1, \ldots, x_n)$ of hermitian elements in $A$. 
Proof. (a) Let 1 denote the unit element in $A^{**}$. For every finite sequence $(x_1, \ldots, x_n)$ of elements in $A$ we have

$$\left\| \sum_{i=1}^{n} L(x_i, x_i) \right\| \geq \left\| \sum_{i=1}^{n} L(x_i, x_i)(1) \right\| = \left\| \sum_{i=1}^{n} |x_i|^p \right\|. $$

To see the other inequality let us denote $S := \sum_{i=1}^{n} L(x_i, x_i)$. Clearly $S$ is a hermitian operator on $A$, Sinclair’s theorem (compare [9, Remark in p. 54]) assures that

$$\|S\| = \sup\{ |\phi(S(z))| : z \in S_A, \phi \in S_{A^{**}}, \phi(z) = 1 \}. $$

It is worth mentioning that $\phi(S(z)) \geq 0$ for any $\phi$ and $z$ in the above setting. Let $z \in S_A$ and $\phi \in S_{A^{**}}$ with $\phi(z) = 1$. We define $\psi(x) := \phi(x \circ z)$, where the symbol $\circ$ denotes the natural Jordan product in $A$. It can be easily seen that $\psi \in S_{A^{**}}$, $\psi(1) = \phi(z) = 1$. Moreover,

$$\psi(L(x, x)(1)) = \phi(L(x, x)(1) \circ z) = \frac{1}{2} \phi(\{x, x, z\} + \{x^*, x^*, z\}) \geq \frac{1}{2} \phi(L(x, x)(z)),$$

for all $x \in A$. Furthermore, $\phi(L(x, x)(z)) = \psi(L(x, x)(1))$, whenever $x = x^*$. In particular $\phi(S(z)) \leq 2\psi(S(1))$, and hence

$$\|S\| \leq 2 \sup \{ \psi(S(1)) : \psi \in S_{A^{**}}, \psi(1) = 1 \} = 2 \sup \left\{ \psi \left( \sum_{i=1}^{n} |x_i|^2 \right) : \psi \in S_{A^{**}}, \psi(1) = 1 \right\} = 2 \left\| \sum_{i=1}^{n} |x_i|^2 \right\|. $$

When $x_1, \ldots, x_n$ are hermitian elements, the constant 2 in the above inequality can be omitted, which in particular gives: $\| \sum_{i=1}^{n} |x_i|^2 \| = \| \sum_{i=1}^{n} L(x_i, x_i) \|.

(b) Every self-adjoint element $x \in A$ admits a decomposition in the form $x = x^+ - x^-$, where $x^+$ and $x^-$ are orthogonal positive elements in $A$. It is not hard to see that $x[\tau] = (x^+[\tau] - x^-[\tau])$. Since $(x^+[\tau])$ and $(x^-[\tau])$ are orthogonal, we have $|x[\tau]|^2 = (x^+)^p + (x^-)^p = \|x\|^p$. Let $x_1, \ldots, x_n$ be self-adjoint elements in $A$. The last paragraph in the proof of the previous statement shows that

$$\left\| \sum_{i=1}^{n} L \left( x_1^{[\tau]}, x_i^{[\tau]} \right) \right\| = \left\| \sum_{i=1}^{n} |x_i^{[\tau]}|^2 \right\| = \sum_{i=1}^{n} |x_i|^p. $$

(c) The formula stated in (b) proves the required statements.

Let $A$ be a $C^*$-algebra and let $X$ be a Banach space. The question is clearly whether $p$-$JB^*$-triple-summing and $p$-$C^*$-summing operators coincide in $L(A, X)$. A strengthening of the inequality in Lemma 3.12, (b) seems to be necessary.

