# Continuity results for a class of variational inequalities with applications to time dependent network problems

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**Abstract**

We consider a class of monotone variational inequalities in which both the operator and the convex set are parametrized by continuous functions. Under suitable assumptions, we prove the continuity of the solution with respect to the parameter. As an important application, we consider the case of finite dimensional variational inequalities on suitable polyhedra. We demonstrate the applicability of our results to time dependent traffic equilibrium problem.

**Keywords:** Variational inequalities; Mosco and Kuratowski convergence; monotonicity; network equilibrium problems.

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## 1 Introduction

Many problems of pure and applied mathematics can be formulated within the framework of variational inequalities. In particular, the connection with partial differential equations has been investigated by G. Stampacchia who also provided many applications to physical sciences such as elasticity problems, fluid flow through porous media, and others (see Kinderlehrer and

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Stampacchia [12]). On the other hand, in the last two decades, a growing number of equilibrium problems, arising from various fields of applied sciences, such as game theory, economic theory and traffic networks (see Giannessi and Maugeri [9], and Nagurney [15]) have been formulated within the framework of finite dimensional variational inequalities (see Facchinei and Pang [7]). However, the most of the above mentioned equilibrium problems can be considered as steady state approximations of the time dependent analogues. Therefore, the question that arises is how to introduce time within their variational inequality formulation. The classical approach to introduce time in finite dimensional setting is to consider parametric variational inequalities. Regularity results close to ours in the framework of either parametric programming or parametric variational inequalities have been obtained by different techniques. For instance, Robinson [18] by using the concept of locally upper Lipschitz function, obtained two continuity results for particular polyhedral multifunctions. On the other hand, Yen [24] proved the continuity of the solution of a variational inequality as a byproduct of Hölder continuity. Recently Daniele et al. [6] proposed a new approach that is based on expressing a time dependent problem as a variational inequality in $L^p$ spaces. Then the connection with some pointwise (in time) equilibrium condition is made and the problem is formulated as a parametric variational inequality. Although the issue of parametric sensitivity analysis has been studied extensively, a general theory of regularity for the solutions of time dependent variational inequalities as mentioned above is not available to the best of our knowledge. In a very recent paper Barbagallo [2] studied the time dependent traffic equilibrium problem, and, under some regularity assumptions on the data and the use of set convergence, proved the continuity of the solution to the related (linear and strongly monotone) variational inequality. In this note we extend the main result of the above mentioned paper in several directions. First of all, our study is performed within the functional setting of a reflexive Banach space (instead of $\mathbb{R}^n$) for $p$–monotone operators. Moreover, we do not require the Lipschitz continuity condition. Then, in the finite dimensional setting, we generalize the Mosco convergence result of [2] to the case of a family of polyhedra parametrized by suitable continuous functions. This makes it possible to study the continuity for all $p$-monotone time-dependent equilibrium problems with linear constraints, and hence our results are not limited only to the traffic equilibrium problem. Evidently, once the continuity of the solution has been established, one can choose among the multitude of algorithms available in the literature. Since we deal mainly with strongly monotone operators, we can refer, for example, to Taji et al. [23] for a relatively recent algorithm for strongly monotone variational inequalities. For the traffic problem, specific algorithms are also available, see

Fukushima [8] and Raciti and Falsaperla [17]. We verify our hypothesis for concrete network problems such as the traffic equilibrium problem. In fact, network problems admit two (in general not equivalent) formulations corresponding to the two ways of representing the flows on the underlying graph. As a matter of fact, in the formulation used in [2], the strong monotonicity assumption is not verified in general. Therefore, to circumvent this problem, one must switch to the other formulation and then adapt the Mosco convergence result to the new framework. We also point out that since our proof (as well as the proof in [2]) is based on the pointwise (i.e. parametric) variational inequality, there is no need of introducing the integral formulation. We refer the reader to Daniele et al. [6] (see also Gwinner and Raciti [10]) for the connections between the two formulations.

# 2 Continuity results

Let $X$ be a real reflexive Banach space with topological dual $X^*$, let $K \subseteq S$ be a subset of a metric space $S$, and let $K \times X \ni (t, x) \to A(t, x) \in X^*$ be a given map. The natural duality pairing between $X$ and $X^*$ will be denoted by $\langle \cdot, \cdot \rangle$, while the norms in $X$ and $X^*$ will be denoted by $\| \cdot \|_X$ and $\| \cdot \|_{X^*}$. We shall denote the weak and the strong convergence in $X$ respectively by "$\rightharpoonup$" and "$\rightarrow$". Moreover, we shall denote by $C_n \stackrel{K}{\rightarrow} C$ and $C_n \stackrel{M}{\rightarrow} C$, the Kuratowski–Painlevé and the Mosco convergence of sets (for the definitions and basic results see Section 3). Finally, we will speak about the weak (resp. strong) topology on $S \times X$ when endowing this product with the metric topology on $S$ times the weak (resp. strong) one on $X$.
We assume that for any $t \in K$, the map $A(t, \cdot)$ is hemicontinuous and monotone. Recall that, for any $t \in K$, $A(t, \cdot)$ is said to be hemicontinuous, if the map defined by the condition

$$\varphi(s) = \langle A(t, x + sy), y \rangle, \quad \forall\, x, y \in X, \forall s \in \mathbb{R},$$

is continuous. Moreover, for any $t \in K$, $A(t, \cdot)$ is said to be monotone, if

$$\langle A(t, x) - A(t, y), x - y \rangle \geq 0, \quad \forall\, x, y \in X.$$

For any $t \in K$, let $C(t) \subseteq X$ be a given non empty closed convex set. For $t \in K$, if the set $C(t)$ is unbounded, we assume that the operator $A(t, \cdot)$ is coercive on $C(t)$. That is, there exists an element $\varphi \in C(t)$ such that

$$\lim_{\|x\|_X \to +\infty} \frac{\langle A(t, x) - A(t, \varphi), x - \varphi \rangle}{\|x - \varphi\|_X} = +\infty.$$

