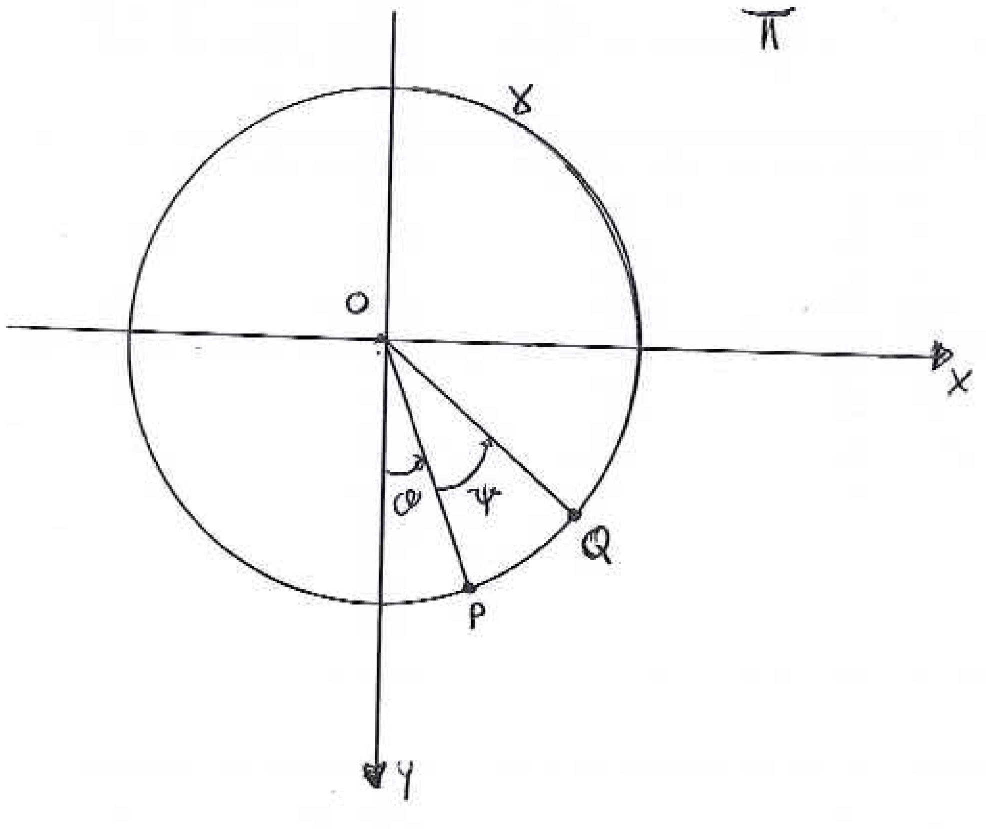
Università degli studi di Catania
Corso di laurea in Fisica
Compito di Meccanica Analitica
Appello del 12.02.2016

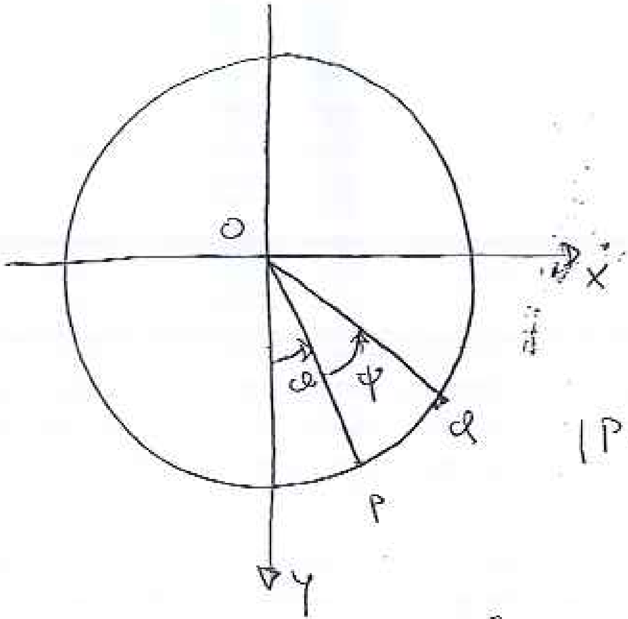
Sia data una guida circolare γ di centro O e raggio R posta in un piano verticale Π dove è stato introdotto un sistema di riferimento cartesiano ortogonale $\{O, x, y\}$ con l'asse delle y verticale discendente. Un sistema materiale S é costituito da due punti P e Q aventi la stessa massa m, vincolati a muoversi senza attrito su γ, ed é soggetto, oltre alla forza peso, alle due forze di mutua repulsione

