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A COMBINATORIAL GENERALIZATION OF THE DURFEE SQUARE

MARIA ARTALE

Introduction.

The Durfee square of a partition λ, $D(\lambda)$, is defined as the largest square contained in the shape of λ.

It was proved in [2] (cf. also [3]) that the size of $D(\lambda)$, $d(\lambda)$, was related to the perfection of a certain module M_λ, an algebro-geometric object (cf. also [1], [4], [5]).

The goal of this note is to propose a generalization of the notion of Durfee square to the case of a pair (α, β) of partitions.

More precisely, given two partitions α and β s.t. the last row of β is shorter then the first row of α, we define in Section 2, a partition $D(\alpha, \beta)$ (not necessarily a square), which we call "generalized Durfee partition of α w.r.t. β". $D(\alpha, \beta)$ is also related to algebro-geometric problems, as it is indicated for instance by the fact (proven in Section 3) that $D(\alpha, \beta)$ allows us to construct Lascoux's rectification of α and β (cf. [8]).

We conjecture that $D(\alpha, \beta)$ might encode important information on the minimal free resolution of significant classes of modules (see, for instance [10]).

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1. Preliminaries.

In this section we recall some background material (cf. [1], [6], [7], [9]).

By a partition λ we mean a weakly increasing sequence of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_t)$.

The non-zero numbers λ_i are called the parts of λ; its length $t = l(\lambda)$ is the number of λ_i which are non zero. We identify two partitions which differ only by zeros added on the left. If $\lambda_1 + \lambda_2 + \dots + \lambda_t = n$, then n is called the weight of λ and denoted by $n = |\lambda|$. The integer d such that $\lambda_{t-i} \geq i + 1$ for $i = 0, 1, \dots, d-1$ and $\lambda_{t-d} \leq d$ is called the size of the Durfee square of λ (the Durfee square of λ is the largest square partition contained in λ), for example, if $\lambda = (1, 3, 4)$, then $d = 2$.

Sometimes we use a notation for λ which indicates the number of times each integer occurs as a part;

$$\lambda = (1^{m_1}, 2^{m_2}, \dots, k^{m_k}, \dots)$$

means that exactly n_k of the parts of λ are equal to k, then the sequence $\{n_1, n_2, \dots, n_i, \dots\}$ is called the multiplicity of λ denoted by $mult(\lambda) = \{n_1, n_2, \dots, n_i, \dots\}$.

Clearly, if some $m_i = 0$, then $i^{m_i} = \emptyset$.

For two partitions λ, μ we write $\lambda \supseteq \mu$ if $\lambda_k \geq \mu_k$ for all k. If the columns of the diagram of λ are of lenghts $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_p$ in a weakly increasing order, then the partition $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_p)$ is called the conjugate of λ and denoted by $\widetilde{\lambda}$.

For two partitions $\lambda \supseteq \mu$, we write λ/μ for the corresponding skew partition. Its diagram is obtained as a set-theoretic difference of the diagrams of λ and μ. The diagram of λ/μ has rows of lengths $\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots$. The weight $|\lambda/\mu|$ of λ/μ is defined as the difference $|\lambda| - |\mu|$. We denote $\widetilde{(\lambda/\mu)} = \widetilde{\lambda}/\widetilde{\mu}$ and call it the conjugate of λ/μ.

The partition $\lambda \times \mu$, product of two partitions $(\lambda_1, ..., \lambda_t)$, $(\mu_1, ..., \mu_q)$, is the skew partition

$$(\lambda_1, \dots, \lambda_t + \mu_1, \lambda_t + \mu_2, \dots, \lambda_t + \mu_q)\Big/(\lambda_t^q).$$

Let $\lambda = (\lambda_1, \dots, \lambda_t)$ and $\mu = (\mu_1, \dots, \mu_q)$ be two sequences in $\mathbb{Z}^t$ and $\mathbb{Z}^q$ respectively, then $P(\lambda, \mu)$ and $m(\lambda, \mu)$ have the following meaning: we consider the set

$$\delta = \left\{\lambda_1 + 1, \lambda_2 + 2, \lambda_3 + 3, \ldots, \lambda_t + t, \mu_1 + t + 1, \ldots, \mu_q + t + q\right\},$$

then two cases are possible; either all these numbers are positives and different, and then this set can be uniquely written as $\delta' = (\delta_1, \ldots, \delta_{t+q})$, with δ' a partition after reordering δ and $m(\lambda, \mu) =$ minimal number of traspositions necessary to get δ' from δ, or otherwise, we let $\delta' = \emptyset, \quad m(\lambda, \mu) = \infty$.

