Università degli studi di Catania
Corso di laurea triennale in Fisica
Esame di Meccanica Analitica
Appello del 28.02.2020

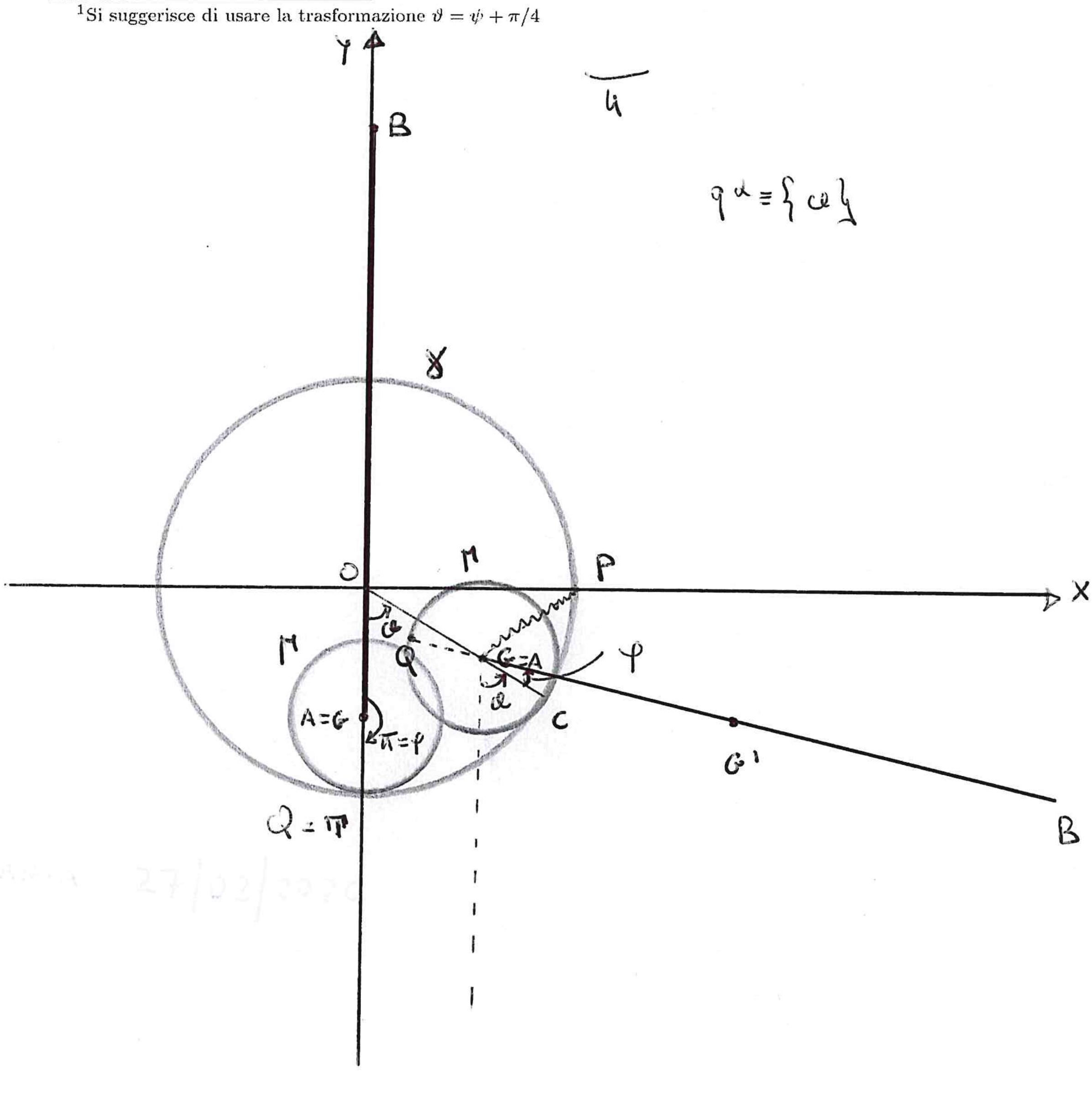
Un sistema materiale rigido, posto in un piano verticale Π, é costituito da una sbarra omogenea pesante AB di massa m e lunghezza L e da un disco omogeneo pesante Γ di massa $2\,m$, centro G e raggio r. L'asta AB é rigidamente saldata sul disco Γ, lungo un suo raggio, in maniera tale che l'estremo A coincida con il centro del disco. Il disco Γ é vincolato a rotolare senza strisciare, lungo il bordo interno di una guida circolare γ di raggio $R = 3\,r$ fissa nel piano verticale, in maniera tale che quando Γ si trova nella posizione piú bassa (vedi figura) il vettore $B - A$ sia verticale ascendente. Sul sistema oltre alla forza peso agisce la forza elastica

$$\{F = -k\,(G - P),\ G\} \qquad \text{con} \quad k > 0$$

essendo P il punto fisso di intersezione della guida circolare γ con l'asse delle x positive, come in figura. Supponendo che valgano le relazioni $k = m\,g/r$ e $L = (12/\sqrt{2})\,r$ e scegliendo come coordinata lagrangiana l'angolo ϑ che la direzione di $\overrightarrow{OG}$ forma con la verticale discendente (vedi figura), si chiede di determinare

1. Le configurazioni di equilibrio[1] del sistema, studiandone la stabilitá.
2. Scrivere l'equazione di moto, determinando gli eventuali integrali primi.
3. Studiare i moti in prima approssimazione attorno ad una configurazione di equilibrio stabile per il sistema.

