A note on weakly compact subsets in the projective tensor product of Banach spaces

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This paper is dedicated to Professor Isaac Namioka, a true gentleman, scholar and fine mathematician

Abstract

Let X and Y be two Banach spaces. In this short note we show that every weakly compact subset in the projective tensor product of X and Y can be written as the intersection of finite unions of sets of the form co(KX ⊗ KY ), where KX and KY are weakly compacts subsets of X and Y, respectively. If either X or Y has the Dunford–Pettis property, then any intersection of sets that are finite unions of sets of the form co(KX ⊗ KY ), where KX and KY are weakly compact sets in X and Y, respectively, is weakly compact.

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1. Preliminaries

Let X and Y be Banach spaces. The projective tensor product X ⊗ Y of X and Y is the completion of the algebraic tensor product X ⊗ Y in the projective norm

\[ \|u\|_\wedge = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i, \quad u \in X \otimes Y, \right\}, \]

where the infimum is taken over all possible representations of u. Grothendieck [7] described members of the projective tensor product of X and Y in the following way: an element \( u \in X \otimes Y \) has the representation

\[ u = \sum_{n=1}^\infty x_n \otimes y_n, \quad \text{with} \quad \sum_{n=1}^\infty \|x_n\| \|y_n\| < \infty \]

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and the projective tensor norm of \( u \) as
\[
\|u\|_\wedge = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\|_Y y_n : u = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}
\]
where the infimum is taken over all possible representations of \( u \) as above. For more about the tensor products of Banach spaces see [7] or [3].

In this short note we show that every weakly compact subset in the projective tensor product of \( X \) and \( Y \) can be written as the intersection of finite unions of sets of the form \( \overline{\sigma(K_X \otimes K_Y)} \), where \( K_X \) and \( K_Y \) are weakly compact subsets of \( X \) and \( Y \), respectively. In case either \( X \) or \( Y \) has the Dunford–Pettis property, then this condition is also sufficient for the weak compactness of a subset of the projective tensor product.

2. Weakly compact sets

The problem of understanding (weak) compactness in Banach spaces is related to many problems in analysis and probability. In the projective tensor product of two Banach spaces, the characterization of relatively norm compactness is due to A. Grothendieck [7], who showed the following

**Proposition 2.1.** Let \( X \) and \( Y \) be Banach spaces. A subset \( K \) of \( X \otimes Y \) is relatively norm compact if and only if there exist compact subsets \( K_X \) of \( X \) and \( K_Y \) of \( Y \) such that \( K \subseteq \overline{\sigma(K_X \otimes K_Y)} \), where \( K_X \otimes K_Y \) is the set \( \{x \otimes y : x \in K_X, y \in K_Y\} \).

Using this fact Grothendieck (see [7, p. 51]) deduced

**Proposition 2.2.** Let \( K \subseteq X \hat{\otimes} Y \) be a compact subset of the projective tensor product of \( X \) and \( Y \). Then there exist two norm null sequences \( \{x_n\}_n \) and \( \{y_n\}_n \) in \( X \) and \( Y \), respectively, and a compact subset \( \overline{K}^{\ell_1} \) of \( \ell_1 \) so that every element of \( u \in K \) can be written as \( u = \sum_{i=1}^{\infty} \lambda^u_i x_i \otimes y_i \) where \( \lambda^u = \{\lambda^u_n\}_n \in \overline{K}^{\ell_1} \).

For the weak compactness in projective tensor product almost nothing is known. First of all a big difference with the norm compact in projective tensor norm is that, if \( K_X \) and \( K_Y \) are norm compact subsets of the Banach spaces \( X \) and \( Y \), respectively, then \( K_X \otimes K_Y \) is a norm compact subset of \( X \otimes Y \). For the weakly compact subsets in projective tensor products the story changes completely.

Indeed, if \( X \) and \( Y \) are reflexive Banach spaces then their closed unit balls, \( B_X \) and \( B_Y \), are weakly compact but since \( B_X \hat{\otimes} Y = \sigma(B_X \otimes B_Y) \), by Grothendieck’s representation theorem, need not be weakly compact. A most important example is found when \( X = \ell_2 = Y \) (see [2]).

In the study of weakly compact subsets of the projective tensor product the singular result of Ülger [12] practically settled the problem in case one was an \( L_1(\mu) \)-space. Ülger’s result was polished into final form in [4].

Previous to Ülger’s work, Michel Talagrand [11] offered a profound analysis of conditionally weakly compact subsets of \( X \hat{\otimes} Y \) when \( X \) is an \( L_1(\mu) \)-space. Talagrand’s work influenced Ülger and so all that has came since. Here is the end result of Ülger, Diestel, Ruess and Schachermayer.