$$\{\mathbf{F}, P\} \qquad \text{e} \qquad \{-\mathbf{F}, Q\} \qquad \text{con} \qquad \mathbf{F} = 4\,\alpha\, m\, g\, R\, \frac{(P-Q)}{|P-Q|^2}$$

essendo α una costante reale positiva.
Il sistema ha ovviamente due gradi di libertá, scelte allora come coordinate Lagrangiane gli angoli ϑ che $(P-O)$ forma con l'asse delle y e ψ che $(Q-O)$ forma con $(P-O)$ ambedue misurati in modo che le rotazioni di P per ϑ crescente, e, per fissato P, quella di Q per ψ crescente siano entrambe in senso antiorario.

1. Dimostrare che la sollecitazione agente su S é conservativa e che per la coppia $\{\vartheta, \psi\}$ delle due variabili lagrangiane $0 \le \vartheta \le 2\pi$ e $0 < \psi < 2\pi$.
2. Determinare le configurazioni di equilibrio del sistema S, studiando la stabilitá delle suddette configurazioni.
3. Determinare le equazioni di moto e gli eventuali integrali primi.
4. Studiare i moti linearizzati, determinando la frequenza dei piccoli moti, attorno ad una configurazione di equilibrio stabile.





$$P \equiv [R \,\mathrm{sen}\,\alpha ,\ R \cos\alpha]$$

$$Q \equiv [R\,\mathrm{sen}(\alpha+\psi),\ R\cos(\alpha+\psi)]$$

$$|P-Q|^2 = R^2[\mathrm{sen}\,\alpha - \mathrm{sen}(\alpha+\psi)]^2 + R^2[\cos\alpha - \cos(\alpha+\psi)]^2 =$$

$$= R^2\{\mathrm{sen}^2\alpha + \mathrm{sen}^2(\alpha+\psi) - 2\,\mathrm{sen}\,\alpha\,\mathrm{sen}(\alpha+\psi) + \cos^2\alpha + \cos^2(\alpha+\psi) - 2\cos\alpha\cos(\alpha+\psi)\}$$

$$= 2R^2\{1 - \underbrace{[\mathrm{sen}\,\alpha\,\mathrm{sen}(\alpha+\psi) + \cos\alpha\cos(\alpha+\psi)]}_{\cos[(\alpha+\psi)-\alpha] = \cos\psi}\}$$

$$\Rightarrow |P-Q|^2 = 2R^2[1-\cos\psi]$$

PROVIAMO CHE LA SOLLECITAZIONE SU S E' CONSERVATIVA, PROVANDO CHE LA FORZA $\underline{F} = 4\alpha mgR \dfrac{(P-Q)}{\rho^2}$ CON $\rho = |P-Q|$

AGENTE SUI PUNTI P, Q $\{F, P\}$ e $\{-F, Q\}$

E' CONSERVATIVA:

RICORDIAMO CHE

$F = f(\rho)\dfrac{(P-Q)}{\rho}$ AVRA' COME POTENZIALE $U = \int f(\rho)\,d\rho$

DA CUI ESSENDO $f(\rho) = \dfrac{4\alpha mgR}{\rho}$ $\Rightarrow U = \int \dfrac{4\alpha mgR}{\rho}\,d\rho =$

$$= 4\alpha mgR \ln(\rho) = 4\alpha mgR \ln[\sqrt{2}R(1-\cos\psi)^{1/2}] + c.$$

$$= 2\alpha mgR \ln(1-\cos\psi) + k_1$$

QUINDI

$$U_F = 2\alpha m g R \ln(1-\cos\psi) + K_1$$

NOTA: ALTRIMENTI SE CALCOLIAMO IL LAVORO

$$dL_{REPULS.} = 4\alpha m g R \frac{(P-Q)}{\rho^2} \cdot d(P-Q) = 2\alpha m g R \frac{d\rho^2}{\rho^2} =$$

$$= 2\alpha m g R \frac{d(1-\cos\psi)}{(1-\cos\psi)} = d\left[2\alpha m g R \ln(1-\cos\psi)\right]$$