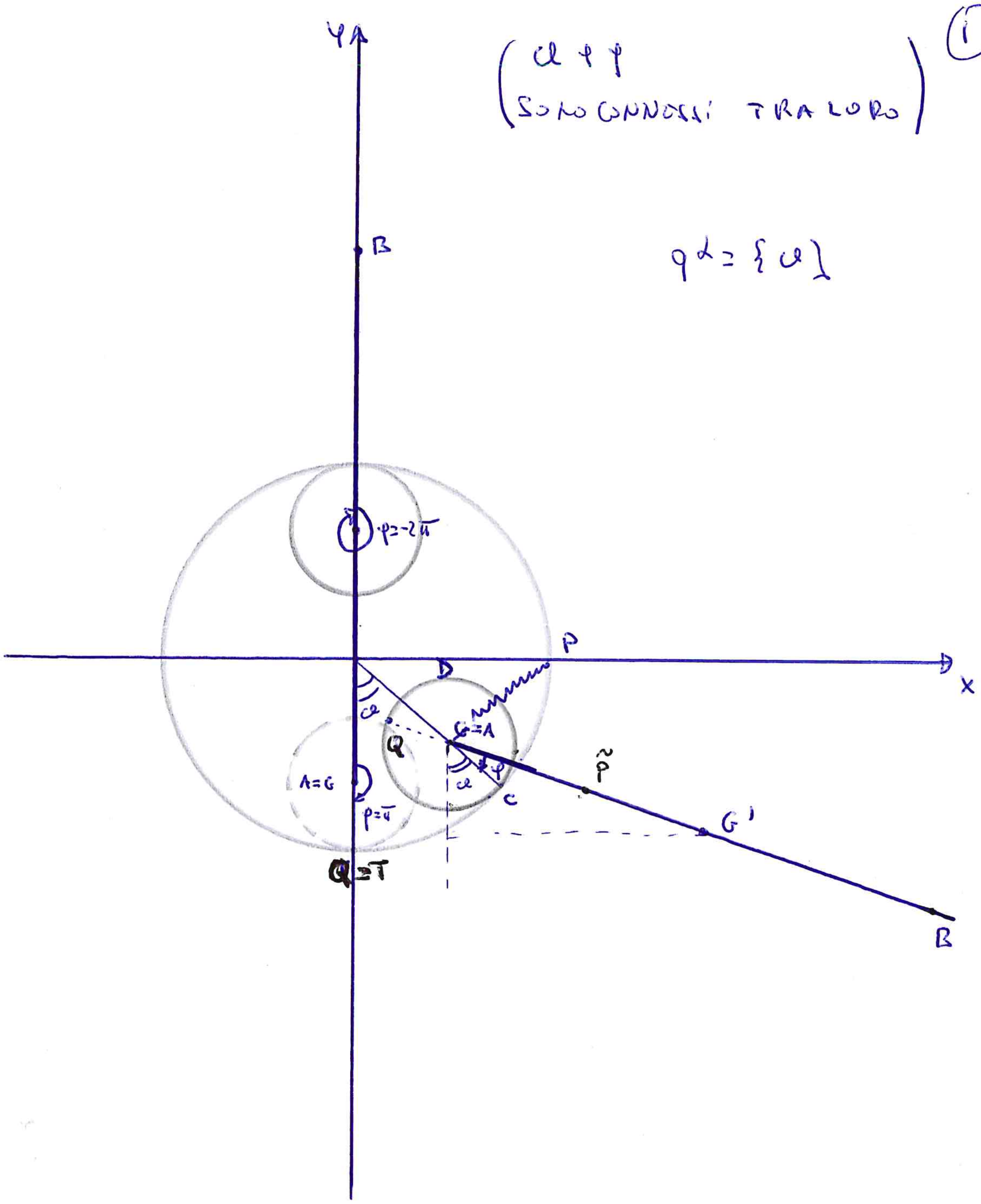
[1] Si suggerisce di usare la trasformazione $\vartheta = \psi + \pi/4$



①

$\left(\begin{array}{l}\alpha \text{ e } \varphi \\ \text{SONO CONNESSI TRA LORO}\end{array}\right)$

$q^L = \{\alpha\}$



$\widehat{TC} = \widehat{QC}$ dove $\widehat{TC} = R\alpha$

$\widehat{QC} = r(\pi - \varphi)$ $\Rightarrow \varphi = \pi - \frac{R}{r}\alpha = \pi - 3\alpha$

$P \equiv \{3r, 0\}$

quindi $\varphi + \alpha = \pi - 2\alpha$

$G = \{2r \operatorname{sen}\alpha, -2r\cos\alpha\}$

$G' \equiv \{x_G + \frac{L}{2}\operatorname{sen}(\alpha+\varphi), y_G - \frac{L}{2}\cos(\alpha+\varphi)\}$

$= 2r\operatorname{sen}\alpha + \frac{L}{2}\operatorname{sen}(\pi - 2\alpha), -2r\cos\alpha - \frac{L}{2}\cos(\pi-2\alpha)\}$

$= \{2r\operatorname{sen}\alpha + \frac{L}{2}\operatorname{sen}(2\alpha), -2r\cos\alpha + \frac{L}{2}\cos(2\alpha)\}$

NOTA:

$\alpha = 0 \Rightarrow \varphi = \pi$;

$\alpha = \frac{\pi}{6} \Rightarrow \varphi = \pi/2$;

$\alpha = \frac{\pi}{4} \Rightarrow \varphi = \frac{\pi}{4}$;

$\alpha = \frac{\pi}{3} \Rightarrow \varphi = 0$;