**Proposition 3.13** Let $A$ be a $C^*$-algebra and let $p \geq 2$. Then the formula

$$\sum_{i=1}^{n} L \left( x_i^{[\tau]}, x_i^{[\tau]} \right)(1) \geq \sum_{i=1}^{n} |x_i|^p,$$

holds for every finite sequence of elements $x_1, \ldots, x_n$ in $A$. In particular $p$-$JB^*$-triple-summing and $p$-$C^*$-summing operators on $A$ coincide.

Proof. Considering $A^{**}$ instead of $A$, we may assume that $A$ is a von Neumann algebra.

Let $e$ be a partial isometry in $A$. It is easy to check that $e[\tau] = e$. Since $ee^* + e^*e$ is a positive element in the closed unit ball of $A$ and $p/2 \geq 1$ we have

$$L \left( e[\tau], e[\tau] \right)(1) = L(e, e)(1) = \frac{ee^* + e^*e}{2} \geq \|e\|^p = \left( \frac{ee^* + e^*e}{2} \right)^{\frac{p}{2}}.$$
Let $a$ be an algebraic element in $A$ when the latter is regarded as a JBW*-triple. That is, $a = \sum_{i=1}^{k} \alpha_{i} e_{i}$, where $\alpha_{i} \in \mathbb{R}^{*}$ and $(e_{i})$ are mutually orthogonal partial isometries (tripotents) in $A$. Since the $e_{i}$’s are mutually orthogonal, we have $a^{\sharp,t} = \sum_{i} \alpha_{i}^{2} e_{i}$. Thus,

$$
L \left( a^{\sharp,t}, a^{\sharp,t} \right)(1) = \sum_{i} \alpha_{i}^{2} L(e_{i}, e_{i})(1) \geq \sum_{i} \alpha_{i}^{2} |e_{i}|^{p} = \left( \sum_{i} \alpha_{i}^{2} |e_{i}|^{2} \right)^{\frac{p}{2}} = |a|^{p}. \tag{3.6}
$$

It is known that the set of tripotents is norm-total in every JBW*-triple, i.e., for every element $x$ in $A$ there exists a sequence $(a_{k})$ of algebraic elements in $A$ converging in norm to $x$ (compare [24, Lemma 3.1(1)]). Since $(a_{k}^{\sharp,t})$ and $(|a_{k}|^{p})$ converge in norm to $x^{\sharp,t}$ and $|x|^{p}$, respectively, inequality (3.6) proves the statement.

From now on, given an element $a$ in a C*-algebra $A$, $\sigma_{A}(a)$ will stand for the spectrum of $a$ in $A$.

**Remark 3.14** The inequality established in the above Proposition 3.13 does not hold for $0 < p < 2$. Indeed, let us consider $A = C([0, 1], M_{2}(\mathbb{C}))$ the C*-algebra of all continuous functions on $[0, 1]$ with values in $M_{2}(\mathbb{C})$. We define $e \equiv e(t) := \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{1-t} \end{pmatrix} \in A$. In this case, we have

$$
(ee^{*} + e^{*}e)(t) = \begin{pmatrix} 1 + t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1 - t \end{pmatrix}.
$$

Since for each $t \in [0, 1]$, the spectrum $\sigma_{M_{2}(\mathbb{C})}(ee^{*} + e^{*}e(t)) = \{1 + \sqrt{t}, 1 - \sqrt{t}\}$, it can be easily seen that $\sigma_{A}(ee^{*} + e^{*}e) = [0, 1]$. We claim that, for $0 < p < 2$, there is no positive constant $C > 0$ satisfying

$$
C L \left( e^{\sharp,t}, e^{\sharp,t} \right)(1) \geq |e|^{p}.
$$

Otherwise, we have

$$
C \frac{ee^{*} + e^{*}e}{2} = C L(e, e)(1) = C L \left( e^{\sharp,t}, e^{\sharp,t} \right)(1) \geq |e|^{p} = \left( \frac{ee^{*} + e^{*}e}{2} \right)^{\frac{p}{2}},
$$

which is impossible, since $Ct \not\equiv t^{\frac{p}{2}}$ in $C[0, 1]$.