Notice that if λ, μ are partitions such that $\lambda_t \leq \mu_1$, then $\delta' = \delta$ and $m(\lambda, \mu) = 0$.

Now if $\delta' \neq \emptyset$, by definition [8] we have

$$P(\lambda, \mu) = (\delta_1 - 1, \delta_2 - 2, \ldots, \delta_{t+q} - t - q).$$

Otherwise $P(\lambda, \mu) = \emptyset$.

$P(\lambda, \mu)$ (if non-empty) is called the rectification of the sequence (λ, μ).

Remark 1.1. Notice that $\delta' \neq \emptyset$ if and only if the set $\delta = \{\lambda_1 + 1, \ldots, \lambda_t + t, \mu_1 + t + 1, \ldots, \mu_q + t + q\}$ consists of $t + q$ elements (positive integers) displayed in a certain order, uniquelty defined by the sequence (λ, μ). More precisely (λ, μ) defines a permutation $\sigma \in \mathcal{S}_{t+q}$ on the set δ such that $m(\lambda, \mu) = l(\sigma) =$ length of the permutation σ.

Clearly $P(\lambda, \mu) = (\lambda, \mu)$ if and only if $m(\lambda, \mu) = 0$, if and only if $\lambda_t \leq \mu_1$.

Example 1.1. For $\lambda = (1, 4, 3), \quad \mu = (1, 5)$, we have $P(\lambda, \mu) = \emptyset, \quad m(\lambda, \mu) = \infty$. If $\lambda = (1, 4), \quad \mu = (2, 0, 5)$, then $\delta = \{2, 6, 5, 4, 10\}$, hence

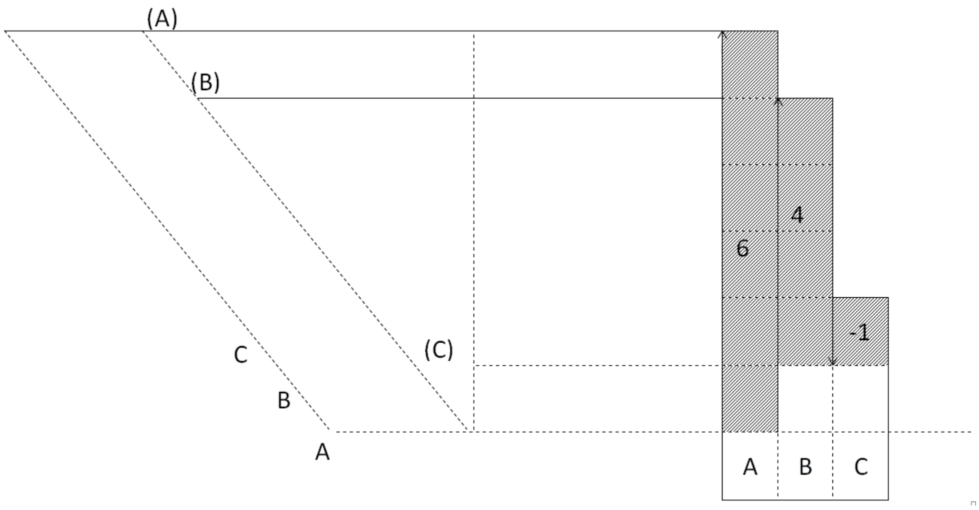
$$\sigma = \begin{pmatrix} 2 & 4 & 5 & 6 & 10 \\ 2 & 6 & 5 & 4 & 10 \end{pmatrix}, \quad l(\sigma) = m(\lambda, \mu) = 3,$$
$$P(\lambda, \mu) = (1, 2, 2, 2, 5).$$

The following examples illustrate the process of "rectification" of all the sequences $(\lambda_1, \ldots, \lambda_s) \in \mathbb{Z}^s$ which define the same partition λ (if non-empty).

