Regularity properties of $H$-convex sets. (English summary)


In the paper under review, the authors study the first- and second-order regularity properties of the boundary of $H$-convex sets in the setting of a real vector space endowed with a noncommutative group law. Their motivation is the recent theory of $H$-convex functions in Carnot groups. They prove that, locally, the noncharacteristic part of the boundary of $H$-convex sets has the intrinsic cone property and that it is “foliated” by intrinsic Lipschitz continuous curves that are twice differentiable almost everywhere.


Jingzhi Tie

References

A Carnot group is a connected simply connected Lie group whose Lie algebra is nilpotent of step two, graded, and stratified. By making use of normal coordinates of the first kind, any point of a Carnot group is associated with a point in $\mathbb{R}^{l+p}$ and the group law, written in these coordinates, is of the second-order polynomial whose coefficients of the highest order form a skew-symmetric matrix. The graded structure of the Lie algebra proposes two types of left-invariant vector fields: $l$ horizontal ones and $p$ vertical ones.

The authors of the paper propose an exact form of the Taylor polynomial for a function of class $C^m_H$ by making use of a special basis of the vector space of all left-invariant differential operators. The class $C^m_H$ is the class of all functions that have $m$ derivatives with respect to horizontal vector fields and for which these derivatives are continuous. The special basis considered by the authors has nice symmetry properties allowing one to find the exact form of the Taylor polynomial that, in addition, imitates the Euclidean form in the horizontal part. In the paper the particular example of the Heisenberg group $H^1$ is considered in detail, and later the general case of step two Carnot groups is presented. The paper is well structured and will please the reader.

Irina G. Markina

References

Caruso, A. O. (I-CATN-MI); Khan, A. A. [Khan, Akhtar Ali] (1-RIT-SM);
Raciti, F. (1-RIT-ACM)

Continuity results for a class of variational inequalities with applications to
time-dependent network problems. (English summary)

The main result concerns the continuity of the solution map of a monotone variational
inequality where the operator and the set are parametrized by time (Section 2). The
authors apply this result to the study of a time-dependent traffic equilibrium problem
(Section 4) and point out how it can be applied to the study of variational inequalities
defined on a polyhedral set in $\mathbb{R}^n$ (Section 3).

Monica Bianchi

References

Parametric Optimization. Birkhäuser Verlag, Basel. MR0701243 (84i:90147)
2. A. Barbagallo (2007). Regularity results for time-dependent variational and quasi-
variational inequalities and application to the calculation of dynamic traffic network.


It is well known that, while spaces of smooth functions are not dense in BMO (John-Nirenberg’s space of functions with bounded mean oscillation), they are dense in VMO (Sarason’s space of functions with vanishing mean oscillation). Starting with the papers by Chiarenza-Frasca-Longo about nonvariational linear elliptic operators with VMO coefficients, the space VMO has become important in the theory of P.D.E.’s; in particular, its analog version in the abstract framework of spaces of homogeneous type (in the sense of Coifman-Weiss) has been used by several authors in connection with degenerate operators modeled on Hörmander’s vector fields. It is in this context, and in particular in that of linear equations or systems in divergence form [see G. Di Fazio and M. S. Fanciullo, Comm. Appl. Nonlinear Anal. 10 (2003), no. 2, 81–95; MR1992309 (2004h:35025); A. O. Caruso, “Local $S_{X,p}^{1}$ estimates for variational hypoelliptic operators with local VMO$_X$ coefficients”, preprint; per bibl.], that the necessity of proving a density result of smooth functions in VMO, in some spaces of Carnot-Carathéodory type, arose. This task is performed in the present paper.

More precisely, let us consider a system of real smooth Hörmander vector fields, defined in some domain of $\mathbb{R}^n$; the well-known technique of “lifting”, introduced by Rothschild-Stein, makes it possible to define a new system of Hörmander free vector fields, defined in a space $\mathbb{R}^N$ of larger dimension, which project on the original ones when restricted to $\mathbb{R}^n$. Introducing the Carnot-Carathéodory distance induced by these lifted vector fields, one sees that the Lebesgue measure of the metric ball behaves like a fixed power of the radius, uniformly with respect to the center; in other words, the Euclidean space endowed with this distance and the Lebesgue measure turns out to be (locally) an Ahlfors regular space of homogeneous type. In this paper, the authors prove the following density result: given two open domains $\Omega'' \Subset \Omega' \Subset \mathbb{R}^N$, and a function $f \in \text{VMO}(\Omega')$ (where the space VMO is defined with respect to the Carnot-Carathéodory metric described above), there exists a sequence of functions $\{f_k\} \subset C^\infty(\Omega'')$ such that $f_k \to f$ in $\text{BMO}(\Omega'')$ and a.e. in $\Omega''$.