However, for each $0 < p < 2$, the question whether every $p$-C*-summing operator on a C*-algebra is automatically $p$-JB*-triple-summing remains open.

Following standard arguments, a Pietsch factorisation theorem for $p$-JB*-triple-summing operators on JB*-triples can be established now.

**Theorem 3.15** Let $T$ be a bounded operator from a JB*-triple $E$ to a Banach space $X$. For each $p > 0$, the following assertions are equivalent.

(a) $T$ is $p$-JB*-triple-summing.

(b) There is a state $\Psi \in D(\mathcal{L}(E), I_{E})$ and a positive constant $C(T)$ such that

$$
\|T(x)\|^{p} \leq C(T) \Psi \left( L \left( x^{\sharp,t}, x^{\sharp,t} \right) \right),
$$

for every $x \in E$.

(c) There exist two norm-one functionals $\varphi_{1}, \varphi_{2} \in E^{*}$ and a positive constant $K(T)$ such that

$$
\|T(x)\|^{p} \leq K(T) \left( \|x^{\sharp,t}\|_{\varphi_{1}}^{2} + \|x^{\sharp,t}\|_{\varphi_{2}}^{2} \right),
$$

for every $x \in E$. 

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Thus, there exists \( \Phi \) such that for every finite sequence \( x_1, \ldots, x_k \in E \), the operator \( S = \sum_{i=1}^{k} L \left( x_i^{[1]}, x_i^{[2]} \right) \) is hermitian, Sinclair’s Theorem (compare [9, Theorem 11.17]) assures that

\[
\| S \| = \sup_{\Phi \in K} | \Phi(S) | = \max_{\Phi \in K} | \Phi(S) |.
\]

(3.7)

Thus, there exists \( \Phi_S \in K \) satisfying that \( \Phi_S(S) = \| S \| \), and hence

\[
f_{x_1, \ldots, x_k} (\Phi_S) = \sum_{i=1}^{k} \| T(x_i) \|^p - C_p(T) \left\| \sum_{i=1}^{k} L \left( x_i^{[1]}, x_i^{[2]} \right) \right\| \leq 0.
\]

By the Ky Fan lemma (see [38, E.4]) there exists an element \( \Psi \in K \) such that \( f_{x_1, \ldots, x_k} (\Psi) \leq 0 \) for every \( f_{x_1, \ldots, x_k} \in \Gamma \), which in particular implies that

\[
\| T(x) \|^p \leq C(T) \Psi \left( L \left( x^{[1]}, x^{[2]} \right) \right),
\]

for every \( x \in E \).

(b) \( \Rightarrow \) (c) Let \( \Psi \in D(L(E), I_E) \), satisfying the assumption (b). The map \( \| . \|_\Psi \) is a prehilbertian seminorm on \( E \). Denoting \( N := \{ x \in E : \| x \|_\Psi = 0 \} \), then the quotient \( E/N \) can be completed to a Hilbert space \( H \). Let us denote by \( Q \) the natural quotient map from \( E \) to \( H \). By [33, Corollary 1] (see also [34, Corollary 1.11]) there exist two norm-one functionals \( \varphi_1, \varphi_2 \in E^* \) such that the inequality

\[
\| Q(x) \|^2 = \| x \|^2_{\Psi} = \Psi \left( L(x, x) \right) \leq 4 \left( \| x \|^2_{\varphi_1} + \| x \|^2_{\varphi_2} \right)
\]

holds for every \( x \in E \). We therefore have:

\[
\| T(x) \|^p \leq 4C(T) \left( \| x \|^2_{\varphi_1} + \| x \|^2_{\varphi_2} \right),
\]

for every \( x \in E \).