DA CUI $\oint dL = 0 \Rightarrow$ FORZA E' CONSERVATIVA

SUL SISTEMA AVREMO ANCHE LE FORZE PESO

$$U_P = +m g \cdot (P-O) = m g(0,1)\cdot(R\sin\alpha, R\cos\alpha) = m g R\cos\alpha$$

$$U_Q = m g \cdot (Q-O) = m g(0,1)\cdot\left[R\sin(\alpha+\psi), R\cos(\alpha+\psi)\right] =$$

$$= m g R\cos(\alpha+\psi)$$

QUINDI IL POTENZIALE TOTALE:

$$U_T = m g R\cos\alpha + m g R\cos(\alpha+\psi) + 2\alpha m g R \ln\left[1-\cos\psi\right] + C$$

PROVIAMO ADESSO CHE $0 < \psi < 2\pi$

INFATTI ESSENDO $E =$ COSTANTE (INTEGRALE PRIMO)

$$T = E + U \geq 0 \Rightarrow \begin{cases} \lim_{\psi\to 0} T = E + \lim_{\psi\to 0} U = -\infty \\ \lim_{\psi\to 2\pi} T = E + \lim_{\psi\to 2\pi} U = -\infty \end{cases} \quad \text{ASSURDO}$$

QUINDI AVREMO CHE $\begin{cases} 0 < \psi < 2\pi \\ 0 \leq \alpha \leq 2\pi \end{cases}$

CALCOLIAMO LE SOLLECITAZIONI DELLE FORZE CONSERVATIVE

$$Q_\alpha = \frac{\partial U}{\partial \alpha} = -mgR\operatorname{sen}\alpha - mgR\operatorname{sen}(\alpha+\psi)$$

$$Q_\psi = \frac{\partial U}{\partial \psi} = -mgR\operatorname{sen}(\alpha+\psi) + 2\delta\, mgR\,\frac{\operatorname{sen}\psi}{1-\cos\psi}$$

LE SOLLECITAZIONI POSSONO ESSERE CALCOLATE ANCHE DIRETTAMENTE DALLA ESPRESSIONE DIRETTA $Q_\alpha = \sum R_i^{(a)} \cdot \frac{\partial P_i}{\partial q_\alpha}$. INFATTI AVREMO:

$$Q_\alpha = m\underline{g}\cdot\frac{\partial P}{\partial \alpha} + m\underline{g}\cdot\frac{\partial Q}{\partial \alpha} + F\cdot\frac{\partial P}{\partial \alpha} - F\,\frac{\partial Q}{\partial \alpha} =$$

$$= mg(0,1)\cdot(R\cos\alpha, -R\operatorname{sen}\alpha) + mg(0,1)\,[R\cos(\alpha+\psi), -R\operatorname{sen}(\alpha+\psi)]$$

$$+ \frac{4\delta\, mgR}{2R^2(1-\cos\psi)}\left\{R\left[\operatorname{sen}\alpha - \operatorname{sen}(\alpha+\psi)\right], R\left[\cos\alpha - \cos(\alpha+\psi)\right]\right\}\cdot$$
$$\cdot\left\{R\cos\alpha, -R\operatorname{sen}\alpha\right\}$$

$$- \frac{4\delta\, mgR}{2R^2(1-\cos\psi)}\cdot\left\{R\left[\operatorname{sen}\alpha - \operatorname{sen}(\alpha+\psi)\right], R\left[\cos\alpha - \cos(\alpha+\psi)\right]\right\}\cdot$$
$$\cdot\left\{R\cos(\alpha+\psi), -R\operatorname{sen}(\alpha+\psi)\right\} =$$

$$= -mgR\operatorname{sen}\alpha - mgR\operatorname{sen}(\alpha+\psi) + \frac{4\delta\, mgR}{2R^2(1-\cos\psi)}\left\{R^2\cancel{\operatorname{sen}\alpha\cos\alpha}\right.$$
$$\left. - R^2\cos\alpha\operatorname{sen}(\alpha+\psi) - R^2\cancel{\operatorname{sen}\alpha\cos\alpha} + R^2\operatorname{sen}\alpha\cos(\alpha+\psi)\right\}$$

$$- \frac{4\delta\, mgR}{2R^2(1-\cos\psi)}\left\{R^2\operatorname{sen}\alpha\cos(\alpha+\psi) - R\,\cancel{\operatorname{sen}(\alpha+\psi)\cos(\alpha+\psi)}\right.$$
$$\left. - R^2\cos\alpha\operatorname{sen}(\alpha+\psi) + R^2\cancel{\operatorname{sen}(\alpha+\psi)\cos(\alpha+\psi)}\right\}$$

