A proof-checking experiment on representing graphs as membership digraphs

Eugenio G. Omodeo

UNIVERSITÀ DEGLI STUDI DI TRIESTE
Dip. Matematica e Geoscienze — DMI

Joint work with
Pierpaolo Calligaris
and with
Alexandru I. Tomescu
University of Helsinki

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A ‘proof pearl’

(a) a connected claw-free graph $G$

(b) a Hamiltonian cycle in $G^2$

(c) a perfect matching in $G$
“Every partial ordering is isomorphic to a set-inclusion ordering.”
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Theory partialOrder(d₀, U ≼ V)

\[
\langle \forall x, y \mid \{x, y\} \subseteq d_0 \rightarrow ((x \preceq y \land y \preceq x) \leftrightarrow x = y) \rangle
\]

\[
\langle \forall x, y, z \mid \{x, y, z\} \subseteq d_0 \rightarrow ((x \preceq y \land y \preceq z) \rightarrow x \preceq z) \rangle
\]

\[\implies (i_\Theta)\]

1−1(i_\Theta) & dom(i_\Theta) = d₀

\[
\langle \forall x, y \mid \{x, y\} \subseteq d_0 \rightarrow (x \preceq y \leftrightarrow i_\Theta \upharpoonright x \subseteq i_\Theta \upharpoonright y) \rangle
\]

\[
i_\Theta = \{[x, \{v \in d_0 \mid v \preceq x\}] : x \in d_0\}
\]

End partialOrder
“Every ring

\[ \mathbb{B} = (\mathcal{B}, \cdot, +, 1_\mathcal{B}, 0_\mathcal{B}) \]

endowed with multiplicative identity, that meets the laws

\[ X + X = 0_\mathcal{B} \]
\[ X \cdot X = X, \]

is isomorphic to a ring of the special form

\[ \mathbb{S} = (\mathcal{S}, \cap, \Delta, \cup, \emptyset), \]

where \( \mathcal{S} \subseteq \mathcal{P}(U) \) and \( \Delta \) designates the symmetric difference operation over sets.”
Have these a bearing on automated deduction?
Remark:

Theorems of the nature exemplified by the celebrated

*Stone representation theorem for Boolean algebras*

enable a transfer of methods from the ‘big’ theory adopted as the foundation to more specialized, ‘niche’ theories.
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Theorems of the nature exemplified by the celebrated *Stone representation theorem for Boolean algebras*

enable a transfer of methods from the ‘big’ theory adopted as the foundation to more specialized, ‘niche’ theories.

E.g., we can exploit a 2-level syllogistic that deals with

\[ \emptyset, \cap, \setminus, \text{ and } \subseteq \]

to reason about Boolean rings and lattices.
Mostowski’s decoration of a digraph

**Question:**

Can we represent the edges of a digraph

\[ \mathcal{D} = (V, D) \]

through set membership in a *multi-level* set universe?
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Can we represent the edges of a digraph

\[ \mathcal{D} = (V, D) \]

to set membership in a _multi-level_ set universe?

**Answer:**
Yes, we can...
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Yes, we can... provided...

...\( \mathcal{D} \) is acyclic...
**Question:**

Can we represent the edges of a digraph

\[ D = (V, D) \]

through set membership in a *multi-level* set universe?

**Answer:**

Yes, we can... provided...

... \( D \) is acyclic... and, more generally (when \( V \) is infinite),...
**Question:**

Can we represent the edges of a digraph

\[ \mathcal{D} = (V, D) \]

through set membership in a *multi-level* set universe?

**Answer:**

Yes, we can... provided...

... \( \mathcal{D} \) has no paths on infinite length.
**Question:**
Can we represent the edges of a digraph

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through set membership in a *multi-level* set universe?

**Answer:**
Yes, we can... provided...

... \( D \) has no paths on infinite length.

Then, in fact, we can label \( v \mapsto \ell \upharpoonright v \) vertices so as to enforce, for each \( v \) in \( V \):

\[ \ell \upharpoonright v = \{ \ell \upharpoonright w : [v, w] \in D \} \]
Decorating a digraph with sets

\[
\begin{align*}
\emptyset, \\
\{\emptyset\} \\
\emptyset
\end{align*}
\]
Decorating a digraph with sets

An injective labeling amounts to extensionality

Extensionality means: no two vertices share the same children.
Decorating a digraph with sets

An injective labeling amounts to **extensionality**

**Extensionality** means: no two vertices share the same children.

If multiple sinks are the only infringement...

...we can manage to get injectivity anyhow, by retouching the labeling function of the sinks.

For each sink $u$ except for one, we put $\ell|_u = \{V\} \cup V \setminus \{u\}$.
On recursion:
A labeling technique à la Mostowski enables one to hook recursive function-definitions to any well-founded digraph,
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A labeling technique à la Mostowski enables one to hook recursive function-definitions to any well-founded digraph,

even with Ref, whose only built-in form of recursion is $\in$-recursion.
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A labeling technique à la Mostowski enables one to hook recursive function-definitions to any well-founded digraph,

even with Ref, whose only built-in form of recursion is \(\in\)-recursion.