For any fixed $t \in K$, consider the following variational inequality: find $x(t) \in C(t)$ such that

$$\langle A(t, x(t)), y - x(t) \rangle \geq 0, \quad \forall y \in C(t) \tag{1}$$

Under the above assumptions (1) admits at least one solution (see [12]). If we assume that for any $t \in K$ there exists a unique solution $x(t) \in C(t)$ of (1), then it makes sense to study the continuity of the map $K \ni t \to x(t) \in X$.

For our main continuity result, we introduce the following conditions:

(1) $A : K \times X \to X^*$ is completely continuous;

(2) $A(t, \cdot)$ is $p$–monotone (strongly monotone if $p = 2$), uniformly with respect to $t \in K$, i.e. there exists $\alpha > 0$ and $p \geq 2$ such that

$$\langle A(t, x) - A(t, y), x - y \rangle \geq \alpha \|x - y\|_X^p \quad \forall t \in K, \ \forall x, y \in X.$$

(3) for any $\{t_n\}_{n \in \mathbb{N}} \subseteq K$ and any $t \in K$ such that $t_n \to t$, then $C(t_n) \stackrel{M}{\to} C(t)$, i.e.:

   (i) for any subsequence $\{k_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that $v_{k_n} \in C(t_{k_n})$ and $v_{k_n} \rightharpoonup v$ for some $v \in X$, then $v \in C(t)$;

   (ii) for any $v \in C(t)$ there exists a sequence $v_n$ such that $v_n \in C(t_n)$ for $n$ large enough and, moreover, $v_n \to v$.

The following is the main result of this section.

**Theorem 2.1** *Under assumption* (1),(2), *and* (3), *the solution map of* (1) $K \ni t \to x(t) \in X$ *is (metric to strong) continuous on* $K$.

**Proof.** For $t \in K$, it suffices to verify that for any $\{t_n\}_{n \in \mathbb{N}} \subseteq K$ such that $t_n \to t$, we get $x(t_n) \to x(t)$. Under our hypothesis, the Minty lemma (see for instance [12]) holds, that is, for any $s \in K$ we have

$$\langle A(s, y), y - x(s) \rangle \geq 0, \quad \forall y \in C(s).$$

Due to (ii), for a given $x(t) \in C(t)$, there exists a sequence $v_n$ such that $v_n \in C(t_n)$ for $n$ large enough, and $v_n \to x(t)$. Consequently, the continuity hypothesis confirms that $A(t_n, v_n) \to A(t, x(t))$. Setting, for $n$ large enough, $v = v_n$ in (1), we have

$$\langle A(t_n, x(t_n)), v_n - x(t_n) \rangle \geq 0.$$

From (2), we obtain

$$\alpha \|x(t_n)-v_n\|_X^p \leq \langle A(t_n, x(t_n))-A(t_n, v_n), x(t_n)-v_n\rangle \leq \|A(t_n, v_n)\|_{X^*} \|x(t_n)-v_n\|_X,$$

implying

$$\|x(t_n) - v_n\|_X \leq \alpha^{\frac{1}{1-p}} \|A(t_n, v_n)\|_{X^*}^{\frac{1}{p-1}}.$$

It follows that

$$\|x(t_n)\|_X \leq \|x(t_n) - v_n\|_X + \|v_n\|_X \leq \alpha^{\frac{1}{1-p}} \|A(t_n, v_n)\|_{X^*}^{\frac{1}{p-1}} + \|v_n\|_X.$$

The above inequality ensures that $\{x(t_n)\}_{n\in\mathbb{N}}$ is bounded. Let $\{x'(t_n)\}_{n\in\mathbb{N}}$ be a subsequence of $\{x(t_n)\}_{n\in\mathbb{N}}$. Since $X$ is reflexive, there are $v \in X$, and $\{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $x'(t_{k_n}) \in C(t_{k_n})$, $x'(t_{k_n}) \rightharpoonup v$. Because of (i), we obtain $v \in C(t)$. We claim that $v = x(t)$. Applying again the Minty lemma to any $x'(t_{k_n})$, we obtain

$$\langle A(t_{k_n}, y), y - x'(t_{k_n})\rangle \geq 0 \quad \forall y \in C(t_{k_n}).$$