$$= -mgR\operatorname{sen}\alpha - mgR\operatorname{sen}(\alpha+\psi) + \cancel{\frac{2\delta\, mgR}{(1-\cos\psi)}\left[\operatorname{sen}\alpha\cos(\alpha+\psi) - \right.}$$
$$\cancel{\left. - \cos\alpha\operatorname{sen}(\alpha+\psi)\right]} - \cancel{\frac{2\delta\, mgR}{(1-\cos\psi)}\left[\operatorname{sen}\alpha\cos(\alpha+\psi) - \cos\alpha\operatorname{sen}(\alpha+\psi)\right]}$$

$$\Rightarrow \boxed{Q_\alpha = -mgR\operatorname{sen}\alpha - mgR\operatorname{sen}(\alpha+\psi)}$$

$$Q_\psi = mg \cdot \cancel{\frac{\partial P}{\partial \psi}} + mg \cdot \frac{\partial Q}{\partial \psi} + F \cdot \cancel{\frac{\partial P}{\partial \psi}} - F \cdot \frac{\partial Q}{\partial \psi}$$

$$= mg(0,1) \cdot [R\cos(\alpha+\psi), -R\sin(\alpha+\psi)]$$

$$- \frac{\cancel{4}\, \alpha\, mgR}{2R^2[1-\cos\psi]} \cdot [R(\sin\alpha - \sin(\alpha+\psi)), R(\cos\alpha - \cos(\alpha+\psi))]$$

$$\cdot [R\cos(\alpha+\psi), -R\sin(\alpha+\psi)]$$

$$= -mgR\sin(\alpha+\psi) - \frac{4\alpha mgR}{2R^2(1-\cos\psi)} \{R^2 \sin\alpha \cos(\alpha+\psi)$$

$$- R^2 \cancel{\sin(\alpha+\psi)} \cos(\alpha+\psi) - R^2\cos\alpha\sin(\alpha+\psi) + R^2 \cancel{\sin(\alpha+\psi)\cos(\alpha+\psi)}$$

$$= -mgR\sin(\alpha+\psi) - \frac{2\alpha mgR}{(1-\cos\psi)} [\underbrace{\sin\alpha\cos(\alpha+\psi) - \cos\alpha\sin(\alpha+\psi)}_{\sin(\cancel{\alpha}-(\cancel{\alpha}+\psi)) = -\sin\psi}]$$

$$\boxed{Q_\psi = -mgR\sin(\alpha+\psi) + 2\alpha mgR\frac{\sin\psi}{1-\cos\psi}}$$

"EQUILIBRIO"

$$\begin{matrix} Q_\alpha = 0 \\ Q_\psi = 0 \end{matrix} \Rightarrow \begin{cases} \sin\alpha + \sin(\alpha+\psi) = 0 & (1) \\ -\sin(\alpha+\psi) + 2\alpha \dfrac{\sin\psi}{1-\cos\psi} = 0 & (2) \end{cases}$$

BASTA STUDIARE IL CASO $k=0,1$

DALLA (1) $\sin(\alpha+\psi) = -\sin(\alpha) = \sin(-\alpha) \Rightarrow$ $\begin{cases} A)\ \alpha+\psi = 2k\pi - \alpha \\ B)\ \alpha+\psi = \pi - (-\alpha) \end{cases}$

A)

DALLA PRIMA $\Rightarrow$ $\psi = 2\pi - 2\alpha$

OSSERVO CHE I VALORI $\alpha = 0, \pi, 2\pi$ NON SONO ACCETTABILI SE $0 < \psi < 2\pi$

INFATTI $\begin{cases} \text{SE } \alpha = 0 & \psi = 2\pi \\ \text{SE } \alpha = \pi & \psi = 0 \\ \text{SE } \alpha = 2\pi & \psi = -2\pi \end{cases}$

SOSTITUENDO NELLA (2) LA CONDIZIONE $\psi = 2\pi - \alpha$

$$-\operatorname{sen}(2\pi-\alpha) + 2d\,\frac{\operatorname{sen}(2\pi-2\alpha)}{1-\cos(2\pi-2\alpha)} = 0$$

$$\operatorname{sen}(2\pi-\alpha) = \operatorname{sen}(-\alpha) = -\operatorname{sen}\alpha$$

$$\operatorname{sen}(2\pi-2\alpha) = \operatorname{sen}(-2\alpha) = -\operatorname{sen}(2\alpha) = -2\operatorname{sen}\alpha\cos\alpha$$

$$\cos(2\pi-2\alpha) = \cos(-2\alpha) = \cos(2\alpha) = \cos^2\alpha - \operatorname{sen}^2\alpha$$

$$+\operatorname{sen}\alpha - \frac{4d\operatorname{sen}\alpha\cos\alpha}{1-\cos^2\alpha+\operatorname{sen}^2\alpha} = \operatorname{sen}\alpha - \frac{4d\,\cancel{\operatorname{sen}\alpha}\cos\alpha}{2\operatorname{sen}^2\alpha} =$$

$$= \operatorname{sen}\alpha - 2d\,\frac{\cos\alpha}{\operatorname{sen}\alpha} = 0 \Rightarrow \frac{\overbrace{\operatorname{sen}^2\alpha}^{1-\cos^2\alpha} - 2d\cos\alpha}{\operatorname{sen}\alpha} = 0$$