Consider the sequence $\lambda = (6, 4, -1) \in \mathbb{Z}^3$, then λ defines uniquely the permutation $\sigma \in \mathcal{S}_3$.

$$\sigma = \begin{pmatrix} A & B & C \\ (C) & (B) & (A) \end{pmatrix}, \qquad l(\sigma) = 3$$

obtained as shown below:



Notice that from the last row of the permutation σ above, we can read directly the partition $\lambda = (1, 4, 4) = P(6, 4, -1)$. In fact $C = -1$ goes to the first place via σ, so $(C) = -1 + 2 = 1$, $B = 4$ remains in the same place via σ, so $(B) = 4$, finally A = 6 goes to the third place via σ, so $(A) = 6 - 2 = 4$.

With the same procedure, considering all the other permutations of $\mathcal{S}_3$, for $\lambda = (1, 4, 4)$ we obtain:

$$\sigma = \begin{pmatrix} A & B & C \\ (A) & (B) & (C) \end{pmatrix} \quad , \quad length = 0 \qquad \longrightarrow \qquad \lambda = (1, 4, 4)$$

$$\sigma = \begin{pmatrix} A & B & C \\ (A) & (C) & (B) \end{pmatrix} \quad , \quad length = 1 \qquad \longrightarrow \qquad \lambda = (1, 5, 3)$$

$$\sigma = \begin{pmatrix} A & B & C \\ (B) & (A) & (C) \end{pmatrix} \quad , \quad length = 1 \qquad \longrightarrow \qquad \lambda = (5, 0, 4)$$

$$\sigma = \begin{pmatrix} A & B & C \\ (B) & (C) & (A) \end{pmatrix} \quad , \quad length = 2 \qquad \longrightarrow \qquad \lambda = (6, 0, 3)$$

$$\sigma = \begin{pmatrix} A & B & C \\ (C) & (A) & (B) \end{pmatrix} \quad , \quad length = 2 \qquad \longrightarrow \qquad \lambda = (5, 5, -1).$$

2. Key definition.

In this section we give our main definition of generalized Durfee square. More precisely:

Definition 2.1. *Let* $\alpha = (\alpha_1, \dots, \alpha_t)$, $\beta = (\beta_1, \dots, \beta_q)$ *be two partitions, by the Durfee square of* α *with respect to* β *we mean the partition*

$$D(\alpha, \beta) = (d_q, d_{q-1}, \dots, d_1) \subseteq \underbrace{(d, \dots, d)}_{q},$$

where d *is the Durfee square of* α *and* d_j, *for* $j = 1, 2, \dots, q$, *are such that*

$$\alpha_{t-i} \geq i + 1 + \beta_j \quad \textit{for} \quad i = 0, 1, \dots, d_j - 1$$

and

$$\alpha_{t-d_j} \leq d_j + \beta_j.$$

Remark 2.1.

i) If follows from the definition that $d_j = 0$ iff $\alpha_t \leq \beta_j$. Clearly $d_j \neq 0$ is the largest integer such that the rectangular partition $(d_j + \beta_j)^{d_j} \subseteq \alpha$. Moreover $\beta_j \leq \beta_{j+1}$ implies that $d_j \geq d_{j+1}$.

ii) Notice that $d_1 \leq d$, hence $D(\alpha, \beta) \subseteq \underbrace{(d, \dots, d)}_{q}$, so that $0 \leq \left|D(\alpha, \beta)\right| \leq d \cdot q$.

iii) From the definition (2.1) if follows that

$$D(\alpha, \underbrace{(0 \dots 0)}_{i}) = \underbrace{(d, \dots, d)}_{i},$$

in particular

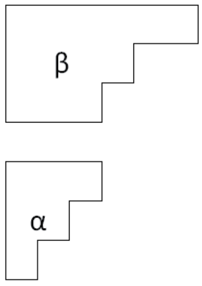
$$D(\alpha, \beta) = \underbrace{(0, \dots, 0)}_{q} \text{ if and only if } \beta_1 \geq \alpha_t,$$