In view of (ii), for any $y \in C(t)$, there exists $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \in C(t_n)$ for $n$ large enough, and $y_n \to y$. Therefore,

$$\langle A(t_{k_n}, y_{k_n}), y_{k_n} - x'(t_{k_n})\rangle \geq 0.$$

Passing the above inequality to limit $n \to \infty$, we obtain

$$\langle A(t, y), y - v\rangle \geq 0, \quad \forall y \in C(t).$$

Applying Minty's lemma once again, we obtain

$$\langle A(t, v), y - v\rangle \geq 0, \quad \forall y \in C(t).$$

By the uniqueness of the solution, it follows that $v = x(t)$, and $x'(t_{k_n}) \rightharpoonup x(t)$. The following inequality

$$\alpha \|x'(t_{k_n}) - v_{k_n}\|_X^p \leq \langle A(t_{k_n}, v_{k_n}), v_{k_n} - x'(t_{k_n})\rangle,$$

in view of the continuity of $A$, and the fact that $(x'(t_{k_n}) - v_{k_n}) \rightharpoonup 0$, completes the proof. □

# 3 Applications to parametric variational inequalities

In this section we study conditions that ensure the stability of Mosco convergence through a given mapping and give a convergence result for a family of polyhedra parametrized by suitable continuous functions. In the case when the parameter represents the time variable, our result can be applied to the time dependent equilibrium problems modeled by variational inequalities with time dependent linear constraints.

## 3.1 Preliminaries on set-convergence

We begin by recalling two notions of convergence for subsets of a metric space $(X, d)$. The first one was introduced in the fifties by Kuratowski [13] (see also [19, 20]), and the second one was introduced in the sixties by Mosco [14] (see also [3] and [11], and the references therein).

First we fix some notations. Let $X$ be a nonempty set endowed with two topologies $\sigma \subseteq \tau$. In this paper we will only deal with nets indexed on numerable set, i.e. with sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq X$. Therefore, in this section we will omit the word *sequentially* when speaking about convergence. We also remark that whenever $X$ denotes a normed space, the underlying field will be $\mathbb{R}$.

Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of $X$. Recall that

$$\begin{aligned}\tau\text{–}\underline{\lim}_n C_n &= \big\{x \in X : \exists\, \{x_n\}_{n\in\mathbb{N}} \text{ eventually in } C_n \text{ such that } x_n \xrightarrow{\tau} x\big\},\\ \sigma\text{–}\overline{\lim}_n C_n &= \big\{x \in X : \exists\, \{x_n\}_{n\in\mathbb{N}} \text{ frequently in } C_n \text{ such that } x_n \xrightarrow{\sigma} x\big\},\end{aligned}$$

where *eventually* means that there exists $\nu \in \mathbb{N}$ such that $x_n \in C_n$ for any $n \geq \nu$, and *frequently* means that there exists an infinite subset $N \subseteq \mathbb{N}$ such that $x_n \in C_n$ for any $n \in N$. We say that $C_n$ $(\sigma,\tau)$–converges to some subset $C \subseteq X$, and we briefly write $C_n \stackrel{(\sigma,\tau)}{\to} C$, if $\tau\text{–}\underline{\lim}_n C_n = \sigma\text{–}\overline{\lim}_n C_n = C$. Thus, in order to verify that $C_n \stackrel{(\sigma,\tau)}{\to} C$, it suffices to check that

(i) $\sigma\text{–}\overline{\lim}_n C_n \subseteq C$, i.e. for any sequence $\{x_n\}_{n\in\mathbb{N}}$ frequently in $C_n$ such that $x_n \xrightarrow{\sigma} x$ for some $x \in S$, then $x \in C$;

(ii) $C \subseteq \tau\text{–}\underline{\lim}_n C_n$, i.e. for any $x \in C$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ eventually in $C_n$ such that $x_n \xrightarrow{\tau} x$.

We now recall the Kuratowski and Mosco convergence of sets. Let $(X, d)$ be a metric space such that $\sigma = \tau = \tau_d$ is the metric topology. In this case the $(\sigma,\tau)$–convergence is called Kuratowski convergence of sets, denoted by $C_n \xrightarrow{K} C$.

Let $X$ be a normed space, and let $\sigma$ and $\tau$, respectively, be the weak and the strong topology on $X$. In this case the $(\sigma,\tau)$–convergence is called Mosco convergence of sets, denoted by $C_n \xrightarrow{M} C$.

Notice that if $X$ is a finite dimensional vector space then Kuratowski convergence and Mosco convergence coincide. In this case we will simply write $C_n \to C$, as well as $\underline{\lim}_n C_n$ and $\overline{\lim}_n C_n$.

**Remark 3.1 (stability of Mosco convergence under mappings)** Let $X$ be a topological space endowed with two topologies $\sigma \subseteq \tau$, and let $S$ be a metric space endowed with the metric topology $\tau_d$. Let $F : X \to S$ be a given mapping. We are interested in whether $F(C_n) \stackrel{K}{\to} F(C)$, whenever $C_n \stackrel{(\sigma,\tau)}{\to} C$. If we assume that $F$ is ($\tau$ to $\tau_d$) continuous, then it is easy to verify that two natural conditions ensuring the Mosco convergence for the image sequence are the following: either $F$ is a ($\sigma$ to $\tau_d$) sequentially proper mapping, or the sequence $C_n$ is eventually contained in a (with respect to $\sigma$) sequentially compact set. This is the case of the traffic equilibrium problem discussed in Section 4 (actually in the example proposed there it is immediately verified that all $C_n$ are uniformly bounded). Recall that a given continuous mapping $F : X \to S$ is said to be ($\sigma$ to $\tau_d$) sequentially proper, if $F^{-1}(K) \subseteq X$ is sequentially compact, for any compact set $K \subseteq S$.