$\Rightarrow$ (ESSENDO $\operatorname{sen}\alpha \neq 0$)

$\Rightarrow \cos^2\alpha + 2d\cos\alpha - 1 = 0$ PONIAMO $\cos\alpha = x$

$\Rightarrow x^2 + 2dx - 1 = 0 \Rightarrow x = \cos\alpha = -d \pm \sqrt{d^2+1}$

SCARTIAMO LA SOLUZIONE $-d - \sqrt{d^2+1} < -1$ $(-1 \leq \cos\alpha \leq 1)$

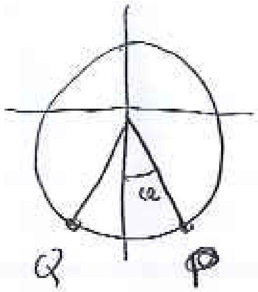
AVREMO $\cos\bar\alpha = -d + \sqrt{d^2+1} > 0$ $(0 < \cos\bar\alpha \leq 1)$ ALLORA $\bar\alpha = \arccos(-d+\sqrt{d^2+1})$

QUINDI $\theta_1 = \arccos(-d+\sqrt{d^2+1}) = \bar\alpha$ $\quad \psi_1 = 2\pi - 2\bar\alpha$

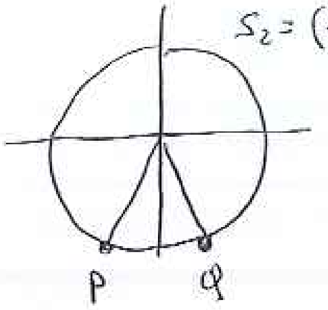
$\theta_2 = -\bar\alpha$ $\quad$ e $\quad \psi_2 = 2\pi + 2\bar\alpha = 2\bar\alpha$

PERCHÉ $0 < \psi < 2\pi$

$s_1 = (\theta_1, \psi_1) = (\bar\alpha, 2\pi - 2\bar\alpha)$ $\qquad s_2 = (\theta_2, \psi_2) = (-\bar\alpha, 2\bar\alpha)$



$S_1 = (\alpha, 2\pi - 2\alpha)$



$S_2 = (-\alpha, 2\alpha)$

B)

Si consideri l'altra condizione

$$\alpha + \psi = \pi + \alpha \quad \Rightarrow \quad \psi = \pi$$

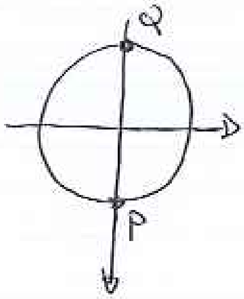
Dalla (2) $\quad \sin(\alpha + \pi) = 0 \quad \Rightarrow \quad -\sin(\alpha) = 0 \quad \Rightarrow \quad \alpha = 0, \pi$

Avremo le due configurazioni

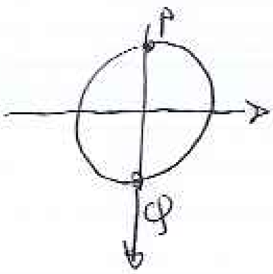
$$S_3 = (\alpha_3, \psi_3) = (0, \pi)$$

$$S_4 = (\alpha_4, \psi_4) = (\pi, \pi)$$

$S_3 = (0, \pi)$



$S_4 = ($



Studiamo la stabilità:

$$\frac{\partial Q_\alpha}{\partial \alpha} = \frac{\partial^2 U}{\partial \alpha^2} = -mgR\cos\alpha - mgR\cos(\alpha + \psi)$$

$$\frac{\partial Q_\psi}{\partial \psi} = \frac{\partial^2 U}{\partial \psi^2} = -mgR\cos(\alpha+\psi) + 2\lambda mgR \, \frac{\overbrace{\cos\psi(1-\cos\psi) - \sin\psi \cdot \sin\psi}^{\cos\psi - 1}}{(1-\cos\psi)^2} =$$