(c) \( \Rightarrow \) (a) Let \( \varphi \in S_{E^*} \) and \( z \in S_{E^{**}} \) with \( \varphi(z) = 1 \). Since for every finite sequence \( (x_1, \ldots, x_n) \) in \( E \) we have

\[
\sum_{i} \| x_i^{[2]} z \|^2_{\varphi} = \sum_{i} \varphi \left( x_i^{[1]}, x_i^{[2]}, z \right) = \varphi \sum_{i} L \left( x_i^{[1]}, x_i^{[2]} \right) (z) = \left\| \sum_{i=1}^{n} L \left( x_i^{[1]}, x_i^{[2]} \right) \right\|,
\]

and hence \( a \) follows from \( c \). \( \Box \)

### 4 w-right-norm continuous holomorphic mappings

Given Banach spaces \( X \) and \( Y \), letting \( m = 1, 2, \ldots \), we shall denote by \( L^{(m)}(X, Y) \) the Banach space of all continuous \( m \)-linear mappings from \( X^m = X \times \ldots \times X \) (\( m \) times) to \( Y \), with respect to the pointwise vector operations and the norm defined by

\[
\| A \| = \sup_{x_1 \neq 0, \ldots, x_m \neq 0} \frac{\| A(x_1, \ldots, x_m) \|}{\| x_1 \| \cdots \| x_m \|}
\]

where \( A \in L^{(m)}(X, Y) \) and \( x_1, \ldots, x_m \in X \).
An element \( A \in \mathcal{L}(^m X, Y) \) is said to be symmetric if
\[
A(x_1, \ldots, x_m) = A(x_{\sigma(1)}, \ldots, x_{\sigma(m)})
\]
for any permutation \( \sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\} \). The symbol \( \mathcal{L}_s(^m X, Y) \) will denote the closed subspace of \( \mathcal{L}(^m X, Y) \) of all symmetric continuous \( m \)-linear mappings.

A continuous \( m \)-homogeneous polynomial \( P \) from \( X \) to \( Y \) is a mapping \( P : X \rightarrow Y \) for which there is a unique \( A \in \mathcal{L}_s(^m X, Y) \) such that
\[
P(x) = A(x, \ldots, x) \quad \text{for any} \quad x \in X.
\]
The \( m \)-linear operator \( A \) is called the generating operator for \( P \) and in the sequel will be denoted by \( \hat{P} \). By a \( 0 \)-homogeneous polynomial we mean a constant function. \( \mathcal{P}(^m X, Y) \) will denote the Banach space of all continuous \( m \)-homogeneous polynomials from \( X \) to \( Y \), with respect the pointwise vector operations and the norm defined by
\[
\| P \| = \sup_{x \neq 0} \frac{\| P(x) \|}{\| x \|}.
\]

Every \( m \)-homogeneous polynomial \( P : X \rightarrow Y \) satisfies the following polarization formula:
\[
\hat{P}(x_1, \ldots, x_m) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m \ P \left( \sum_{i=1}^{m} \varepsilon_i x_i \right), \tag{4.1}
\]

Jointly \( w \)-right-norm continuous multilinear operators have been studied in [22, 30] and [37]. A multilinear operator \( T : X_1 \times \ldots \times X_m \rightarrow X \) is jointly \( w \)-right-to-norm continuous if and only if it is jointly \( w \)-right-to-norm continuous at \( 0 \) if and only if there exist reflexive Banach spaces \( R_1, \ldots, R_m \) and bounded linear operators \( T_i : X_i \rightarrow R_i \) satisfying, for each \( x_i \in X_i \),
\[
\| T(x_1, \ldots, x_m) \| \leq \| x_1 \|_{T_1} \cdots \| x_m \|_{T_m}.
\]
(compare [22, Theorem 4] and [37, Proposition 3.11] or [30, Theorem 1]).

The polarization formula (4.1) guarantees that an \( m \)-homogeneous polynomial \( P \) is \( w \)-right-norm continuous if and only if its generating multilinear operator is jointly \( w \)-right-norm continuous (at \( 0 \)) if and only if \( P \) is \( w \)-right-norm continuous at \( 0 \). The corresponding affirmation for the strong* topology is also true.