**Theorem 2.3** (Ülger, Diestel, Ruess, Schachermayer). Let \((\Omega, \Sigma, \mu)\) be a finite measure space, and let \( X \) be a Banach space. Let \( A \) be a bounded subset of \( L_1(\mu, X) \). Then the following are equivalent:

(i) \( A \) is relatively weakly compact;
(ii) \( A \) is uniformly integrable, and, given any sequence \( (f_n)_n \subseteq A \) there exists a sequence \( (g_n)_n \) with \( g_n \in \text{co}\{f_k, k \geq n\} \) such that \( (g_n(\omega))_n \) is norm convergent in \( X \) for a.e. \( \omega \in \Omega \);
(iii) \( A \) is uniformly integrable, and, given any sequence \( (f_n)_n \subseteq A \) there is a sequence \( (g_n)_n \) with \( g_n \in \text{co}\{f_k, k \geq n\} \) such that \( (g_n(\omega))_n \) is weakly convergent in \( X \) for a.e. \( \omega \in \Omega \).

**Definition 2.4.** Let \( X \) and \( Y \) be Banach spaces. A bounded linear operator \( T : X \rightarrow Y \) is called completely continuous if \( T \) maps weakly convergent sequences to norm convergent sequences.
A Banach space $X$ has the **Dunford–Pettis property** (DPP) if, for every Banach space $Y$, every weakly compact operator from $X$ to $Y$ is completely continuous (see [5]).

The following are alternative formulations of the Dunford–Pettis property:

(i) Every weakly compact linear operator from $X$ into $c_0$ is completely continuous.
(ii) For every sequence $(x_n)_n$ in $X$ converging weakly to some $x$ and every sequence $(x^*_n)_n$ in $X^*$ converging weakly to some $x^*$, the sequence $(x^*_n(x_n))_n$ converges to $x^*(x)$.
(iii) For every sequence $(x_n)_n$ in $X$ converging weakly to 0 and every sequence $(x^*_n)_n$ in $X^*$ converging weakly to 0, the sequence $(x^*_n(x_n))_n$ converges to 0.

The following can be found in [5]; we include its proof for the sake of completeness and to highlight the result of Section 3 below.

**Proposition 2.5.** Let $X$, $Y$ be Banach spaces, with $X$ having the Dunford–Pettis property. If $W_X \subseteq X$ and $W_Y \subseteq Y$ are weakly compact subsets then $W_X \otimes W_Y$ is a weakly compact of $X \widehat{\otimes} Y$

**Proof.** By the Eberlein–Smuliàn theorem it suffices to show that $W_X \otimes W_Y$ is weakly sequentially compact. Let $(u_i = x_i \otimes y_i)_i$ be a sequence in $W_X \otimes W_Y$. Let $(n_k)_k$ be a strictly increasing sequence of positive integers such that for some $x \in W_X$ and $y \in W_Y$

$$x = \text{weak-} \lim_{k \to \infty} x_{n_k} \quad \text{and} \quad y = \text{weak-} \lim_{k \to \infty} y_{n_k}.$$  

We need to test $(u_{n_k})_k$ vis-à-vis members of $(X \widehat{\otimes} Y)^*$. Since $(X \widehat{\otimes} Y)^* = B(X, Y)$, the space of bilinear continuous functionals on $X \times Y$, take a continuous bilinear functional $F$ on $X \times Y$. If $x^*_k = F(\cdot, y_{n_k})$, then $x^*_k \in X^*$ and $x^* = F(\cdot, y) \in X^*$. Define $T_F : Y \to X^*$ by

$$T_F(y)(x) = F(x, y),$$

$T_F$ is a bounded linear operator and $T_F(y_{n_k}) = x^*_k$ as well as $T_F(y) = x^*$. Since $T_F$ is also weak-to-weak continuous, the fact that $(y_{n_k})_k$ converges weakly to $y$ soon reveals that $(x^*_k)_k$ converges weakly to $x^*$. Now we are in business: $x = \text{weak-} \lim_{k \to \infty} x_{n_k}$ and $x^* = \text{weak-} \lim_{k \to \infty} x^*_k$. Hence, thanks to $X$’s enjoyment of the Dunford–Pettis property, $F(x, y) = T_F(y)(x) = x^*(x) = \lim_k x^*_k(x_{n_k}) = T_F(y_{n_k})(x_{n_k}) = F(x_{n_k}, y_{n_k})$, which is as it should be. \(\Box\)

By the previous proposition we easily have

**Corollary 2.6.** Let $X_1$, $X_2$, $Y_1$, $Y_2$ be Banach spaces. Let $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ be two weakly compact operators. Suppose either $Y_1$ or $Y_2$ has the Dunford–Pettis property, then the projective tensor product $T_1 \hat{\otimes} T_2 : X_1 \widehat{\otimes} X_2 \to Y_1 \widehat{\otimes} Y_2$, of $T_1$ and $T_2$, is weakly compact.