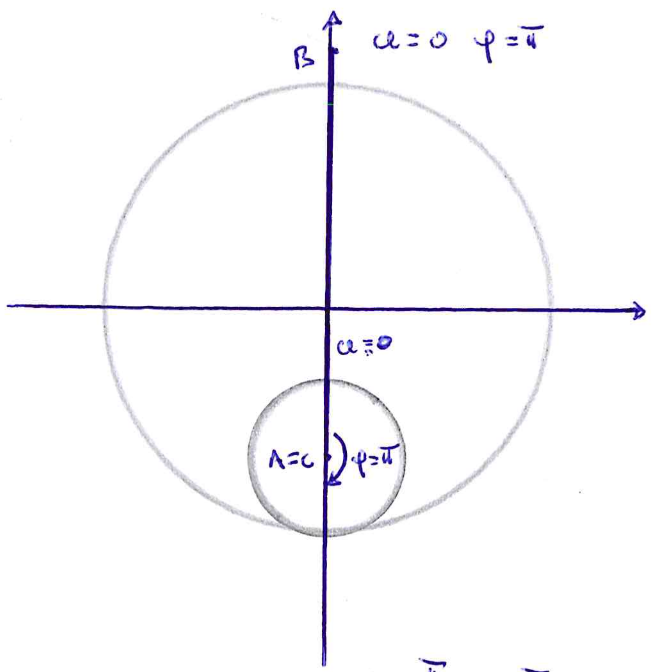
$\alpha = \frac{\pi}{2}$ $\varphi = -\frac{\pi}{2}$;

$\alpha = \frac{2}{3}\pi \Rightarrow \varphi = -\pi$

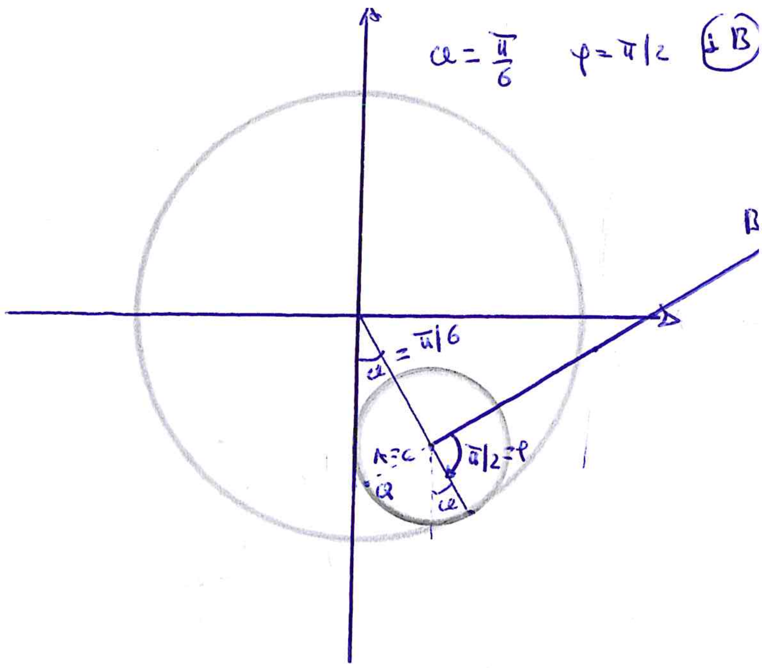
$\alpha = \frac{3}{4}\pi \Rightarrow \varphi = -$

$\alpha = \frac{5}{6}\pi \Rightarrow \varphi = -\frac{3}{2}$

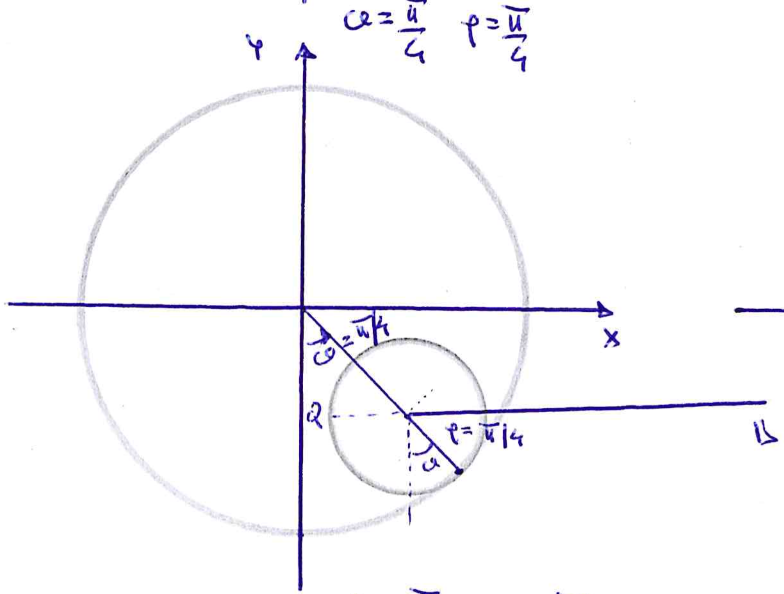
$\alpha = \pi$ $\varphi = -2\pi$



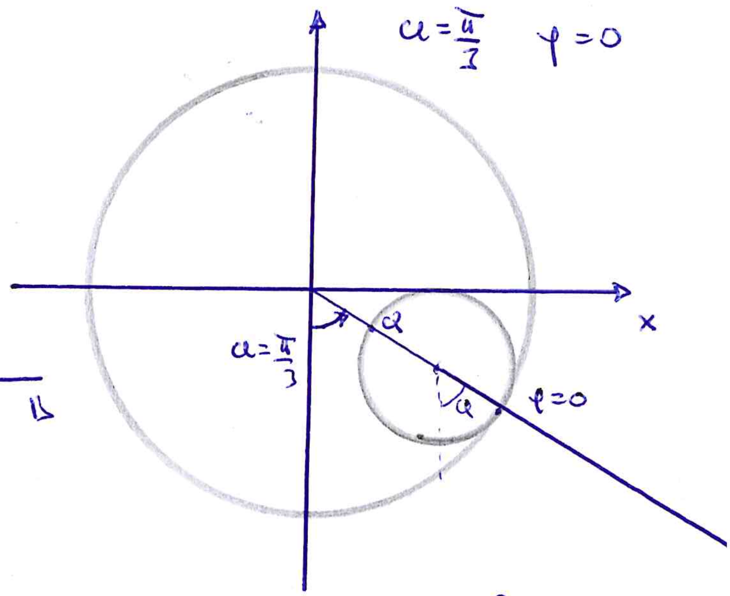
$\alpha = 0 \quad \varphi = \pi$
B
$\alpha = 0$
$A = C$
$\varphi = \pi$