$$= -mgR\cos(\alpha+\psi) - 2\lambda mgR \, \frac{\cancel{(1-\cos\psi)}}{(1-\cos\psi)^{\cancel{2}}}$$

$$= -mgR\left\{\cos(\alpha+\psi) + \frac{2\lambda}{1-\cos\psi}\right\}$$

$$\frac{\partial^2 U}{\partial\alpha\,\partial\psi} = -mgR\cos(\alpha+\psi)$$

SE CALCOLIAMO QUESTE QUANTITÀ NEI 4 PUNTI

A)

$$\left.\frac{\partial^2 U}{\partial \varphi^2}\right|_{S_1,S_2} = -mgR\cos(\bar\varphi) - mgR(\cos\bar\varphi) = -2mgR\cos\bar\varphi < 0$$

PERCHÉ $0 < \cos\bar\varphi \leq 1$

$$\left.\frac{\partial^2 U}{\partial \psi^2}\right|_{S_1 S_2} = -mgR\left[\cos\bar\varphi + \frac{2\lambda}{1-\cos(2\bar\varphi)}\right] < 0$$

$$\left.\frac{\partial^2 U}{\partial\varphi\partial\psi}\right|_{S_1 S_2} = -mgR\cos(\bar\varphi) < 0$$

DA CUI:

$$H\big|_{S_1,S_2} = 2(mgR)^2\left[\cos\varphi^2 + \frac{2\lambda\cos(\bar\varphi)}{1-\cos(2\bar\varphi)}\right] - (mgR)^2\cos^2\varphi =$$

$$= (mgR)^2\left[\cos^2\varphi + \frac{4\lambda\cos(\bar\varphi)}{1-\cos(2\bar\varphi)}\right] > 0$$

QUINDI: S_1, S_2 SONO CONF. DI MASSIMO PER U $\Rightarrow$ $S_1\ S_2$ SONO STABILI

B)

$$\left.\frac{\partial^2 U}{\partial\varphi^2}\right|_{S_3,S_4} = -mgR(\pm 1) - mgR(\mp 1) = 0$$

$$\left.\frac{\partial^2 U}{\partial\psi^2}\right|_{S_3,S_4} = -mgR\{\ \cdots\ \}$$

$$\left.\frac{\partial^2 U}{\partial\varphi\partial\psi}\right|_{S_3,S_4} = -mgR(\mp 1)$$

QUINDI: $H_{S_3,S_4} = -(mgR)^2 < 0$

QUINDI NON SONO MASSIMI $\Rightarrow$ S_3, S_4 SONO INSTABILI

CALCOLIAMO L'ENERGIA CINETICA

$$T = \frac{1}{2} m \dot{P}^2 + \frac{1}{2} m \dot{Q}^2$$

$$\dot{P} = [R \dot{\varphi} \cos\varphi , -R\dot{\varphi} \sin\varphi] \Rightarrow \dot{P}^2 = R^2 \dot{\varphi}^2$$

$$\dot{Q} = [R(\dot{\varphi}+\dot{\psi}) \cos(\varphi+\psi), -R(\dot{\varphi}+\dot{\psi}) \sin(\varphi+\psi)]$$

$$\dot{Q}^2 = R^2(\dot{\varphi}+\dot{\psi})^2 = R^2(\dot{\varphi}^2+\dot{\psi}^2+2\dot{\varphi}\dot{\psi})$$

$$T = \frac{1}{2} m R^2 (2\dot{\varphi}^2+\dot{\psi}^2+2\dot{\varphi}\dot{\psi}) = m R^2 [\dot{\varphi}^2+\dot{\varphi}\dot{\psi}+\frac{1}{2}\dot{\psi}^2]$$

$$\frac{\partial T}{\partial \dot{\varphi}} = m R^2 [2\dot{\varphi}+\dot{\psi}] \Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} = m R^2 [2\ddot{\varphi}+\ddot{\psi}]$$

$$\frac{\partial T}{\partial \varphi} = 0 \qquad \frac{\partial T}{\partial \psi} = 0$$

$$\frac{\partial T}{\partial \dot{\psi}} = m R^2 [\dot{\psi}+\dot{\varphi}] \Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} = m R^2 [\ddot{\psi}+\ddot{\varphi}]$$

<u>DA CUI:</u>

$$\begin{cases} \cancel{m} R^{\cancel{2}} [2\ddot{\varphi}+\ddot{\psi}] = -\cancel{m} g \cancel{R} \{\sin\varphi + \sin(\varphi+\psi)\} \\ \cancel{m} R^{\cancel{2}} [\ddot{\varphi}+\ddot{\psi}] = -\cancel{m} g \cancel{R} \left\{ \sin(\varphi+\psi) - 2\alpha \frac{\sin\psi}{1-\cos\psi} \right\} \end{cases}$$

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INTEGRALI PRIMI:

1) FORZE CONSERVATIVE $E = T - V = \text{costante}$

2) NON ABBIAMO VARIABILI CICLICHE

3) LE EQUAZIONI SONO ACCOPPIATE.