A particular class of proper mappings is given by the completely continuous, one to one, linear mappings defined on a reflexive Banach space. Indeed, let $X$ and $Y$ be two reflexive Banach spaces, and let $L : X \to Y$ be completely continuous, one to one, and linear. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be such that $F(x_n)$ converges strongly to some $y \in Y$. Writing $x_n = a_n\,\hat{x}_n$, where the hat denotes the unit vector and $a_n = \|x_n\|_X$, let $\{\hat{x}_{k_n}\}_{n\in\mathbb{N}} \subseteq \{\hat{x}_n\}_{n\in\mathbb{N}}$ and $\hat{x} \in X$ be such that $\hat{x}_{k_n} \rightharpoonup \hat{x}$. Therefore, from $\|F(x_{k_n})\|_Y = a_{k_n}\|F(\hat{x}_{k_n})\|_Y$, letting $x = \frac{\|y\|_Y}{\|F(\hat{x})\|_Y}\hat{x}$ we get $x_{k_n} \rightharpoonup x$.

The following example show that the hypothesis stated in the above remark is essential.

**Example 3.1** Let $C_n, C \subset\mathbb{R}^2$ be the unbounded polyhedra defined by $C_n = \big\{(x,y) \in \mathbb{R}^2 : (n+1)x - ny = 0\big\}$, for any $n \in \mathbb{N}$, and $C = \big\{(x,y) \in \mathbb{R}^2 : x - y = 0\big\}$. Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a surjective linear mapping defined by $L(x,y) = y - x$, for any $(x,y) \in \mathbb{R}^2$. Then, $C_n \to C$, however, for any $n \in \mathbb{N}$, we have $L(C_n) = \mathbb{R} \to \mathbb{R} \neq L(C)$ because $\mathrm{Ker}(L) = C$.

Finally, according to the second sufficient condition, we recall that when the space $X$ is a finite dimensional normed space, the eventual uniform boundedness is ensured by Proposition 3 of [19], whenever $C_n$ is connected and closed (at least for $n \in \mathbb{N}$ large enough) and $C$ is compact. Nevertheless, if $X$ is not finite dimensional this proposition is no longer true. Consider for instance, on $X = l_2$, any arbitrary sequence of subset in $\mathbb{R}$, say $\{B_n\}_{n\in\mathbb{N}}$, each of them containing 0. Moreover, let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of $l_2$. Define $C_n = B_n e_n \subset l_2$, for any $n \in \mathbb{N}$. Then clearly $C_n \stackrel{M}{\to} \{0\}$, but on the sequence $C_n$ nothing can evidently be proved in general.

### 3.2 A convergence result for polyhedra in $\mathbb{R}^N$

Now we present a convergence result for suitable sequences of polyhedra in $\mathbb{R}^N$. A convergence result of similar type can also be found in [1], where very different techniques have been employed. Let $m$ be a positive integer, and let $\{a_{ij}(t)\}_{\substack{i=1,\dots,m \\ j=1,\dots,N}}$ and $\{b_i(t)\}_{i=1,\dots,m}$ be continuous functions defined on some subset $K$ of a metric space $S$. For any $i = 1, \dots, m$ and for any $t \in K$, define $a_i(t) = (a_{i1}(t), \dots, a_{iN}(t))$. We shall denote by $P_t$ the polyhedron in $\mathbb{R}^N$ defined by the following system of linear inequalities

$$\langle a_i(t), x\rangle \leq b_i(t), \qquad i = 1, \dots, m. \tag{2}$$

Let $\{t_n\} \subseteq K$ be a sequence converging to some fixed $t \in K$. We denote by $P_n$ and $P$ the polyhedra $P_{t_n}$ and $P_t$, respectively.

We remark that the continuity of coefficients $a_{ij}$ and $b_j$ is not enough to avoid undesirable situations. By means of the following example, we show that the assumption $P \neq \emptyset$ does not, in general, imply $P_n \neq \emptyset$ for $n$ large enough.

**Example 3.2** Consider $P = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and, for any $t \neq 0$, define $P_t = \{(x, y) \in \mathbb{R}^2 : y + t \leq 0, -y + t \leq 0\}$. Then, by taking the limit of the coefficients in $P_{1/n}$ we obtain the polyhedron $P$ which obviously is not the Kuratowski limit of $P_{1/n}$.

The following example also shows that the continuity of the coefficients does not imply that $P_{t_n}$ converges in Kuratowski sense.

**Example 3.3** Consider $P = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and, for any $t \neq 0$, define $P_t = \{(x, y) \in \mathbb{R}^2 : y + tx \leq 0,\ -y \leq 0\}$. Since a point of $P$ with $x > 0$ can not be the limit of a sequence of points of $P_{1/n}$, it follows that $P$ can not be the Kuratowski limit of $P_{1/n}$.

In order to avoid situations depicted in the above examples, it suffices to assume that the combinatorial structure of the parametrized polyhedra does not change in a neighborhood of $t$. The following is the main result of this section.