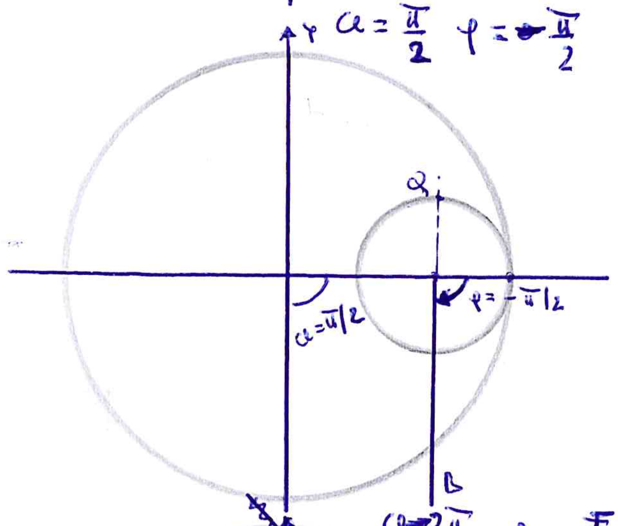
$\alpha = \frac{\pi}{6} \quad \varphi = \pi/2$
B
$\alpha = \pi/6$
$A = C$
$\pi/2 = \varphi$
α



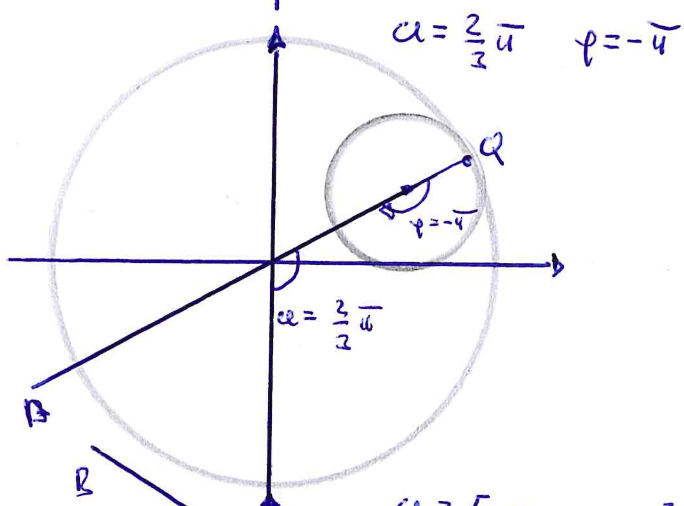
$\alpha = \frac{\pi}{4} \quad \varphi = \frac{\pi}{4}$
y
x
$\alpha = \pi/4$
Q
$\varphi = \pi/4$
α
B



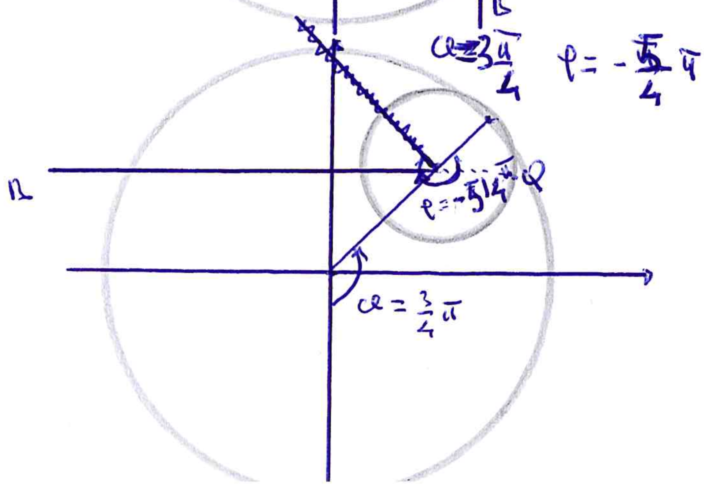
$\alpha = \frac{\pi}{3} \quad \varphi = 0$
x
$\alpha = \frac{\pi}{3}$
Q
α
$\varphi = 0$



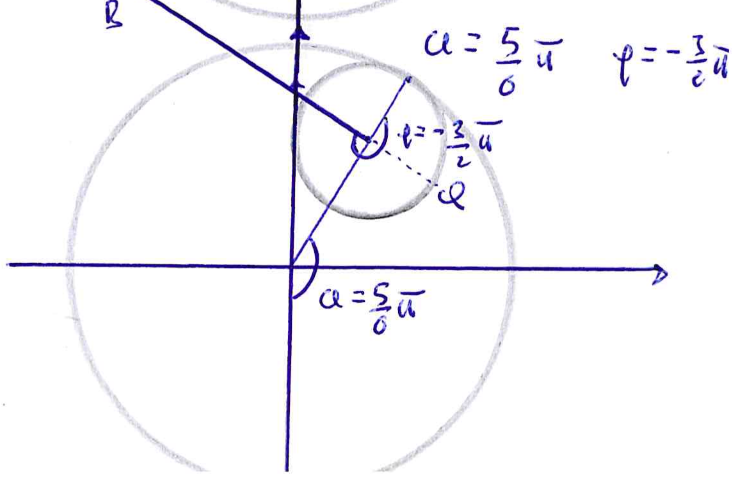
y $\alpha = \frac{\pi}{2} \quad \varphi = -\frac{\pi}{2}$
Q
$\alpha = \pi/2$
$\varphi = -\pi/2$
B



$\alpha = \frac{2}{3}\pi \quad \varphi = -\pi$
Q
$\varphi = -\pi$
$\alpha = \frac{2}{3}\pi$
B



$\alpha = \frac{3\pi}{4} \quad \varphi = -\frac{5}{4}\pi$
B
Q
$\varphi = -5\pi/4$
$\alpha = \frac{3}{4}\pi$



