The rotating Bénard problem and its optimal Lyapunov function

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The rotating Bénard problem is a classical fluid-dynamics system of PDEs that describes an infinite layer of fluid heated from below in a reference frame uniformly rotating along a vertical axis. In this setup, a central question is the stability of the motionless solution with constant temperature gradient under stress-free and fixed temperature boundary conditions. The spectral analysis of this system is straightforward, and it produces a critical threshold of stability for the Rayleigh number $R$ depending on the rotation parameter $T$: the faster is the rotation, the higher becomes the critical threshold [1, 2], which hence exceeds the stability threshold for the simple Bénard problem (stabilizing effect of the rotation).

Some authors have tried to obtain the same results by means of optimal Lyapunov functions (a family of functions depending on the parameters of the problem that turn out to prove Lyapunov stability every time the parameters are chosen in the linear stability region) [3, 4, 5]. The upshot of this approach is that a Lyapunov function allows one to estimate the size of the basin of attraction of the equilibrium. Unfortunately, the construction of Lyapunov functions is not algorithmic, and it requires an ad-hoc analysis of the particular problem at hand. Nonetheless, for reaction-diffusion systems the method of reduction to canonical coordinates has proven to be effective in many different systems [6, 7, 8, 9, 10, 11], so that it has been conjectured that this technique should typically yield an optimal Lyapunov function.

Despite this optimism, for the rotating Bénard problem there is no proof that the Lyapunov function $L$ computed with this method is optimal, and many authors have shown its optimality only up to a certain value of the rotation parameter $T$ [12, 13]. It is hence possible —although surprising— that the strongly stable modes of the system contribute to make the function not Lyapunov when $T$ is big and $R$ close to the criticality.

We show that indeed such Lyapunov function is optimal only up to a certain value of $T$, while for very high values of the rotational parameter there are choices of subcritical Rayleigh number and of particular solutions along which the function $L$ is not monotonically decreasing (such solutions do converge to the motionless solution, we are hence not dealing with subcritical instabilities). There is hence a very good reason for meeting difficulties in demonstrating that the function $L$ is optimal for all values of the rotation parameter $T$.