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Moti in prima approssimazione:

Consideriamo la configurazione di equilibrio statico

$$S_1 = (\bar{\theta},\ 2\bar{u} - 2\bar{\theta}) = (\bar{\theta},\ -2\bar{\theta}) \qquad \text{con } \cos\bar{\theta} = -\alpha + \sqrt{\alpha^2+1}$$

e linearizziamo le due equazioni attorno.

$$S_1 \equiv (\bar{\theta}, -2\bar{\theta}, 0, 0, 0, 0)$$

$$2R\ddot{\theta} + R\ddot{\psi} + g\left[\sin\theta + \sin(\theta+\psi)\right] = F(\theta,\psi,\dot{\theta},\dot{\psi},\ddot{\theta},\ddot{\psi}) = 0$$

$$R\ddot{\theta} + R\ddot{\psi} + g\left[\sin(\theta+\psi) - 2\alpha\frac{\sin\psi}{1-\cos\psi}\right] = G(\theta,\psi,\dot{\theta},\dot{\psi},\ddot{\theta},\ddot{\psi}) = 0$$

$$F\Big|_{S_1} = 0 \qquad \frac{\partial F}{\partial\ddot{\theta}} = 2R \qquad \frac{\partial F}{\partial\ddot{\psi}} = R \qquad \frac{\partial F}{\partial\dot{\theta}} = \frac{\partial F}{\partial\dot{\psi}} = 0$$

$$\frac{\partial F}{\partial\theta}\Big|_{S_1} = g\left[\cos\theta + \cos(\theta+\psi)\right]\Big|_{S_1} = g\left[\cos(\bar{\theta}) + \cos(-\bar{\theta})\right] = 2g\cos\bar{\theta}$$

$$= 2g\left[\sqrt{\alpha^2+1} - \alpha\right]$$

$$\frac{\partial F}{\partial\psi} = g\cos(\theta+\psi)\Big|_{S_1} = g\cos(-\bar{\theta}) = g\cos\bar{\theta} = g\left[\sqrt{\alpha^2+1} - \alpha\right]$$

da cui

$$2R\ddot{\theta} + R\ddot{\psi} + \overbrace{2g\left[\sqrt{\alpha^2+1} - \alpha\right]}^{K}(\theta - \bar{\theta}) + \overbrace{g\left[\sqrt{\alpha^2+1} - \alpha\right]}^{H}(\psi + 2\bar{\theta}) = 0$$

$$G\Big|_{S_1} = +\frac{1}{\sin(\bar{\theta})}\left[\cos^2\bar{\theta} + 2\alpha\cos(\bar{\theta}) + 1\right] = 0$$

$$\frac{\partial G}{\partial\ddot{\theta}} = R\ ; \quad \frac{\partial G}{\partial\ddot{\psi}} = R\ ; \quad \frac{\partial G}{\partial\dot{\theta}} = \frac{\partial G}{\partial\dot{\psi}} = 0$$

$$\frac{\partial G}{\partial\theta} = g\left\{\cos(\theta+\psi)\right\}\Big|_{S_1} = g\cos(-\bar{\theta}) = g\cos(\bar{\theta})$$

$$\frac{\partial G}{\partial\psi} = g\left\{\cos(\theta+\psi) - 2\alpha\frac{\overbrace{\cos\psi(1-\cos\psi) - \sin\psi\sin\psi}^{(\cos\psi - 1)}}{(1-\cos\psi)^2}\right\}\Bigg|_{S_1} =$$

$$\frac{\partial G}{\partial \psi} = g\left\{ \cos(\theta+\psi) + \frac{2\alpha}{1-\cos\psi}\right\}_{S_1} = g\left[\cos(-\bar\theta) + \frac{2\alpha}{1-\cos(2\bar\theta)}\right]$$

$$= g\left[\cos(\bar\theta) + \frac{2\alpha}{1-\cos^2(\bar\theta)+\sin^2(\bar\theta)}\right] = g\left[\cos(\bar\theta) + \frac{\cancel{2}\alpha}{\cancel{2}\sin^2(\bar\theta)}\right] =$$

$$= g\left[\cos(\bar\theta) + \frac{\alpha}{1-\cos^2(\bar\theta)}\right]$$

NOTA: essendo $\cos\bar\theta = -\alpha + \sqrt{\alpha^2+1} \Rightarrow \cos^2\bar\theta = \alpha^2+\alpha^2+1-2\alpha\sqrt{\alpha^2+1} =$