Notice that

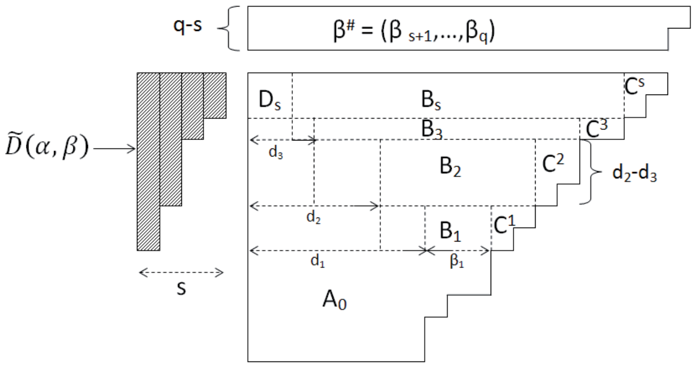
$$D(\alpha, \beta) \text{ is a rectangle if and only if } d_1 = d_2 = \cdots = d_q.$$

The pictures below illustrate the various cases:

a) $D(\alpha,\beta) = \underbrace{(0,\ldots,0)}_{q} \quad \Rightarrow$

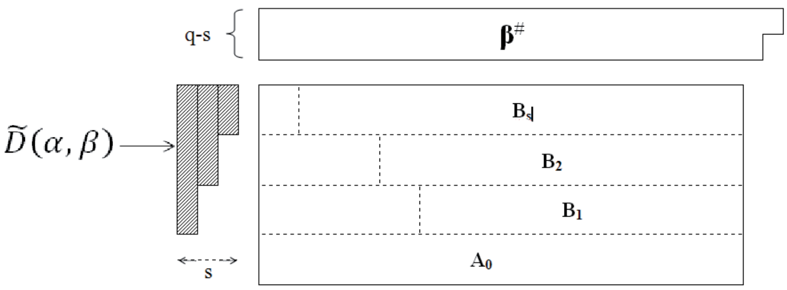


b) $D(\alpha,\beta) = (0^{q-s}, d_s, \ldots, d_1) \quad \Rightarrow$

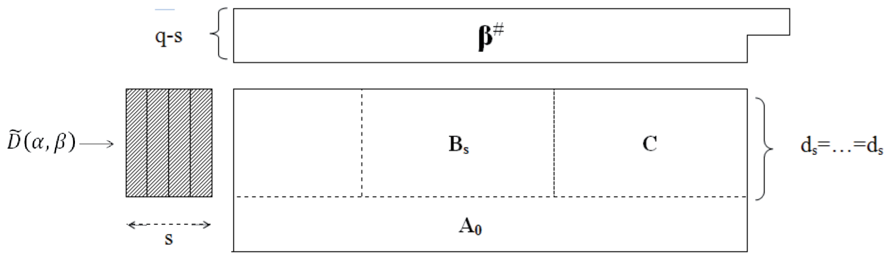


in particular we have the following cases

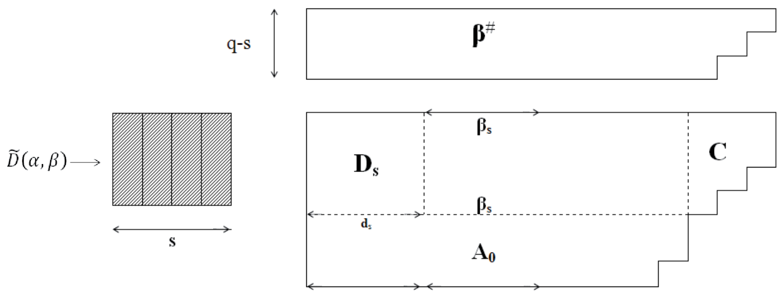
c) α rectangle with $D(\alpha,\beta)$ not a rectangle $\quad \Rightarrow$



d) α rectangle with $D(\alpha, \beta)$ rectangle $\Rightarrow$



e) α not a rectangle with $D(\alpha, \beta)$ a rectangle $\Rightarrow$



iv) Notice that:

$$\beta \subseteq \beta' \quad \rightarrow \quad D(\alpha, \beta') \subseteq D(\alpha, \beta)$$

$$\alpha \subseteq \alpha' \quad \rightarrow \quad D(\alpha, \beta) \subseteq D(\alpha', \beta).$$

If $\beta = \emptyset$, we let by definition $D(\alpha, \emptyset) = \emptyset$.