Arens [3, 4] was the first author in considering extensions of bilinear operators to the product of the biduals. For multilinear operators, Aron and Berner introduced, in [5], a method to extend \( k \)-linear mappings to the product of the biduals that can be described as follows: Let \( X_1, \ldots, X_k \) and \( X \) be Banach spaces and \( T : X_1 \times \cdots \times X_k \rightarrow X \) a \( k \)-linear operator. Let \( \pi : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\} \) (denoted \( i \mapsto \pi_i \)) be a permutation. We define the Aron-Berner extension of \( T \) associated to \( \pi \)
\[
AB(T)_{\pi} : X_1^{**} \times \cdots \times X_k^{**} \rightarrow X^{**}
\]
by
\[
AB(T)_{\pi}(z_1, \ldots, z_k) = \text{weak*} - \lim_{\alpha \rightarrow \pi_1} \cdots \text{weak*} - \lim_{\alpha \rightarrow \pi_k} T(z_1^{\alpha_1}, \ldots, z_k^{\alpha_k}),
\]
where \( (z_1, \ldots, z_k) \in X_1^{**} \times \cdots \times X_k^{**} \) and, for \( 1 \leq i \leq k \), \( (x_i^{\alpha_i})_{\alpha_i} \subset X_i \) is a net weak* convergent to \( z_i \). \( AB(T)_{\pi} \) is bounded and has the same norm as \( T \). For each \( k \)-linear operator there are \( k! \) possibly different extensions. However, for each symmetric \( k \)-linear operator \( T \) the restriction of \( AB(T)_{\pi} \) to the diagonal does not depend on the permutation \( \pi \).

Given an \( m \)-homogeneous polynomial \( P : X \rightarrow Y \), the \( m \)-homogeneous polynomial \( AB(P) : X \rightarrow Y \),
\[
AB(P)(x) := AB(P)_{\pi}(x, \ldots, x) \quad \text{(where \( \pi \) is any permutation of the set \( \{1, \ldots, k\} \))},
\]
will be called the Aron-Berner extension of \( P \).
A continuous polynomial $P$ from $X$ to $Y$ is a finite sum of continuous homogeneous polynomials. We shall denote by $\mathcal{P}(X, Y)$ the space of all continuous polynomials from $X$ to $Y$ with respect to pointwise vector operations. Following [31], a polynomial $P : X \rightarrow Y$ is said to be weakly compact if $P$ maps bounded sets in $X$ into relatively weakly compact sets in $Y$.

We have already noticed that a bounded linear operator $T : X \rightarrow Y$ is weakly compact if and only if $T$ is w-right-norm continuous. The following examples show that none of these implications holds for continuous polynomials in general Banach spaces.

**Example 4.1** Let $P : \ell_2 \rightarrow \ell_1$ be the 2-homogeneous polynomial whose generating operator is defined by

$$
\hat{P} : \ell_2 \times \ell_2 \rightarrow \ell_1,
$$

$$
\hat{P}(x, y) = x \cdot y,
$$

where $x \cdot y$ denotes the pointwise multiplication. It follows by Hölder’s inequality that $\hat{P}$ is well defined with $\|\hat{P}\| \leq 1$. Since $\ell_2$ is a reflexive Banach space, and for any reflexive Banach space the w-right topology coincides with the norm topology, we trivially have that $P$ is w-right-norm continuous. However, $P$ cannot be weakly compact because $P$ maps the canonical basis of $\ell_2$ to the canonical basis of $\ell_1$ and the latter admits no weakly convergent subsequences.

A weakly compact polynomial on a Banach space $X$ need not be w-right-norm continuous, even when $X$ satisfies the Dunford-Pettis property.