$\alpha = \frac{5}{6}\pi \quad \varphi = -\frac{3}{2}\pi$
B
$\varphi = -\frac{3}{2}\pi$
Q
$\alpha = \frac{5}{6}\pi$

FORZE : $\begin{cases} \text{FORZA PESO} \\ \text{FORZA ELASTICA.} \quad F = -K(G-P) \end{cases}$

$\boxed{\text{METODO DEL POTENZIALE}}$

$$U = 2mg(0,-1)\cdot(G-O) + mg(0,-1)\cdot(G'-O) - \frac{1}{2}K(G-P)^2$$

ESSENDO

$$G = \{2\ell \operatorname{sen}\alpha, -2\ell\cos\alpha\}$$

$$G' = \{2\ell\operatorname{sen}\alpha + \frac{L}{\cancel{2}}\cancel{2}\operatorname{sen}\alpha\cos\alpha, -2\ell\cos\alpha + \frac{L}{2}(\cos^2\alpha - \operatorname{sen}^2\alpha)\}$$

$$P = \{3\ell, 0\}$$

AVREMO

$$U = 4mg\ell\cos\alpha + mg[2\ell\cos\alpha - \frac{L}{2}\underbrace{\overbrace{(\cos^2\alpha}^{1-\operatorname{sen}^2\alpha} - \operatorname{sen}^2\alpha)}_{(1-2\operatorname{sen}^2\alpha)}]$$

$$-\frac{1}{2}K\{2\ell\operatorname{sen}\alpha - 3\ell, -2\ell\cos\alpha\}^2 =$$

$$= 6mg\ell\cos\alpha + mgL\operatorname{sen}^2\alpha - \frac{1}{2}K(-12\ell^2\operatorname{sen}\alpha) + \cancel{\tilde{K}}$$

$$= 6mg\ell\cos\alpha + 6K\ell^2\operatorname{sen}\alpha + mgL\operatorname{sen}^2\alpha$$

DA CUI PONENDO $\boxed{mg = K\ell}$ E $\boxed{L = \frac{12}{\sqrt{2}}\ell}$

$$U = 6K\ell^2\cos\alpha + 6K\ell^2\operatorname{sen}\alpha + \frac{12}{\sqrt{2}}K\ell^2\operatorname{sen}^2\alpha + \tilde{K}$$

$$= 6K\ell^2\{\cos\alpha + \operatorname{sen}\alpha + \frac{2}{\sqrt{2}}\operatorname{sen}^2\alpha\} + \tilde{K}$$

DA CUI LA SOLLECITAZIONI

$$Q_\alpha = \frac{\partial U}{\partial\alpha} = 6K\ell^2\{\cos\alpha - \operatorname{sen}\alpha + \frac{4}{\sqrt{2}}\operatorname{sen}\alpha\cos\alpha\}$$

METODO DELLE SOLLECITAZIONI

$$Q_a = 2mg\,(0,-1)\,\frac{\partial G}{\partial a} + mg\,(0-1)\cdot\frac{\partial G'}{\partial a} - K(G-P)\cdot\frac{\partial G}{\partial a}$$

ESSENDO $\frac{\partial G}{\partial a} = \{2\ell\cos a\ ,\ 2\ell\,\mathrm{sen}\,a\}$ AVREMO

$$\frac{\partial G'}{\partial a} = \{\ ''\ \ \ ''\ ,\ 2\ell\,\mathrm{sen}\,a + 2L\,\mathrm{sen}\,a\cos a\}$$

$$Q_a = -4mg\ell\,\mathrm{sen}\,a - 2mg\ell\,\mathrm{sen}\,a + 2mgL\,\mathrm{sen}\,a\cos a$$

$$-K\{2\ell\,\mathrm{sen}\,a - 3\ell\ ,\ -2\ell\cos a\}\cdot\{2\ell\cos a\ ,\ 2\ell\,\mathrm{sen}\,a\}$$

$$= -6mg\ell\,\mathrm{sen}\,a + 2mgL\,\mathrm{sen}\,a\cos a - K[4\ell^2\cancel{\mathrm{sen}\,a\cos a}$$

$$-6\ell^2\cos a - 4\ell^2\cancel{\mathrm{sen}\,a}\cos a] =$$

$$= -6mg\ell\,\mathrm{sen}\,a + 2mgL\,\mathrm{sen}\,a\cos a + 6K\ell^2\cos a$$

PONENDO $mg = K\ell$ $\quad L = \frac{12}{\sqrt{2}}\ell$ AVREMO

$$Q_a = -6K\ell^2\,\mathrm{sen}\,a + 6K\ell^2\cos a + \frac{24}{\sqrt{2}}K\ell^2\,\mathrm{sen}\,a\cos a.$$

$$Q_a = 6K\ell^2\{\cos a - \mathrm{sen}\,a + \frac{4}{\sqrt{2}}\,\mathrm{sen}\,a\cos a\}$$

DA CUI INTEGRANDO IL POTENZIALE

$$U = \int Q_a\,da = 6K\ell^2\Big\{\int\cos a\,da - \int\mathrm{sen}\,a\,da$$

$$+\frac{4}{\sqrt{2}}\int\mathrm{sen}\,a\cos a\,da\Big\} = 6K\ell^2\{\mathrm{sen}\,a + \cos a + \frac{2}{\sqrt{2}}\,\mathrm{sen}^2 a\} + \tilde{u}$$

PONIAMO $\alpha = \varphi + \frac{\pi}{4}$

$$\operatorname{sen}\alpha = \operatorname{sen}\left(\varphi + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos\varphi + \operatorname{sen}\varphi\right)$$

$$\cos\alpha = \cos\left(\varphi + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos\varphi - \operatorname{sen}\varphi\right)$$

DA CUI: $\operatorname{sen}\alpha - \cos\alpha = \frac{2}{\sqrt{2}}\operatorname{sen}\varphi$

$$\operatorname{sen}\alpha\cos\alpha = \frac{1}{2}\left[\cos^2\varphi - \operatorname{sen}^2\varphi\right] = \frac{1}{2}\left[1 - 2\operatorname{sen}^2\varphi\right]$$