3. Main results.

What follows shows how the generalized notion of Durfee square is used to compute $P(\alpha, \beta)$. More precisely, we will show that if α is “large enough”, then $\left|D(\alpha, \beta)\right| = l(\sigma) = m(\alpha, \beta)$, where $\sigma \in \mathcal{S}_{t+q}$ is the permutation defined by the sequence $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_q\}$ (cf. Remark 1.1). To simplify the notations, given α, β, with $D(\alpha, \beta) = (d_q, \ldots, d_1)$, we write $\alpha = (A_0, A_1, \ldots, A_q)$, where:

$A_0 = (\alpha_1, \ldots, \alpha_{r-d_1})$,

$$A_k = ((d_k+\beta_k)^{d_k-d_{k+1}} + C^k) = (\alpha_{r-d_k+1}, \dots, \alpha_{r-d_{k+1}}),$$
$$C^k \subseteq ((d_{k+1}+\beta_{k+1}-d_k-\beta_k)^{d_k-d_{k+1}}),$$
$$C^k = (c_1^k, \dots, c^k_{d_k-d_{k+1}}).$$

Notice that, if $d_i = 0$, then for all $k \geq i$, $A_k = \emptyset$.

Theorem 3.1. *Let* $\alpha = (\alpha_1, \dots, \alpha_r)$, $\beta = (\beta_1, \dots, \beta_q)$, $D(\alpha, \beta) = (d_q, \dots, d_1)$. *Consider the partitions*

$$\beta' = (\beta_1, \beta_2+1, \dots, \beta_q+q-1), \qquad D(\alpha, \beta') = (d'_q, \dots, d'_1)$$

then

$*$ $P(\alpha, \beta) \neq \emptyset$ if and only if $\alpha \supseteq (A_0, \overline{A}_1, \overline{A}_2, \dots, \overline{A}_q)$, where

$$\overline{A}_k = ((d_k+\beta_k)^{d_k-d'_k}, (d'_k+\beta'_k+1)^{d'_k-d_{k+1}}).$$

In this case $m(\alpha, \beta) = \left|D(\alpha, \beta')\right| = l(\sigma)$ and

$**$ $P(\alpha, \beta) = (A_0, \widehat{A}_1, \dots, \widehat{A}_q)$

with

$***$ $\widehat{A}_k = ((d_k+\beta_k-k+1)^{d_k-d'_k} + \widehat{C}^k, d'_k+\beta_k, (d'_k+\beta_k)^{d'_k-d_{k+1}} + \overline{C}^k)$

where $\overline{C}^k$ and $\widehat{C}^k$ are such that

$$C^k \Big/ ((k-d_k+d'_k)^{d'_k-d_{k+1}}) = \widehat{C}^k \times \overline{C}^k;$$

otherwise $P(\alpha, \beta) = \emptyset$.

Proof. Let $q = 1$, then the process of rectification of the sequence $\{A_0, d_1+\beta_1+c_1^1, d_1+\beta_1+c_2^1, \dots, d_1+\beta_1+c^1_{d_1}, \beta_1\}$ leads, after d_1 steps, to the sequence

$$\{A_0, d_1+\beta_1, d_1+\beta_1+c_1^1-1, d_1+\beta_1+c_2^1-1, \dots, d_1+\beta_1+c^1_{d_1}-1\}$$

which is a partition if $c_1^1 \geq 1$. In this case $m(\alpha, \beta) = d_1$, otherwise $P(\alpha, \beta_1) = \emptyset$ and $m(\alpha, \beta) = \infty$. Now the claim follows by induction since

$$P(\alpha, (\beta_1, \beta_2)) = P(P(\alpha, \beta_1), \beta_2) \quad \text{and} \quad D(P(\alpha, \beta_1), \beta_2) = (d'_2).$$

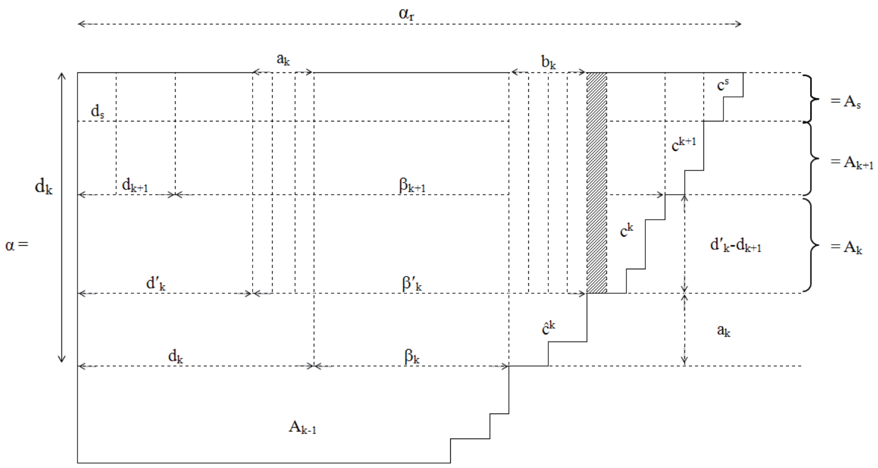
Notice that $0 \leq d_k - d'_k \leq k-1$. In particular