**Example 4.2** Since the interval $[\frac{1}{2}, 1]$ is not scattered, there is a continuous surjective linear map $q : C\left([\frac{1}{2}, 1]\right) \rightarrow \ell_2$ (compare [13, Corollary 4.16]). By the open mapping theorem, we can pick $f_n \in C\left([\frac{1}{2}, 1]\right)$ with $\|f_n\| = 1$ such that $q(f_n) = e_n$, for every $n \in \mathbb{N}$, where $(e_n)$ denotes the canonical basis of $\ell_2$. We can define a sequence $(g_n)$ in $C([0, 1])$ satisfying that $g_n\left(\frac{1}{2}\right) = f_n$ and $g_n\left(\frac{1}{4}\right) = 0$.

On the other hand, the assignment $f \mapsto (f\left(\frac{1}{4n}\right) - f(\frac{1}{2}))_n \in \ell_1$ defines a linear operator $p : C([0, 1]) \rightarrow c_0$.

Finally, we define a symmetric bilinear map

$$
V : C([0, 1]) \times C([0, 1]) \rightarrow \ell_2
$$

given by $V(f, g) := p(f) \cdot q\left(g\left|_{\frac{1}{2}, 1}\right.\right) + p(g) \cdot q\left(f\left|_{\frac{1}{2}, 1}\right.\right)$, where for $a \in c_0$ and $b \in \ell_2$, $a \cdot b \in \ell_1$ is defined by $(a \cdot b)_n = a_n b_n$. It is clear that $V$ is weakly compact. We claim that $V$ is not jointly w-right-norm continuous. Indeed, let us pick a sequence $(x_n)$ of mutually orthogonal continuous functions in $C([0, 1])$ satisfying $\|x_n\| = x_n\left(\frac{1}{4n}\right) = 1$. By definition, $(x_n)$ is a w-right-null sequence in $C([0, 1])$ (compare [35, Lemma 13]), while $(g_n)$ is a bounded sequence in $C([0, 1])$. Thus, if $V$ were jointly w-right-norm continuous, then Proposition 3.11 in [37], would imply that

$$
1 = \|e_n\| = \|V(x_n, g_n)\| \rightarrow 0,
$$

which is impossible.

Let us recall that an operator is said to be pseudo weakly compact if it is sequentially w-right-norm continuous. A Banach space is called sequentially right if every pseudo weakly compact operator from $X$ to another Banach space is weakly compact. C*-algebras, JB*-triples and Banach spaces satisfying Pelczynski’s Property $(V)$ are examples of sequentially right spaces (compare [35]).

It is also known that a bounded linear operator $T : X \rightarrow Y$ is weakly compact if and only if its bitranspose remains $Y$-valued. In the multilinear setting, a similar question has been recently considered in [37]. We first recall the following definition introduced in [37]: Given Banach spaces $X_1, \ldots, X_k$, $X$, a multilinear operator $T : X_1 \times \cdots \times X_k \rightarrow X$ is right quasicompletely continuous (RQCC) if for arbitrary w-right Cauchy sequences $(x^n_i)_n \subset X_i$ ($1 \leq i \leq k$), the sequence $(T(x^n_1, \ldots, x^n_k))_n$ converges in norm, equivalently, for every sequence $(x^n_i) \subset X_i$ which is w-right-convergent to $x_i \in X_i$ ($1 \leq i \leq k$) we have

$$
\lim_n \|T(x^n_1, \ldots, x^n_k) - T(x_1, \ldots, x_k)\| = 0,
$$

that is $T$ is jointly sequentially w-right-norm continuous. The following result follows from Proposition 3.3 and Theorem 3.8 in [37]. Let $X_1, \ldots, X_k$ be non zero sequentially right Banach spaces and let $T : X_1 \times \cdots \times X_k \rightarrow X$
be a multilinear operator. Then $T$ is RQCC if and only if all of the Aron-Berner extensions of $T$ are $X$-valued if and only if $T$ has an $X$-valued Aron-Berner extension. We shall study this equivalence in the case of holomorphic mappings between complex Banach spaces.