The proof of this result consists in two steps which are performed at different levels of generality. First, the authors prove that in any Ahlfors regular space of homogeneous type, the spaces BMO and VMO, defined by means of metric balls, can be equivalently defined by means of “cubes”. This fact relies on an abstract construction of dyadic cubes in spaces of homogeneous type, previously introduced by G. David and M. Christ. Then, the authors shift to Carnot-Carathéodory spaces shaped on free vector fields; in this context, exploiting in particular the so-called “ball-box theorem” by Nagel-Stein-
Weinger, they prove the density of smooth functions in the space VMO defined by means of cubes.

References

4. Caruso, A. O.: Local $S^1_{p,p}$ estimates for variational hypoelliptic operators with local $VMO$ coefficients. - Preprint.


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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In 1982, G. B. Folland and E. M. Stein [Hardy spaces on homogeneous groups, Princeton Univ. Press, Princeton, N.J., 1982; MR0657581 (84h:43027)] defined the Taylor polynomial for functions \( f \in C^n(\mathbb{G}) \), where \( \mathbb{G} \) is a connected and simply connected real Lie group whose algebra is endowed with a family of nonisotropic dilations. However, despite the extensive efforts made to develop analysis and geometry in this setting, an explicit representation of the \( n \)-th (homogeneous) Taylor polynomial seems to be lacking in the particularly interesting case of Carnot groups.

In this note, starting from the usual Taylor expansion for analytic functions in analytic
groups, the authors provide a way to write explicitly the Taylor polynomial for functions in Carnot groups. For computational simplicity, all results are announced for the most important example of a Carnot group, i.e., the first Heisenberg group $H^1$. The authors are able to exhibit the $n$-th homogeneous Taylor polynomial for $C^n$ (in an intrinsic sense) functions. Applications to a Whitney-type extension theorem for $C^2$ functions in $H^1$ are also provided.

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MR2159975 (2006c:35044) 35H10 42B35 46E15

Caruso, A. O. (I-CATN-MI); Fanciullo, M. S. (I-CATN-MI)

A density result on the space $VMO_\omega$. (English summary)


Summary: “In Carnot-Carathéodory metric spaces related to a family of free Hörmander vector fields $X_1,\ldots,X_q$, we prove that the space $C^\infty$ is locally dense in $VMO_\omega$ with respect to the $BMO_\omega$ norm.”

{For the entire collection see MR2160743 (2006c:49002)}

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MR2107477 (2005g:93011) 93B05 93C20

Caruso, A. [Caruso, Andrea O.] (I-CATN-MI)

On the complete controllability of a distributed parameter nonlinear control system with partially $C^1$ coefficients. (English summary)


In this paper the author considers a distributed control problem governed by a system of quasilinear hyperbolic equations on the rectangular domain $[0,a] \times [0,b]$ with continuous coefficients and a nonlinear term which is Lipschitz continuous in the derivatives of the state. The system is discussed in the space of continuously differentiable functions. The main result is the following exact controllability property: for all boundary conditions $\varphi \in C^{1}([0,a])$ and $\psi \in C^{1}([0,b])$ satisfying the compatibility condition $\varphi(0) = \psi(0)$ (one would also expect the condition $\varphi'(0) = \psi'(0)$) and for all $z \in \mathbb{R}^n$ there exists a continuous control such that the solution $u$ of the system satisfies $u(a,b) = z$. The proof of this result is based on an implicit integral representation of the solution which involves the Riemann function. This allows one to reformulate the control problem as
a fixed point problem. The existence of a fixed point follows from an application of Darbo’s fixed point theorem.

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MR1951735 (2003m:46006) 46A03 46B20
Caruso, A. [Caruso, Andrea O.] (I-CATN-MI)
A short remark on Kolmogoroff normability theorem. (English summary)

Summary: “The Kolmogorov normability theorem turns out to be a characterization for the complete normability of a topological vector space by replacing the convexity hypothesis with the σ-convexity one. In particular, the well-known theorem that characterizes completeness of a normed vector space by means of an absolutely convergent series is obtained as an easy consequence of the theorem given in the present paper.”

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MR1997734 (2004g:46043) 46E30
Caruso, Andrea (I-CATN-MI)
Two properties of norms in Orlicz spaces. (English summary)

In this paper a well-known characterization of the inclusion between $L^p$-spaces over a finite measure space (i.e., $L^p$-norms tend to the $L^\infty$-norm as $p \to +\infty$) is extended to the class of Orlicz spaces generated by some Young functions.

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