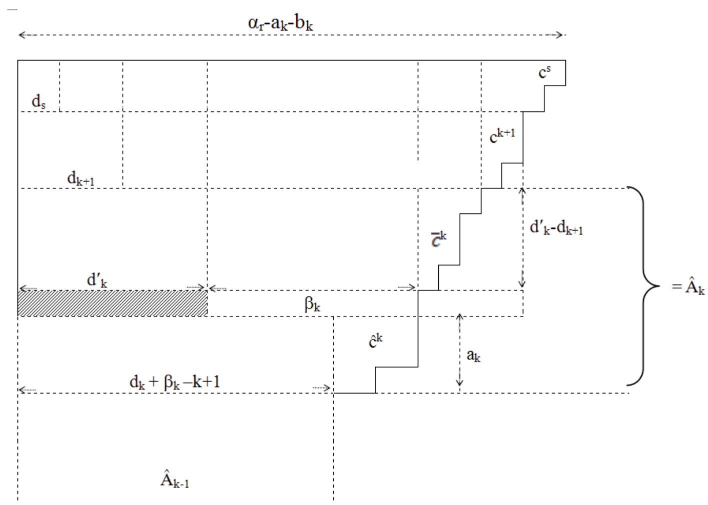
$$D(\alpha, \beta') \subseteq D(\alpha, \beta), \quad \text{so that} \quad 0 \leq m(\alpha, \beta) = \left|D(\alpha, \beta')\right| \leq \left|D(\alpha, \beta)\right|.$$

$\square$

Remark 3.1. The following pictures illustrate the k-step of induction in (Theorem 3.1).

Let us denote
$a_k = d_k - d'_k$, $0 \le a_k \le k-1$, $b_k + a_k = k-1$, $g_k = d_{k+1} - d'_k + \beta_{k+1} - \beta_k - k$ (see pictures below),





then it is clear from above that $\widehat{C}^k \subseteq \underbrace{(b_k, \ldots, b_k)}_{a_k}$ and

$\overline{C}^k \subseteq \underbrace{(g_k, \ldots, g_k)}_{d'_k - d_{k+1}}$.

Corollary 3.1. *Let α be a rectangle, $D(\alpha,\beta) \neq (0^q)$, then $P(\alpha,\beta) \neq \emptyset$ if and only if*

i) $D(\alpha,\beta) = (d_q, d_{q-1}, \dots, d_1)$ *is a rectangle,*

ii) $d_1 = l(\alpha)$ *(notice that in general $d_1 \leq l(\alpha)$),*

iii) $l(\widetilde{\alpha}) \geq \beta_s + d_1 + s, \qquad s = l(D(\alpha,\beta))$.

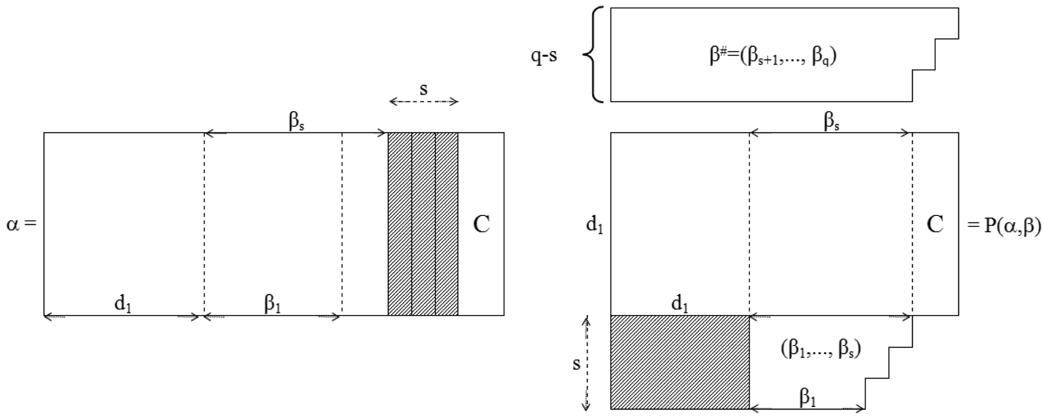
If the above conditions are satisfied, then