We now consider weakly compact holomorphic mappings. Let $X, Y$ be two Banach spaces, a mapping $f : X \rightarrow Y$ is said to be a holomorphic map if for each $x \in X$ there exists a sequence of polynomials

$$\hat{d}^n f(x) \in \mathcal{P}(n, X, Y)$$

and a neighborhood $V_x$ of $x$ such that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(x)(y - x)$$

converges uniformly to $f(y)$ for every $y \in V_x$.

A holomorphic function $f : X \rightarrow Y$ is said to be of bounded type if it is bounded on all bounded subsets of $X$. The polynomial series at zero $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(y)$ of such a function have infinite radius of uniform convergence, i.e., $\limsup ||\frac{1}{n!} \hat{d}^n f(0)||^\frac{1}{n} = 0$ (compare [16, Section 6.2]).

If $f : X \rightarrow Y$ is a holomorphic function of bounded type and $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(y)$ ($y \in X$) is its Taylor series at 0, it follows by [21, Section 2] or [16, Proposition 6.16] that the assignment

$$y \mapsto AB(f)(y) = \sum_{n=0}^{\infty} \frac{1}{n!} AB(\hat{d}^n f)(0)(y), \quad (y \in X^{**})$$

defines a holomorphic function of bounded type, $AB(f) : X^{**} \rightarrow Y^{**}$, called the Aron-Berner extension of $f$.

A holomorphic map $f : X \rightarrow Y$ is said to be weakly compact if for every $x \in X$ there exists a neighborhood $V_x$ of $x$ such that $f(V_x)$ is a relatively weakly compact set of $Y$. See [28] or [17] for details about holomorphic maps. The Examples 4.1, and 4.2 show that weak compactness is not the correct property to guarantee Aron-Berner extensions valued in the same codomain space.

We shall now show that w-right-norm continuity of a holomorphic mapping $f$ implies w-right-norm continuity of its derivatives at every point.

**Proposition 4.3** Let $f : X \rightarrow Y$ be a holomorphic mapping between two Banach spaces. Then the following statements hold:

(a) If $f$ is w-right-norm continuous (respectively, strong*-norm continuous), then the polynomial $\hat{d}^n f(x)$ is w-right-norm continuous (respectively, strong*-norm continuous) for every $n \in \mathbb{N}$ and every $x \in X$.

(b) If $f$ is sequentially w-right-norm continuous (respectively, strong*-norm continuous), then the polynomial $\hat{d}^n f(x)$ is sequentially w-right-norm continuous (respectively, strong*-norm continuous) for every $n \in \mathbb{N}$ and every $x \in X$.

**Proof.** We shall only include here the proof of the statements concerning the w-right topology, the proofs of those affirmations concerning the strong* topology follow similarly.

(a) Let us fix $x \in X$. By hypothesis, there exist reflexive spaces $R_1, \ldots, R_k$, bounded linear operators $T_i : X \rightarrow R_i$ ($i \in \{1, \ldots, k\}$) and $\delta > 0$ satisfying that $f(W) \in f(x) + B(Y)$, where

$$W = \{y \in X : ||x - y||_{R_i} < \delta, \forall i \in \{1, \ldots, k\}\}.$$

Since $W_0 := \{y \in X : ||y||_{R_i} < \delta, \forall i \in \{1, \ldots, k\}\}$ is a balanced set, it follows by [41, Lemma 3.1] (compare also the proof of [6, Proposition 3.4]),

$$\frac{1}{n!} \hat{d}^n f(x)(W_0) \subseteq \overline{co}(f(x) + W_0) \subseteq \overline{co}(f(x) + B(Y)),$$

where $\overline{co}A$ denotes the convex balanced hull of $A$. In particular there exists a constant $M_n > 0$ satisfying that

$$\|\hat{d}^n f(x)(y)\| \leq M_n, \quad \text{for all} \quad y \in W_0.$$
Taking $R = \oplus^f_z R_i$ and $T : X \rightarrow R, x \mapsto (\delta/2)^{-1}(T_i(x))$, it can be easily seen that, for each $y \in X$ with $T(y) \neq 0$, we have $\frac{\hat{y}}{W_0} \in W_0$, and hence,