**Theorem 3.1 (Kuratowski convergence for polyhedra)** *Let $P_n$ and $P$ be polyhedra as defined above. Assume that for any distinct $i_1, \dots, i_k \in \{1, \dots, m\}$, the matrices formed with the vector functions $\{a_i(s)\}_{i=i_1,\dots,i_k}$ are of constant ranks in some neighborhood of $t$. Then $P_n \to P$.*

**Proof.** We will prove (i) and (ii) of page 6. For (i), it suffices to show that for any $\{x_n\}_{n\in\mathbb{N}}$, frequently in $P_n$, and converging to some $x \in \mathbb{R}^N$, one gets that $x \in P$. From $\langle a_i(t_n), x_n\rangle \le b_i(t_n)$, by taking the limit, we obtain $\langle a_i(t), x\rangle \le b_i(t)$, which implies that $x \in P$.

We proceed to prove (ii). Without loss of generality, we suppose that $P_t$ has a nonempty interior. Consider first an element $x$ that belongs to the interior of $P$, that is,

$$\langle a_i(t), x\rangle < b_i(t), \quad \text{for all } i \in \{1, 2, \dots, m\}.$$

If for some $i$, it were frequently $\langle a_i(t_n), x\rangle \ge b_i(t_n)$, then letting $n \to \infty$, we easily obtain a contradiction. Hence, it is eventually

$$\langle a_i(t_n), x\rangle < b_i(t_n), \quad \text{for all } i \in \{1, 2, \dots, m\},$$

and, therefore, we can choose $x_n = x$ for all $n \in N$.

Let $I = \{1, 2, \dots, m\}$, and let $I'$ be a subset of $I$. To each $I'$, there is a face $F_{I'}(t)$ of $P_t$, given by

$$F_{I'}(t) = \begin{cases} A_{I'}(t) & = & b_{I'}(t) \\ A_{I\setminus I'}(t) & \le & b_{I\setminus I'}(t) \end{cases}. \tag{3}$$

Suppose that the face $F_{I'}(t)$ is nonempty, and suppose that the dimension $\ell$ of $F_{I'}(t)$ is less than $N$. Due to the hypothesis, there exists $n_0$ such that $\dim(F_{I'}(t_n)) = \ell$ for all $n > n_0$. Without loss of generality, we also assume that $\mathrm{card}(I') \le N$.

Let $x \in F_{I'}(t)$, where $F_{I'}(t)$ is the face of minimal dimension that contains $x$. Let, for each $n$, $x_n$ be the orthogonal projection of $x$ on the minimal affine manifold $M_n$ which contains $F_{I'}(t_n)$. By employing constructive argument we will show that $x_n \to x$. To compute $x_n$ we can exploit the fact that $x_n$ is the intersection of $M_n$ and the linear affine manifold passing through $x$ and orthogonal to $M_n$.

Without loss of generality, we can assume that the equations in (3) are such that vector functions $\{a_i(s)\}_{i\in I'}$ are linearly independent in some neighborhood of $t$. Let the cardinality of $I'$ be $k$ with $k \le N$.

Let $n$ be a sufficiently large integer. We shall construct two bases $\{v_{1,n}, \dots, v_{N-k,n}\}$ of $M_n$, and $\{v_1, \dots, v_{N-k}\}$ of $M$ such that $v_{h,n} \to v_h$, for any $h = 1, \dots, N-k$.

For this, we need to fix some notations. We set

$$\begin{aligned} A_n &= \big(a_{ij}(t_n)\big)_{\substack{i=1,\dots,k \\ j=1,\dots,N}} \\ A &= \big(a_{ij}(t)\big)_{\substack{i=1,\dots,k \\ j=1,\dots,N}} \end{aligned}$$

Moreover, for any set $J = \{j_1, \dots, j_k\} \subseteq \{1, \dots, N\}$ with $1 \leq j_1 < \dots < j_k \leq N$, assume that $J' = \{1, \dots, N\} \setminus J$ is the complementary ordered set $J' = \{j'_1, \dots, j'_{N-k}\}$. Denote by $A_{J_{r,s},n}$ and $A_{J_r,B,n}$ the minors obtained from $A_{J,n} = \big(a_{ij_l}(t_n)\big)_{\substack{i=1,\dots,k \\ l=1,\dots,k}}$ by replacing $a_{ij_r}(t_n)$ respectively with $a_{ij'_s}(t_n)$ and $b_i(t_n)$, for some $r = 1, \dots, k$, some $s = 1, \dots, N-k$ and for any $i = 1, \dots, k$. We will adopt the same notation when referring to analogous minors obtained by the matrix $A$, just by suppressing the index $n$.

From $\operatorname{rank}(A) = k$ it follows that there exist an ordered set $J = \{j_1, \dots, j_k\}$ such that $(a_{1j_l}(t), \dots, a_{kj_l}(t))$, $l = 1, \dots, k$. Consequently $(a_{1j_l}(t_n), \dots, a_{kj_l}(t_n))$, $l = 1, \dots, k$, are eventually linearly independent. In particular, for any $v = (v^1, \dots, v^N) \in \mathbb{R}^N$, we can write

$$v \in M_n \iff \sum_{j \in J} a_{ij}(t_n)\, v^j = b_i(t_n) - \sum_{j' \in J'} a_{ij'}(t_n)\, v^{j'}, \;\; i = 1, \dots, k.$$

We have an analogous form for $v \in M$. Therefore, by the Cramer's rule, we get

$$v^{j_l} = \frac{|A_{J_l,B,n}| - \sum_{m=1}^{N-k} v^{j'_m} |A_{J_{l,m},n}|}{|A_{J_n}|}, \qquad l = 1, \dots, k,$$

and a similar form by replacing $A_n$ with $A$.