DA CUI:

$$Q_\alpha = 6Kr^2\left\{\frac{2}{\sqrt{2}}\left(1 - 2\operatorname{sen}^2\varphi\right) - \frac{2}{\sqrt{2}}\operatorname{sen}\varphi\right\} =$$

$$= -\frac{12}{\sqrt{2}}Kr^2\left\{2\operatorname{sen}^2\varphi + \operatorname{sen}\varphi - 1\right\} = 0$$

EQUILIBRIO $Q_\alpha = Q_\varphi = 0 \Rightarrow 2\operatorname{sen}^2\varphi + \operatorname{sen}\varphi - 1 = 0$

PONENDO $z = \operatorname{sen}\varphi$ $\quad 2z^2 + z - 1 = 0 \quad z = \frac{-1 \pm \sqrt{1+8}}{4} = \begin{cases} -1 \\ 1/2 \end{cases}$

$$\operatorname{sen}\varphi = \begin{cases} -1 & \Rightarrow \varphi = -\pi/2 \\ 1/2 & \Rightarrow \varphi = \frac{\pi}{6},\ \frac{5}{6}\pi \end{cases}$$

DA CUI $\varphi = -\frac{\pi}{2} \Rightarrow \alpha = -\frac{\pi}{2} + \frac{\pi}{4} = -\pi/4$

$$\varphi = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5}{12}\pi$$

$$\varphi = \frac{5}{6}\pi \Rightarrow \alpha = \frac{5}{6}\pi + \frac{\pi}{4} = \frac{13}{12}\pi$$

QUINDI AVREMO TRE CONFIGURAZIONI DI EQUILIBRIO

$\alpha = -\pi/4$ $\left(\varphi = -\pi/2\right)$; $\alpha = \frac{5}{12}\pi$ $\left(\varphi = \frac{\pi}{6}\right)$; $\alpha = \frac{13}{12}\pi$ $\left(\varphi = \frac{5}{6}\pi\right)$

STABILITÀ: STUDIAMO LA Q_φ DA CUI (OPPURE IN TERMINI DI α ⇒ VEDI RETRO)

$$\frac{\partial Q_\varphi}{\partial \varphi} = -\frac{12}{\sqrt{2}} K l^2 \left\{ 4 \operatorname{sen}\varphi \cos\varphi + \cos\varphi \right\}$$

DA CUI:

1) $$\left.\frac{\partial Q_\varphi}{\partial \varphi}\right|_{\varphi=-\pi/2} = 0 \qquad \text{NON POSSIAMO DIRE NULLA}$$

DA CUI $$\frac{\partial^2 Q_\varphi}{\partial \varphi^2} = -\frac{12}{\sqrt{2}} K l^2 \left\{ 4\left[\cos^2\varphi - \operatorname{sen}^2\varphi\right] - \operatorname{sen}\varphi \right\}\Big|_{\varphi=-\pi/2} =$$

$$= -\frac{12}{\sqrt{2}} K l^2 \left\{ -4 + 1 \right\} > 0$$

$\boxed{\text{NO MAX} \Rightarrow \text{INSTABILE.}}$

2) $$\left.\frac{\partial Q_\varphi}{\partial \varphi}\right|_{\varphi=\frac{\pi}{6}} = -\frac{12}{\sqrt{2}} K l^2 \left\{ \cancel{4} \cdot \frac{1}{\cancel{2}} \frac{\sqrt{3}}{\cancel{2}} + \frac{\sqrt{3}}{2} \right\} < 0$$

$\boxed{\text{MAX} \Rightarrow \text{STABILE}}$

2) $$\left.\frac{\partial Q_\varphi}{\partial \varphi}\right|_{\varphi=\frac{5}{6}\pi} = -\frac{12}{\sqrt{2}} K l^2 \left\{ 4 \cdot \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} \right\} > 0$$

$\boxed{\text{NO MAX} \Rightarrow \text{INSTABILE.}}$

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ENERGIA CINETICA:

IL DISCO SI MUOVE CON MOTO DI PURO ROTOLAMENTO QUINDI

$$T_M = \frac{1}{2}(2m)\dot{G}^2 + T_M' \qquad T_M' = \frac{1}{2} I^M_{z,G}\, \Omega^2$$

$$\dot{G} \equiv \left\{ 2l\cos\alpha\,\dot{\alpha},\ 2l\operatorname{sen}\alpha\,\dot{\alpha} \right\} \Rightarrow \dot{G}^2 = 4l^2\dot{\alpha}^2$$

SE STUDIAMO LA STABILITA' IN TERMINI DI α

$$\frac{\partial Q_\alpha}{\partial \alpha} = 6K z^2 \left\{ -\operatorname{sen}\alpha - \cos\alpha + \frac{4}{\sqrt{2}} (\cos^2\alpha - \operatorname{sen}^2\alpha) \right\}$$

1) PER $\alpha = -\pi/4$ $\quad \left.\frac{\partial Q_\alpha}{\partial \alpha}\right|_{\alpha=-\pi/4} = 0$

DA CUI DERIVIAMO ULTERIORMENTE

$$\frac{\partial^2 Q_\alpha}{\partial \alpha^2} = 6K z^2 \left\{ -\cos\alpha + \operatorname{sen}\alpha - \frac{8}{\sqrt{2}} \operatorname{sen}\alpha \cos\alpha \right\}$$

$$\left.\frac{\partial^2 Q_\alpha}{\partial \alpha^2}\right|_{\alpha=-\pi/4} = 6K z^2 \left\{ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{8}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \right\} =$$

$$6K z^2 \left\{ + \frac{2}{\sqrt{2}} \right\} > 0 \Rightarrow \text{NO MAX} \Rightarrow \text{INSTABILE}$$

2) PER $\alpha = \frac{5}{12}\pi$ $\quad \left(\psi = \frac{\pi}{6}\right)$

$$\begin{cases} \operatorname{sen}\left(\frac{5}{12}\pi\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{(1+\sqrt{3})}{2\sqrt{2}} \\ \cos\left(\frac{5}{12}\pi\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}} \end{cases}$$

DA CUI

$$\left.\frac{\partial Q_\alpha}{\partial \alpha}\right|_{\alpha=\frac{5}{12}\pi} = 6K z^2 \left\{ -\frac{\sqrt{3}}{\sqrt{2}} + \frac{4}{\sqrt{2}} \left(-\frac{\sqrt{3}}{2}\right) \right\} = -18K z^2 \frac{\sqrt{3}}{\sqrt{2}} < 0$$

MAX $\Rightarrow$ STABILE.