$$\|\hat{d}_n f(x)(y)\| \leq M_n \|y\|^n.$$ 

For each $y \in \ker(T)$, and $t > 0$, $ty$ lies in $W_0$, thus $t^n \|\hat{d}_n f(x)(ty)\| \leq M_n$, which implies that $\hat{d}_n f(x)(y) = 0$. We have then shown that

$$\|\hat{d}_n f(x)(y)\| \leq M_n \|y\|^n,$$

for all $x \in X$. This proves that $\hat{d}_n f(x)$ is w-right-norm continuous at 0, which gives the desired statement.

(b) We assume that $f$ is sequentially w-right-norm continuous. Let $(y_k)$ be a sequence in $X$ converging in the w-right topology to $y \in X$. Let us fix $x \in X$ and $\varphi$ in the closed unit ball of $Y^*$. Defining $g_k(\lambda) := \varphi f(x + \lambda y_k)$ and $g(\lambda) := \varphi f(x + \lambda y)$, it follows by Cauchy’s integral formula that

$$\left| \frac{1}{n!} \varphi(\hat{d}_n f(x)(y_k) - \hat{d}_n f(x)(y)) \right| = \left| \left( g_k^{(n)}(0) - g^{(n)}(0) \right) / n! \right| \leq \sup\{|g_k - g| : |\lambda| = 1\} \leq \sup\{|f(x + \lambda y_k) - f(x + \lambda y)| : |\lambda| = 1\}.$$

Taking supreme over all $\varphi$ in the closed unit ball of $Y^*$, we have

$$\|\hat{d}_n f(x)(y_k) - \hat{d}_n f(x)(y)\| \leq n! \sup\{|f(x + \lambda y_k) - f(x + \lambda y)| : |\lambda| = 1\}.$$

Finally, since $f$ is sequentially w-right-norm continuous, it can be easily seen that

$$\lim_{k \to \infty} \sup\{|f(x + \lambda y_k) - f(x + \lambda y)| : |\lambda| = 1\} = 0.$$

\[\Box\]

**Theorem 4.4** Let $X$ be a sequentially right space, $Y$ a Banach space and let $f : X \rightarrow Y$ be a holomorphic function of bounded type. Then $f$ is sequentially w-right-norm continuous if and only if $AB(f)$ is $Y$-valued.

\[\textbf{Proof.}\] Let $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}_n f(0)(y), (y \in X)$ and $AB(f)(y) = \sum_{n=0}^{\infty} \frac{1}{n!} AB(\hat{d}_n f(0))(y), (y \in X^*)$ be the Taylor series of $f$ and $AB(f)$ at zero, respectively. If $f$ is sequentially w-right-norm continuous, then Proposition 4.3 b) implies that, for each natural $n$, $\hat{d}_n f(0)$ is sequentially w-right-norm continuous. The polarization formula (4.1) implies that, for each natural $n$, the generating multilinear operator of $\hat{d}_n f(0)$ is jointly sequentially w-right-norm continuous or RQCC. Theorem 3.8 in [37] guarantees that $AB(\hat{d}_n f(0))$ is $Y$-valued for all natural $n$. The uniform convergence of the Taylor series at zero of the function $AB(f)$ assures that $AB(f)(X^*) \subseteq Y$.

Assume now that $AB(f)(X^*) \subseteq Y$. Since $X^*$ is a balanced set, it follows by [41, Lemma 3.1] (compare also the proof of [6, Proposition 3.4]), that

$$\frac{1}{n!} AB(\hat{d}_n f(0))(X^*) \subseteq \overline{\partial} AB(f)(X^*) \subseteq Y.$$

It follows again from Theorem 3.8 in [37] that $\hat{d}_n f(0)$ is sequentially w-right-norm continuous. The desired statement will finally follow from the uniform convergence of the Taylor series. \[\Box\]

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