Turing pattern formation in the Brusselator system with anomalous diffusion

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In this work we investigate the Turing pattern formation for the following reaction-diffusion system:

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\begin{align*}
\frac{\partial u}{\partial t} &= D_u \frac{\partial}{\partial x} \left( \left( \frac{u}{u_0} \right)^m \frac{\partial u}{\partial x} \right) + \Gamma \left( a - (b + 1)u + u^2v \right), \\
\frac{\partial v}{\partial t} &= D_v \frac{\partial}{\partial x} \left( \left( \frac{v}{v_0} \right)^n \frac{\partial v}{\partial x} \right) + \Gamma \left( bu - u^2v \right),
\end{align*}
\]  

(1)

The nonlinear diffusion terms, when \(m, n \geq 1\), model the tendency of the species to diffuse faster or slower, depending whether \(u, v\) are larger or smaller than the thresholds \(u_0, v_0\), than predicted by classical diffusion. The nonlinear coupling between \(u\) and \(v\) is represented by the Brusselator autocatalytic reaction term. The stability of the homogeneous solution of system like (1) has been extensively studied, see e.g. [8, 7]. In this talk we focus on the relationship between the possible destabilization mechanisms. In fact, since the Brusselator kinetics supports Hopf bifurcation, whether destabilization occurs through Turing bifurcation, depends on the location of the Hopf and the Turing instability boundaries [6, 9]. We shall see that the effect of nonlinear diffusion is to make Turing pattern formation easier than it would be for classical diffusion [2]; moreover Turing instability occurs even if the diffusion coefficient of the inhibitor \(D_v\) is smaller than the coefficient \(D_u\) of the activator.

Through a weakly nonlinear multiple scales analysis [1] we derive the equations for the amplitude of the stationary pattern, both in the supercritical and in the subcritical case. To describe the traveling front enveloping a pattern which invades a large spatial domain we obtain the Ginzburg-Landau equation [4]. In a 2D domain more complex pattern forming phenomena occur, as degeneracy leads to the interaction of different modes even close to threshold. We derive the amplitude equations appropriate for each case (rolls, squares, mixed-mode and hexagonal patterns) [3]. We also mention the effects of the competition between space and time instabilities near a codimension-two Turing-Hopf bifurcation [5].