3) PER $\alpha = \frac{13}{12}\pi$ $\quad \left(\psi = \frac{5}{6}\pi\right)$

$$\begin{cases} \operatorname{sen}\left(\frac{13}{12}\pi\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = -\frac{(\sqrt{3}-1)}{2\sqrt{2}} \\ \cos\left(\frac{13}{12}\pi\right) = \frac{1}{\sqrt{2}}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = -\frac{(\sqrt{3}+1)}{2\sqrt{2}} \end{cases}$$

DA CUI

$$\left.\frac{\partial Q_\alpha}{\partial \alpha}\right|_{\alpha=\frac{13}{12}\pi} = 6K z^2 \left\{ \frac{\sqrt{3}}{\sqrt{2}} + \frac{4}{\sqrt{2}} \left(\frac{\sqrt{3}}{2}\right) \right\} = 18K z^2 \frac{\sqrt{3}}{\sqrt{2}} > 0$$

NO MAX $\Rightarrow$ INSTABILE.

$$I^{\Gamma}_{z,G} = \int \rho^2 \, dm = \frac{2m}{\pi r^2} \cdot \iint \rho^2 \rho \, d\rho \, d\alpha =$$

$$= \frac{2m}{\pi r^2} \int_0^r \rho^3 d\rho \int_0^{2\pi} d\alpha = \frac{2m}{\cancel{\pi} r^2} \cdot \frac{r^4}{\cancel{4}} \cdot \cancel{2\pi} = m r^2$$

PER CALCOLARE Ω^2 CONSIDERIAMO LA CONDIZIONE DI PURO ROTOLAMENTO:

$$\underline{V}_C = \underline{V}_G + \Omega \wedge (G-C) = 0 \quad \Rightarrow |V_G| = \Omega r$$

DOVE $|\underline{V}_G| = 2r\dot{\varphi}$ DA CUI $\Omega^2 = \left(\frac{2\cancel{r}\dot{\varphi}}{\cancel{r}}\right)^2 = 4\dot{\varphi}^2$

QUINDI $T_\Gamma = m(4r^2\dot{\varphi}^2) + \frac{1}{2}(mr^2)4\dot{\varphi}^2$

$$\boxed{T_\Gamma = 6mr^2\dot{\varphi}^2}$$

PER L'ASTA POSSIAMO UTILIZZARE DUE METODI:

1) METODO PUNTUALE:

UN GENERICO PUNTO $\tilde{P} \in AB$ A PARTIRE DALL'ESTREMO A AVRÀ COORDINATE:

$$\tilde{P} \equiv \{2r \operatorname{sen}\varphi + s \operatorname{sen}(2\varphi), -2r\cos\varphi + s\cos(2\varphi)\}$$

DA CUI $\dot{\tilde{P}} = \{2r\cos\varphi\,\dot{\varphi} + s\cos(2\varphi)\cdot 2\dot{\varphi},$
$2r\operatorname{sen}\varphi\,\dot{\varphi} - s\operatorname{sen}(2\varphi)\cdot 2\dot{\varphi}\}$

DA CUI $\dot{\tilde{P}}^2 = \{4r^2\cos^2\varphi + 4s^2\cos^2(2\varphi) + 8rs\cos\varphi\cos(2\varphi)$
$+4r^2\operatorname{sen}^2\varphi + 4s^2\operatorname{sen}^2(2\varphi) - 8rs\operatorname{sen}\varphi\operatorname{sen}(2\varphi)\}\dot{\varphi}^2$

$\dot{\tilde{P}}^2 = \{4r^2 + 4s^2 + 8rs[\cos\varphi\cos(2\varphi) - \operatorname{sen}\varphi\operatorname{sen}(2\varphi)]\}\dot{\varphi}^2$

DA CUI INTEGRAMO

$$T_{AB} = \frac{1}{2}\int \dot{\tilde{p}}^2 dm = \frac{1}{2}\sigma \cdot \Big\{ 4z^2 \int_0^L dy + 4\int_0^L y^2 dy +$$

$$+ 8z\left[\cos\alpha \cos(2\alpha) - \operatorname{sen}\alpha \operatorname{sen}(2\alpha)\right] \int_0^L y\, dy \Big\} \dot{\alpha}^2$$

$$= \frac{1}{2}\frac{m}{L}\Big\{ 4z^2 L + 4\frac{L^3}{3} + 8z\left[\cos\alpha\cos(2\alpha) - \operatorname{sen}(\alpha)\operatorname{sen}(2\alpha)\right] \cdot$$

$$\cdot \frac{L^2}{2}\Big\} \dot{\alpha}^2$$

$$T_{AB} = \frac{1}{2} m \Big\{ 4z^2 + \frac{4}{3}L^2 + 4zL \underbrace{\left[\cos\alpha\cos 2\alpha - \operatorname{sen}\alpha \operatorname{sen}(2\alpha)\right]}_{\cos(\alpha+2\alpha)} \Big\} \dot{\alpha}^2$$

$$T_{AB} = m\Big\{ 2z^2 + \frac{2}{3}L^2 + 2zL\cos(3\alpha)\Big\}\dot{\alpha}^2$$

$$\boxed{T_{AB} = 2m\Big\{ z^2 + \frac{1}{3}L^2 + zL\cos(3\alpha)\Big\}\dot{\alpha}^2}$$