For any $l = 1, \dots, N-k$, by choosing

$$v^{j'_1} = \delta^l_1, \dots, v^{j'_{N-k}} = \delta^l_{N-k}\,,$$

we obtain the bases $\{v_{1,n}, \dots, v_{N-k,n}\}$ of $M_n$, and $\{v_1, \dots, v_{N-k}\}$ of $M$.

Due to the construction, we get $v_{h,n} \to v_h$ for any $h = 1, \dots, N-k$. With an abuse of notation, for $v = (v^1, \dots, v^N) \in \mathbb{R}^N$, we write

$$v = (\underbrace{v^{j_1}, \dots, v^{j_k}}_{J-\text{tuple}}) \cup (\underbrace{v^{j'_1}, \dots, v^{j'_{N-k}}}_{J'-\text{tuple}}).$$

Then we choose the bases $\{v_{1,n}, \dots, v_{N-k,n}\}$ as follows

$$v_{1,n} = (\underbrace{|A_{J_1,B,n}| - |A_{J_{1,1},n}|, \dots, |A_{J_k,B,n}| - |A_{J_{k,1},n}|}_{J-\text{tuple}}) \cup (\underbrace{|A_{J_n}|, \dots, 0}_{J'-\text{tuple}})\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$v_{N-k,n} = (\underbrace{|A_{J_1,B,n}| - |A_{J_{1,N-k},n}|, \dots, |A_{J_k,B,n}| - |A_{J_{k,N-k},n}|}_{J-\text{tuple}}) \cup (\underbrace{0, \dots, |A_{J_n}|}_{J'-\text{tuple}})\,;$$

Similar considerations hold for $\{v_1, \ldots, v_{N-k}\}$. It follows that the point $x_n$ is the unique solution of the system

$$\begin{cases} \langle a_i(t_n), v\rangle = b_i(t_n), & i = 1, \ldots, k \\ \langle w_h, v\rangle = \langle w_h, x\rangle, & h = 1, \ldots, N-k \end{cases}, \tag{$S_n$}$$

where

$$w_{1,n} = (\underbrace{|A_{J_{1,1},n}|, \ldots, |A_{J_{k,1},n}|}_{J-\text{tuple}}) \cup (\underbrace{-|A_{J_n}|, \ldots, 0}_{J'-\text{tuple}}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$w_{N-k,n} = (\underbrace{|A_{J_{1,N-k},n}|, \ldots, |A_{J_{k,N-k},n}|}_{J-\text{tuple}}) \cup (\underbrace{0, \ldots, -|A_{J_n}|}_{J'-\text{tuple}}).$$

Notice that the system $(S_n)$, as well the one, say $(S)$, associated to $A$, is of Cramer type due to the orthogonality construction.

By setting

$$(D_n|E_n) = \left(\begin{array}{ccc|c} a_{11}(t_n) & \ldots & a_{1N}(t_n) & b_1(t_n) \\ \ldots & \ldots & \ldots & \ldots \\ a_{k1}(t_n) & \ldots & a_{kN}(t_n) & b_k(t_n) \\ w^1_{1,n} & \ldots & w^N_{1,n} & \langle w_{1,n}, x\rangle \\ \ldots & \ldots & \ldots & \ldots \\ w^1_{N-k,n} & \ldots & w^N_{N-k,n} & \langle w_{N-k,n}, x\rangle \end{array}\right),$$

we get

$$\lim_n \det(D_n) = D \neq 0.$$

Therefore, eventually $\det(D_n) \neq 0$, with evident meaning of notation.

It remains to verify that $x_n \to x$ in $\mathbb{R}^N$. For this, it suffices to observe that by the construction, we also have $w_{h,n} \to w_h$, for any $h = 1, \ldots, N-k$, with evident meaning of notation.

Thus, for any arbitrarily fixed $j \in \{1, \ldots, N\}$, we obtain

$$x^j_n = \frac{\begin{vmatrix} a_{11}(t_n) & \ldots & b_1(t_n) & \ldots & a_{1N}(t_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k1}(t_n) & \ldots & b_k(t_n) & \ldots & a_{kN}(t_n) \\ w^1_{1,n} & \ldots & \langle w_{1,n}, x\rangle & \ldots & w^N_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ w^1_{N-k,n} & \ldots & \langle w_{N-k,n}, x\rangle & \ldots & w^N_{N-k,n} \end{vmatrix}}{|\det(D_n)|} \to x^j.$$

We still have to show that the points $x_n$ are eventually feasible, that is, also the inequalities are satisfied. Assume by contradiction that for some $i \in I \setminus I'$, it results frequently

$$\langle a_i(t_n), x_n \rangle > b_i(t_n).$$

Letting $n \to \infty$, we obtain

$$\langle a_i(t), x \rangle \geq b_i(t),$$

for such $i$, that is evidently false. □

# 4 The time dependent traffic equilibrium problem

In this section we discuss the applicability of our results to network equilibrium problems by considering as a paradigmatic example the celebrated traffic equilibrium problem. A common characteristic of these problems is that they admit two different formulations based either on link variables or on path variables. These two formulations are actually related to each other by a linear transformation. We remark that these two approaches are not equivalent in general. Moreover, in the path approach, which is used for instance in [2], the strong monotonicity assumption is not reasonable. However, due to a remark at page 6 of Section 2, we are able to fill this gap by working in the link-group of variables. To be more precise, we need first some preliminary notations commonly used to state the standard traffic equilibrium problem from the user's point of view in the stationary case (see [22, 5, 16]).