The above corollary was also discovered by G. Racher [10].

3. **Weakly compact subsets in projective tensor products**

In order to study this question let us introduce a topology in $X \widehat{\otimes} Y$, inspired by the work [6] of Godefroy and Kalton, which we will call in the sequel the $\tau$-topology. A base of neighborhoods for the $\tau$-topology has the form

$$A = X \widehat{\otimes} Y \setminus \bigcup_{i=1}^n \overline{\sigma}(U_i \otimes V_i)$$

where $U_i$ and $V_i$ are weakly compact subsets of $X$ and $Y$, respectively, for $i = 1, \ldots, n$. As the reader can note $\tau$ is the coarsest topology so that the sets $\overline{\sigma}(U \otimes V)$ (with $U$ and $V$ weakly compact subsets of $X$ and $Y$, respectively) are $\tau$-closed. Since such subsets are weakly closed (because every convex norm closed set in a Banach space is weakly closed) then the weak topology is finer than the $\tau$-topology on $X \widehat{\otimes} Y$ (recall that if $\theta_1, \theta_2$ are two topologies in $X$ then...
\( \theta_2 \) is finer than \( \theta_1 \) if \( \theta_1 \subseteq \theta_2 \). At first glance the \( \tau \)-topology does not look very beautiful (because it is not Hausdorff in general), but the key idea is to study the restriction of \( \tau \) to certain bounded subsets of \( X \hat{\otimes} Y \) (especially the weak compact subsets) to get a “reasonable” topology (in particular we are interested to see when such a restriction \( \tau \) is Hausdorff). We will not study the \( \tau \)-topology on \( X \hat{\otimes} Y \) in detail, but we will use it only to derive the result. Note that for the topology \( \tau \) we have:

1. For fixed \( v \in X \hat{\otimes} Y \) the map \( u \mapsto u + v \) is \( \tau \)-continuous.
2. For fixed \( \lambda > 0 \) the map \( u \mapsto \lambda u \) is \( \tau \) continuous.
3. The map \( u \mapsto -u \) is \( \tau \)-continuous.

A topology which satisfies (1) and (2) is called a prelinear topology (see [6]). So \( \tau \) is a prelinear topology.

**Theorem 3.1.** Let \( X \) and \( Y \) be two Banach spaces. Every weakly compact subset in \( X \hat{\otimes} Y \) can be written as the intersection of a finite union of sets of the form \( \mathcal{C}(U \otimes V) \), where \( U \) and \( V \) are weakly compact subsets of \( X \) and \( Y \), respectively.

**Proof.** Let \( W \) be a weakly compact subset of \( X \hat{\otimes} Y \). Since the weak topology is finer than the topology \( \tau \), our theorem will be proved once it is shown that the restriction of \( \tau \) to \( W \) is a Hausdorff topology; that means that \( W \) is closed for the topology \( \tau \), and so \( W \) will be as wished.

Let \( u, v \in W \) so that \( u \neq v \). Without loss of generality we can assume \( u = 0 \) (otherwise consider \( \{u - w: w \in W \} \) which is still weakly compact in \( X \hat{\otimes} Y \), and by (1) and (3) above, the translation is a \( \tau \)-homeomorphism). Moreover using (2) we can assume \( \|v\| = 1 \).

We need to distinguish two cases:

**Case 1.** \( v = \sum_{k=1}^{n} \lambda_k x_k \otimes y_k \) with \( \sum_{k=1}^{n} \lambda_k = 1 \) and \( \|x_k\|, \|y_k\| = 1 \) for all \( 1 \leq k \leq n \); i.e. \( v \) is a simple vector of \( X \hat{\otimes} Y \). Now using the Hahn–Banach theorem there exist \( x^* \in X^* \) and \( y^* \in Y^* \) so that

\[
x^* \otimes y^*(0) = 0 < \delta^2 < x^* \otimes y^*(v).
\]

Since \( X \) and \( Y \) are norm one complemented in \( X \hat{\otimes} Y \), let \( P_X, P_Y \) be the projections from \( X \hat{\otimes} Y \) to \( X \) and \( Y \), respectively. Define

\[
\begin{align*}
K_1^u &= [x^* \geq \delta] \cap P_X(W), \\
K_2^u &= [x^* \geq \delta] \cap P_Y(W), \\
K_1^v &= [y^* \geq \delta] \cap P_X(W), \\
K_2^v &= [y^* \geq \delta] \cap P_Y(W),
\end{align*}
\]