$= 2\alpha^2+1-2\alpha\sqrt{\alpha^2+1} \Rightarrow 1-\cos^2\bar\theta = 2\alpha\left[-\alpha+\sqrt{\alpha^2+1}\right] = 2\alpha\cos(\bar\theta)$

QUINDI:

$$\frac{\partial G}{\partial \psi}\Big|_{S_1} = g\left[\cos(\bar\theta) + \frac{\cancel{\alpha}}{2\cancel{\alpha}\cos\bar\theta}\right] = g\left[\cos(\bar\theta) + \frac{1}{2\cos(\bar\theta)}\right]$$

DA CUI AVREMO:

$$2R\ddot\theta + R\ddot\psi + \left[2g\cos\bar\theta\right](\theta-\bar\theta) + (g\cos\bar\theta)(\psi+2\bar\theta) = 0$$

$$R\ddot\theta + R\ddot\psi + (g\cos\bar\theta)(\theta-\bar\theta) + g\left[\cos\bar\theta + \frac{1}{2\cos(\bar\theta)}\right](\psi+2\bar\theta) = 0$$

DA CUI CERCHIAMO SOLUZIONI

$\psi + 2\bar\theta = \psi_0 e^{\lambda t}$; $\theta - \bar\theta = \theta_0 e^{\lambda t}$

$$2R\lambda^2\theta_0 + R\lambda^2\psi_0 + (2g\cos\bar\theta)\theta_0 + (g\cos\bar\theta)\psi_0 = 0$$

$$R\lambda^2\theta_0 + R\lambda^2\psi_0 + (g\cos\bar\theta)\theta_0 + g\left(\cos\bar\theta + \frac{1}{2\cos\bar\theta}\right)\psi_0 = 0$$

DA CUI LE DUE EQUAZIONI ALGEBRICHE NELLE VARIABILI θ_0, ψ_0

$$2\left[R\lambda^2 + g\cos\bar\theta\right]\theta_0 + \left[R\lambda^2 + g\cos\bar\theta\right]\psi_0 = 0$$

$$\left[R\lambda^2 + g\cos\bar\theta\right]\theta_0 + \left\{R\lambda^2 + g\left[\cos\bar\theta + \frac{1}{2\cos\bar\theta}\right]\right\}\psi_0 = 0$$

Da cui l'equazione secolare

$$2\left[R\lambda^2+g\cos\bar{\vartheta}\right]\left\{R\lambda^2+g\left[\cos\bar{\vartheta}+\frac{1}{2\cos\bar{\vartheta}}\right]\right\}-$$

$$-\left[R\lambda^2+g\cos\bar{\vartheta}\right]^2=0$$

$$\Rightarrow\left[R\lambda^2+g\cos\bar{\vartheta}\right]\left\{2R\lambda^2+2g\left[\cos\bar{\vartheta}+\frac{1}{2\cos\bar{\vartheta}}\right]-\left[R\lambda^2+g\cos\bar{\vartheta}\right]\right\}=0$$

$$\Rightarrow\boxed{\lambda^2=-\frac{g}{R}\cos\bar{\vartheta}}$$

$$\Rightarrow\cancel{2}R\lambda^2+g\left[\cancel{2}\cos\bar{\vartheta}+\frac{1}{\cos\bar{\vartheta}}-\cancel{\cos\bar{\vartheta}}\right]=0$$

$$\Rightarrow\boxed{\lambda^2=-\frac{g}{R}\left[\cos\bar{\vartheta}+\frac{1}{\cos\bar{\vartheta}}\right]}$$

Due moti armonici di pulsazione

$$\omega_1=\sqrt{\frac{g}{R}\cos\bar{\vartheta}}\qquad\omega_2=\sqrt{\frac{g}{R}\left(\cos\bar{\vartheta}+\frac{1}{\cos\bar{\vartheta}}\right)}$$

con $\lambda_{1,2}=\pm i\,\omega_1$ $\qquad\lambda_{3,4}=\pm i\,\omega_2$

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