CON $L = 12/\sqrt{2}\, z$

$$T_{AB} = 2mz^2\Big\{ 25 + \frac{12}{\sqrt{2}}\cos(3\alpha)\Big\}\dot{\alpha}^2$$

2° METODO: TEOREMA DI KONIG.

$$T_{AB} = \frac{1}{2}(4m)\dot{G}'^2 + T'_{AB} \qquad \text{DOVE} \quad T'_{AB} = \frac{1}{2} I^{AB}_{G',z} \big[\underbrace{(\varphi+\alpha)^{\circ}}_{(-2\dot{\alpha})}\big]^2$$

$$\dot{G}' \equiv \Big\{ 2z\cos\alpha\,\dot{\alpha} + \frac{L}{\cancel{2}}\cos(2\alpha)\cancel{2}\dot{\alpha},\ +2z\operatorname{sen}\alpha\,\dot{\alpha} - \frac{L}{\cancel{2}}\operatorname{sen}(2\alpha)\cancel{2}\dot{\alpha}\Big\}$$

$$\dot{G}'^2 = \Big\{ 4z^2 + L^2 + 4zL\big[\underbrace{\cos\alpha\cos 2\alpha - \operatorname{sen}\alpha\operatorname{sen}(2\alpha)}_{\cos(3\alpha)}\big]\Big\}\dot{\alpha}^2$$

$$\dot{G}'^2 = \{4 z^2 + L^2 + 4 z L \cos(3\alpha)\} \dot{\alpha}^2$$

$$I^{AB}_{G',z} = \int_{-L/2}^{L/2} s^2 \, dm = \frac{m}{L} \int_{-L/2}^{L/2} s^2 ds = \frac{m}{L} \left[\frac{s^3}{3}\right]_{-L/2}^{L/2} =$$

$$= \frac{2}{3} \frac{m}{L} \cdot \frac{L^3}{8} = \frac{1}{12} m L^2$$

DA CUI

$$T_{AB} = \frac{1}{2} m \left[4 z^2 + L^2 + 4 z L \cos(3\alpha)\right] \dot{\alpha}^2$$

$$+ \frac{1}{2} \cdot \left(\frac{1}{12} m L^2\right) 4 \dot{\alpha}^2$$

$$= \frac{1}{2} m \left\{4 z^2 + \left(L^2 + \frac{1}{3} L^2\right) + 4 z L \cos(3\alpha)\right\} \dot{\alpha}^2$$

$$\boxed{T_{AB} = 2 m \left\{z^2 + \frac{1}{3} L^2 + z L \cos(3\alpha)\right\} \dot{\alpha}^2}$$

DA CUI $T_{TOT} = T_M + T_{AB} = 2 m \left\{4 z^2 + \frac{1}{3} L^2 + z L \cos(3\alpha)\right\} \dot{\alpha}^2$

EQUAZIONI DI LAGRANGE: (NOTA: CHE POTREMO SCRIVERE IN TERMINI DI α O IN TERMINI DI φ)

IN TERMINI DI α AVREMO:

$$\frac{\partial T}{\partial \dot{\alpha}} = 4 m z^2 \left\{28 + \frac{12}{\sqrt{2}} \cos(3\alpha)\right\} \dot{\alpha}$$

$$\frac{\partial T}{\partial \alpha} = - \frac{6 (12)}{\sqrt{2}} m z^2 \operatorname{sen}(3\alpha) \dot{\alpha}^2$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\alpha}} = - \frac{(12)^2}{\sqrt{2}} m z^2 \operatorname{sen}(3\alpha) \dot{\alpha}^2 + 4 m z^2 \left\{28 + \frac{12}{\sqrt{2}} \cos(3\alpha)\right\} \ddot{\alpha}$$

DA QUI:

$$4 m r^2 \left\{ 28 + \frac{12}{\sqrt{2}} \cos(3\alpha) \right\} \ddot{\alpha} - \frac{72}{\sqrt{2}} m r^2 \operatorname{sen}(3\alpha) \dot{\alpha}^2 =$$

$$= Q_\alpha = 6 K r^2 \left\{ \cos\alpha - \operatorname{sen}\alpha + \frac{4}{\sqrt{2}} \operatorname{sen}\alpha \cos\alpha \right\}$$

"PICCOLI MOTI" ATTORNO ALLA CONFIGURAZIONE $\alpha = \frac{5}{12}\pi$

LINEARIZZANDO IL 1° MEMBRO AVREMO:

$$4 m r^2 \left(28 - \frac{12}{\sqrt{2}} \frac{1}{\sqrt{2}} \right) \ddot{\alpha} = Q_\alpha$$

$$\Rightarrow 88 m r^2 \ddot{\alpha} = \cancel{Q_\alpha \Big|_{\alpha = \frac{5}{12}\pi}} + \overbrace{\frac{\partial Q_\alpha}{\partial \alpha}\Big|_{\alpha = \frac{5}{12}\pi}}^{-18 K r^2 \sqrt{\frac{3}{2}}} \left(\alpha - \frac{5}{12}\pi \right)$$

$$88 m \cancel{r^2} \ddot{\alpha} = - \frac{18\sqrt{3}}{\sqrt{2}} K \cancel{r^2} \left(\alpha - \frac{5}{12}\pi \right)$$

$$\ddot{\alpha} = A \left(\alpha - \frac{5}{12}\pi \right) \qquad \text{CON } A = - \frac{9\sqrt{3}}{44\sqrt{2}} \frac{K}{m} < 0$$

DA QUI PONENDO $\alpha - \frac{5}{12}\pi = \alpha_0 e^{\lambda t}$

AVREMO $\lambda^2 = A < 0$ $\qquad \lambda_1, \lambda_2 = \pm i \sqrt{|A|}$

MOTI ARMONICI.