where if \( \alpha \in \mathbb{R} \) we are denoting by \( [x^* \leq \alpha] = \{x \in X: x^*(x) \leq \alpha \} \) and \( [y^* \geq \alpha] = \{y \in Y: y^*(y) \geq \alpha \} \). Then \( K_1^u, K_2^u \) are weakly compact subsets of \( X \), and \( K_1^v, K_2^v \) are weakly compact subsets of \( Y \). By construction and by the definition of the topology \( \tau \), we get that \( W \setminus \mathcal{C}(K_1^u \otimes K_2^v) \) is a \( \tau \)-neighborhood of 0 and \( W \setminus \mathcal{C}(K_1^v \otimes K_2^u) \) is a \( \tau \)-neighborhood of \( v \). Since

\[
W \subseteq \mathcal{C}(K_1^u \otimes K_2^v) \cup \mathcal{C}(K_1^v \otimes K_2^u)
\]

we get

\[
[W \setminus \mathcal{C}(K_1^u \otimes K_2^v)] \cap [W \setminus \mathcal{C}(K_1^v \otimes K_2^u)] = \emptyset
\]

hence when \( v \) is a simple tensor we can always separate 0 and \( v \) by two disjoint \( \tau \)-neighborhoods in \( W \).

**Case 2.** Suppose that \( 0 \neq v = \sum_{k=1}^{\infty} \lambda_k x_k \otimes y_k \); we can assume that for all \( n \geq N, \sum_{k=1}^{n} \lambda_k x_k \otimes y_k \neq 0 \), as well.

Suppose \( v \) and 0 cannot be separated by disjoint \( \tau \)-open sets; this means that for any \( \tau \)-open sets \( U, V \) with \( 0 \in U \) and \( v \in V \) we have \( U \cap V \neq \emptyset \). (*)
By Case 1 we know that for each $n \geq N$ there are $\tau$-open sets $U_n$, $V_n$ containing 0 so that

$$(U_n) \cap \left( \sum_{k=1}^{n} \lambda_k x_k \otimes y_k + V_n \right) = \emptyset.$$ 

But $U_n$ and $V_n$, being $\tau$-open, are norm open so there is $n_0 > N$ so

$$v - \sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k \in U_{n_0} \cap V_{n_0}$$

or

$$v \in \left( \sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k + (U_{n_0} \cap V_{n_0}) \right). \quad (**)$$

In tandem (⋆) and (**) tell us that

$$\emptyset \neq \{U_{n_0} \cap V_{n_0}\} \cap \left( \sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k + (U_{n_0} \cap V_{n_0}) \right)$$

(after all, $U_{n_0} \cap V_{n_0}$ is $\tau$-open and contains 0 while $\sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k + (U_{n_0} \cap V_{n_0})$ is $\tau$-open and contains $v$, so (⋆) is in effect)

$$\subseteq U_{n_0} \cup \left( \sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k \right) + V_{n_0} = \emptyset.$$

OOPS! □

4. Some loose ends

To summarize, we have the following

**Theorem 4.1.** If either $X$ or $Y$ has the Dunford–Pettis property, then a subset $K$ of $X \otimes Y$ is relatively weakly compact if and only if $K$ is contained in a finite union of sets of the form $\overline{\text{co}}(K_X \otimes K_Y)$, where $K_X$ is a relatively weakly compact subset of $X$ and $K_Y$ is a relatively weakly compact subset of $Y$.

More precise results would unquestionably be of considerable use in case $X$ is a $C(K)$ space or the disk algebra. In particular we ask

**Question 1.** Find necessary and sufficient conditions that a sequence $(u_n)_n$ in $X \hat{\otimes} Y$ be weakly null, if $X$ is a $C(K)$ space or the disk algebra.

It is hoped that the condition will rely on the behavior of the values $(u_n(\omega))_n$ the sequence takes for points $\omega$ of the domain.

The level of ignorance in affairs of weak compactness in $X \hat{\otimes} Y$ is so high that the following rushes to the front demanding an answer.

**Question 2.** Suppose $1 < p < \infty$. What are the weakly compact subsets of $L_p[0, 1] \hat{\otimes} X$?

We rush to point out that Q. Bu [1] has showed that the natural inclusion of $L_p[0, 1] \hat{\otimes} X$ into $L_p([0, 1], X)$, the Lebesgue–Bochner space of (equivalence classes of) strongly measurable $X$-valued functions on $[0, 1]$ is a semi-embedding; the Ülger, Diestel–Ruess–Schachermayer result does extend to characterize relatively weakly compact subsets of $L_p([0, 1], X)$.

There is, naturally, an infinite list of things we do not know about weak compactness in this gorgeously complicated space $X \hat{\otimes} Y$. We include one more because surprisingly much is known about $X \hat{\otimes} Y$ in the situation of interest.
Question 3. If $X$ is a $C^*$-algebra with unit, then what do weakly compact subsets of $X \hat{\otimes} Y$ look like?

Of course, the work of Pfitzner (see [9]) comes to mind as well as that of Kaijser and Sinclair [8].

References