Let $p, n, m, k \in \mathbb{N}$, $p, n, m, k \geq 1$. A traffic network consists of a triple $(N, A, W)$, where $N = \{N_1, \ldots, N_p\}$ is the set of nodes, $A = (A_1, \ldots, A_n)$ represents the set of the directed arcs connecting couples of nodes and $W = \{W_1, \ldots, W_m\} \subset N \times N$ is the set of the origin–destination $(O, D)$ pairs. The flow on the arc $A_i$ is denoted by $f_i$, $f = (f_1, \ldots, f_n)$. For the sake of simplicity we shall consider arcs with infinite capacity. We call a set of consecutive arcs a path, and assume that each $(O_j, D_j)$ pair $W_j$ is connected by $r_j$, $r_j \in \mathbb{N}$, $r_j \neq 0$, paths whose set is denoted by $P_j$, $j = 1, \ldots, m$. All the paths in the network are grouped in a vector $(R_1, \ldots, R_k)$. We can describe the arc structure of the paths by using the arc–path incidence matrix $\Delta = \left(\delta_{ir}\right)_{\substack{i=1,\ldots,n \\ r=1,\ldots,k}}$, whose entries take the value

$$\delta_{ir} = \begin{cases} 1 & \text{if } A_i \in R_r \\ 0 & \text{if } A_i \notin R_r \end{cases}.$$

To each path $R_r$ there corresponds a flow $F_r$. The path flows are grouped in a vector $(F_1, \dots, F_k)$ which is called the path (network) flow. The flow $f_i$ on the arc $A_i$ is equal to the sum of the flows on the paths which contain $A_i$, so that $f = \Delta F$. Let us now introduce the unit cost of going through $A_i$ as a real function $c_i(f) \geq 0$ of the flows on the network, so that $c(f) = (c_1(f), \dots, c_n(f))$ denotes the arc cost vector on the network. Analogously one can define a cost on the paths as $C(F) = (C_1(F), \dots, C_k(F))$. Usually $C_r(F)$ is just the sum of the costs on the arcs which build that path: $C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f)$ or in compact form,

$$C(F) = (\Delta^T c \, \Delta)(F). \tag{C}$$

For each pair $W_j$ there is a given traffic demand $D_j \geq 0$, so that $(D_1, \dots, D_m)$ is the demand vector. Feasible flows are nonnegative flows which satisfy the demands, i.e. which belong to the set

$$K = \big\{ F \in \mathbb{R}^k : \ F_r \geq 0 \text{ for any } r = 1, \dots, k \text{ and } \Phi F = D \big\},$$

where $\Phi$ is the pair–path incidence matrix whose elements, say $\varphi_{jr}$, $j = 1, \dots, m$, $r = 1, \dots, k$, are

$$\varphi_{jr} = \begin{cases} 1 & \text{if the path } R_r \text{ connects the pair } W_j \\ 0 & \text{elsewhere} \end{cases}.$$

A path flow $H$ is called an equilibrium flow, or *Wardrop Equilibrium*, if $H \in K$ and for any $W_j \in W$ and any $R_q, R_s \in P_j$ there holds

$$C_q(H) < C_s(H) \quad \Longrightarrow \quad H_s = 0. \tag{E}$$

This statement is equivalent to (see [5, 22]) to

$$H \in K \quad \text{and} \quad [C(H)]^T (F - H) \geq 0, \quad \forall F \in K. \tag{EVI}$$

Roughly speaking, the meaning of Wardrop Equilibrium is that the road users choose minimum cost paths, and the meaning of the cost is usually that of travel time. Let us notice that condition (E) implies that all the used paths of a given pair have the same cost.

Although the Wardrop equilibrium principle is expressed in the path variables, it is clear that the *physical* (and measured) quantities are expressed in the link variables. Moreover, the strong monotonicity hypothesis on $c(f)$ is quite common, but as noticed for instance in [4] this does not imply the strong monotonicity of $C(F)$ in (C), unless the matrix $\Delta^T \Delta$ is nonsingular. Although one can give a procedure for buildings networks preserving the strong monotonicity property (see [17]), the condition fails for a generic network, even for a very simple one as we shall illustrate in the sequel. Thus, it is useful to consider the following variational inequality problem:

$$\text{Find } f^* \in \Delta K \text{ such that } [c(f^*)]^T (f^* - f) \geq 0 \quad \forall f \in \Delta K. \tag{EVI'}$$

If $c$ is strongly monotone each solution $H$ of (EVI) is such that $f^* = \Delta H$ and one can prove that for each solution $H$ of EVI, $C(H)$ =const., i.e. the (possibly) infinite solutions of EVI share the same cost. From an algorithmic point of view, it is worth noting that one advantage in working in the path variables is the simplicity of the corresponding convex set but the price to be paid is that, the number of paths grows exponentially with the size of the network.