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E.G. Omodeo, J.T. Schwartz.
Theory finAcycLabeling\((v_0, d_0, h(S, X))\)

\[
\begin{align*}
\text{Acyclic}(v_0, d_0) & \land \text{Finite}(v_0) \\
\Rightarrow (\ell_\Theta) \\
\text{Svm}(\ell_\Theta) & \land \text{dom}(\ell_\Theta) = v_0 \\
\langle \forall x \in v_0 \mid \ell_\Theta | x = h \left(\{\ell_\Theta | p^{[2]} : p \in d_0 | \{x\} \mid p^{[2]} \in v_0\}, x\right) \rangle \\
\end{align*}
\]

End finAcycLabeling
Theory finAcycLabeling\((v_0, d_0, h(S, X))\)

\[ \text{Acyclic}(v_0, d_0) \land \text{Finite}(v_0) \]
\[ \implies (\ell_{\Theta}) \]
\[ \text{Svm}(\ell_{\Theta}) \land \text{dom}(\ell_{\Theta}) = v_0 \]
\[ \langle \forall x \in v_0 \mid \ell_{\Theta}\lvert x = h(\{\ell_{\Theta}\lvert p[2] : p \in d_0\lvert \{x\} \mid p[2] \in v_0\}, x) \rangle \]

End finAcycLabeling

Theory finMostowskiDecoration\((v_0, d_0)\)

\[ v_0 \times v_0 \supseteq d_0 \land v_0 \neq \emptyset \land \text{Finite}(v_0) \land \text{Acyclic}(v_0, d_0) \land \text{WExtensional}(v_0, d_0) \]
\[ \implies (m_{\Theta}) \]
\[ 1-1(m_{\Theta}) \land \text{dom}(m_{\Theta}) = v_0 \]
\[ \langle \forall w \mid w \in \text{dom}(d_0) \rightarrow m_{\Theta}\lvert w = \{m_{\Theta}\lvert p[2] : p \in d_0\lvert \{w\}\} \land m_{\Theta}\lvert w \neq \emptyset \rangle \]
\[ \emptyset \in \text{range}(m_{\Theta}) \land \langle \forall y \mid y \in \text{range}(m_{\Theta}) \rightarrow \text{Finite}(y) \rangle \]
\[ \langle \forall y \mid y \in v_0 \rightarrow (m_{\Theta}\lvert y \in m_{\Theta}\lvert x \leftrightarrow [x, y] \in d_0) \rangle \]

End finMostowskiDecoration
When it comes to (undirected) graphs...

For any graph (finite, devoid of self-loops) 
\[ G = (V, E) \]

there is a \( D \subseteq V \times V \) s.t.

- \( E = \{\{x, y\} : [x, y] \in D\} \),
- \((V, D)\) is acyclic and weakly extensional
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This entails a representation theorem based on a bijection

\[ v \mapsto \ell \upharpoonright v \]

s.t.

\[ \{x, y\} \text{ is an edge of } G \text{ iff: either } \ell \upharpoonright x \in \ell \upharpoonright y \text{ or } \ell \upharpoonright y \in \ell \upharpoonright x \]
A graph \((V, E)\) is said to be \textit{claw-free} if none of its subgraphs induced by 4 vertices has the shape of a ‘Y’.
A Favorable Case: Connected Claw-Free Graphs

**Definition**

A graph \((V, E)\) is said to be **claw-free** if none of its subgraphs induced by 4 vertices has the shape of a ‘Y’.

**Figure:** Forbidden claw \(K_{1,3}\)
**Definition**

A graph \((V, E)\) is said to be **claw-free** if none of its subgraphs induced by 4 vertices has the shape of a ‘\(Y\)’

![claw-free graph](image_url)

**Figure:** A claw-free graph
For any *connected* claw-free graph (finite, devoid of self-loops)

\[ G = (V, E) \]

there is a \( D \subseteq V \times V \) s.t.

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- \((V, D)\) is acyclic and *extensional*
For any *connected* claw-free graph (finite, devoid of self-loops)

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- \((V, D)\) is acyclic and *extensional*

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M. Milanič and A. I. Tomescu.
Set graphs. I. Hereditarily finite sets and extensional acyclic orientations.
For any *connected* claw-free graph (finite, devoid of self-loops)

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there is a \( D \subseteq V \times V \) s.t.

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For any connected claw-free graph (finite, devoid of self-loops) $G = (V, E)$ there is a $D \subseteq V \times V$ s.t.
- $E = \{\{x, y\} : [x, y] \in D\}$,
- $(V, D)$ is acyclic and extensional

Now the representing injection $\nu \mapsto \ell|\nu$

is onto a set $\nu_G$ such that $\nu_G \supseteq \bigcup \nu_G$. 
“Every connected claw-free graph admits a perfect matching and has Hamiltonian cycle in its square”.

(1970s / 1980s)
A fully formal reconstruction of the said results on connected claw-free graphs has been achieved by means of Ref.

E. G. Omodeo and A. I. Tomescu.
Set graphs. III. Proof Pearl: Claw-free graphs mirrored into transitive hereditarily finite sets.
DOI: 10.1007/s10817-012-9272-3.
Thank you for your attention!