Now, it is clear that the assumption that the traffic demand is, in general, time dependent, as well as the cost functions, so one can consider the dynamic version of (EVI) and (EVI'):

$$H(t) \in K(t) \quad \text{ and } [C(t, H(t))]^T (F(t) - H(t)) \geq 0, \quad \forall F(t) \in K(t), \tag{DVI}$$

where, for any $t \in [0, T]$,

$$K(t) = \big\{ F(t) \in \mathbb{R}^k : F_r \geq 0 \text{ for any } r = 1, \dots, k \text{ and } \Phi F = D(t) \big\},$$

and we look for $f^*(t) \in \Delta K(t)$ such that

$$[c(t, f^*(t))]^T (f(t) - f^*(t)) \geq 0, \quad \forall f(t) \in \Delta K(t) \,. \tag{DVI'}$$

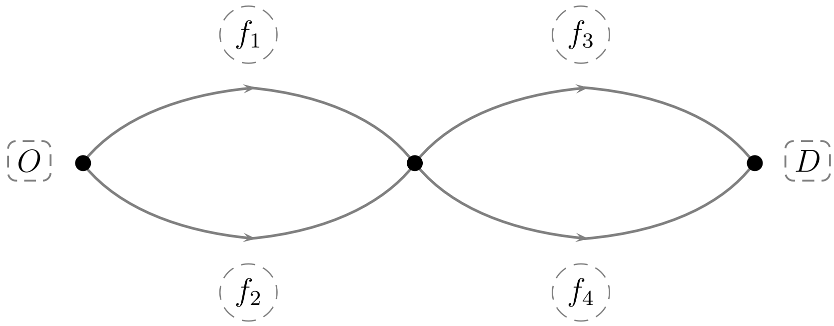
Moreover, (DVI) is equivalent to the dynamic Wardrop principle: for any $t \in [0, T]$, for any $H(t) \in K(t)$ and for any $W_j \in W$, $R_q, R_s \in P_j$, there holds

$$C_q(t, H(t)) < C_s(t, H(t)) \quad \Longrightarrow \quad H_s(t) = 0.$$

Under the hypothesis made in Section 2, in particular the uniform strong monotonicity of $c$, we get that the unique solution $f^*(t)$ to (DVI') is a continuous function of $t$. We also have applied Remark at page 6 in Section 2 to get $\Delta K(t_n) \xrightarrow{K} \Delta K(t)$ from $K(t_n) \xrightarrow{K} K(t)$. In the following simple example we show that the original strong monotonicity of $c$ is not preserved after the transformation to the path flows variables and, as a result, there are discontinuous as well as continuous solutions to the variational inequality in the path flows variables.

**Example 4.1** We consider the simple network of Figure 1 below which consists of four arcs and one origin–destination pair, which can be connected by four different paths.

Figure 1: Lost of strong monotonicity through a linear mapping.




Let us assume that the traffic demand between $O$ and $D$ is given by a continuous function in the interval $[0,T] \ni t \to d(t) \in \mathbb{R}$, and that the link cost functions are given by $c_1 = 2f_1, c_2 = 3f_2, c_3 = f_3, c_4 = f_4$. The link flows belong to the set

$$\{f(t) \in \mathbb{R}^4 : \exists F(t) \in K(t),\ f(t) = \Delta F(t)\},$$

where $K(t)$ is the feasible set in the path flow variables

$$K(t) = \{F_1(t), F_2(t), F_3(t), F_4(t) \geq 0 \text{ s.t. } F_1(t)+F_2(t)+F_3(t)+F_4(t) = d(t)\},$$

and $\Delta$ is the link–path matrix. Hence, if $F(t)$ is known, one can derive $f$ from the conditions

$$\begin{aligned} f_1(t) &= F_1(t) + F_2(t), \\ f_2(t) &= F_3(t) + F_4(t), \\ f_3(t) &= F_1(t) + F_3(t), \\ f_4(t) &= F_2(t) + F_4(t). \end{aligned}$$

The path–cost functions are given by the other conditions

$$\begin{aligned} C_1(t) &= c_1(t) + c_3(t) = 3F_1(t) + 2F_2(t) + F_3(t), \\ C_2(t) &= c_1(t) + c_4(t) = 2F_1(t) + 3F_2(t) + F_4(t), \\ C_3(t) &= c_2(t) + c_3(t) = F_1(t) + 4F_3(t) + 3F_4(t), \\ C_4(t) &= c_2(t) + c_4(t) = F_2(t) + 3F_3(t) + 4F_4(t). \end{aligned}$$

The solution of the variational inequality, expressed in term of the second path variables yields

$$\Big(\frac{3d(t)}{5} - H(t), H(t), H(t) - \frac{d(t)}{10}, -H(t) + \frac{d(t)}{2}\Big)$$

where $H(t) \in \left[\frac{d(t)}{10}, \frac{d(t)}{2}\right]$ is a given function, *not necessarily continuous.* Let us observe that for each feasible value of $H(t)$, the cost at the solution is equally constant to $\frac{17}{10}\, d(t)\big(1\,,1\,,1\,,1\big)$. One can also solve the variational inequality in the link variables by using the relations

$$\begin{aligned} f_1(t) + f_2(t) &= d(t), \\ f_3(t) + f_4(t) &= d(t). \end{aligned}$$

We are then left with the problem:

$$\big(c_2(t) - c_1(t)\big)\,\big(f_2(t) - f_2^*(t)\big) + \big(c_4(t) - c_3(t)\big)\,\big(f_4(t) - f_4^*(t)\big) \geq 0,$$

which yields

$$f^*(t) = d(t)\left(\frac{3}{5}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right).$$

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