$$P(\alpha,\beta) = ((d_1^s) + (\beta_1, \dots, \beta_s), (\alpha_1 - s)^{d_1}, \beta^{\#})$$

and

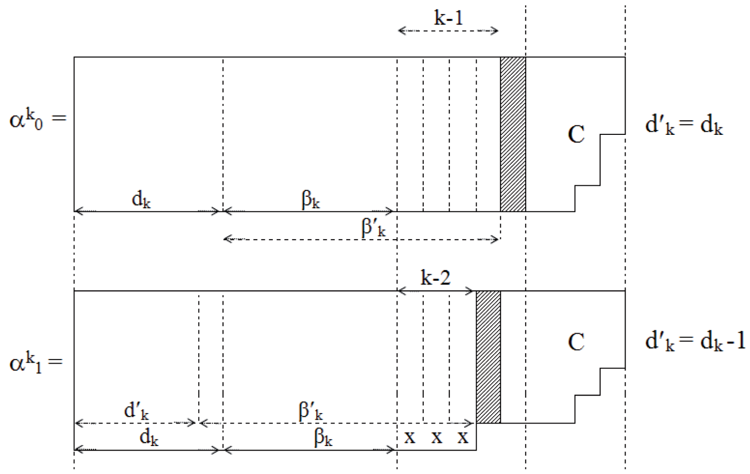
$$m(\alpha,\beta) = d_1 \cdot s.$$

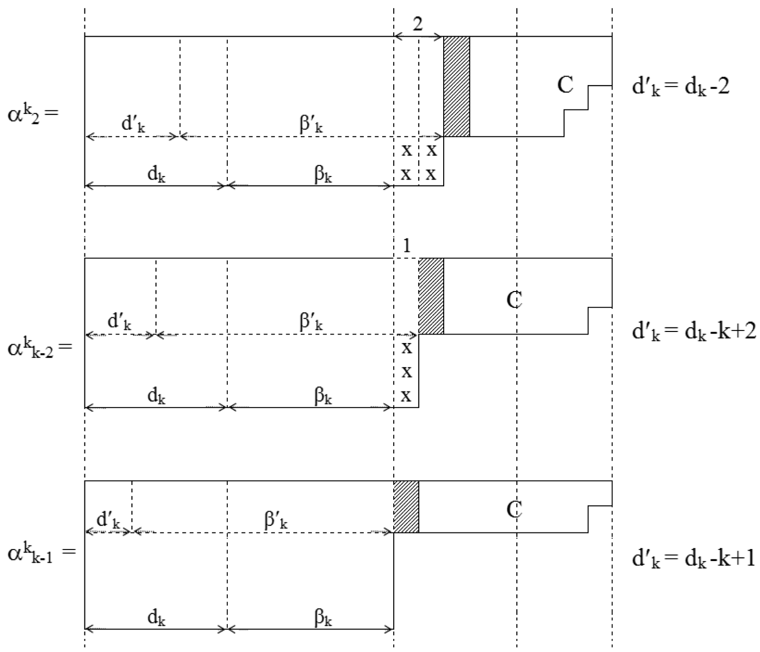
Proof. Let $q = 2$, $D(\alpha,(\beta_1,\beta_2)) = (d_2,d_1)$. Assume $d_2 < d_1$, then $\alpha_1 = d_2 + \beta_2 = d_1 + \beta_1$, $\quad d_2' = d_2 - 1$ so $P(\alpha,\beta) = \emptyset$. The general case is clear, see pictures below:



4. Examples of $P(\alpha,\beta)$.

Example 4.1. Let $s = l(D(\alpha,\beta))$, then $P(\alpha,\beta) \neq \emptyset$ if and only if the diagram of $\alpha^k = (\alpha_{r-d_k+1}, \dots, \alpha_r)$, for $k = 1, \dots, s$, is one of the diagrams shown below:



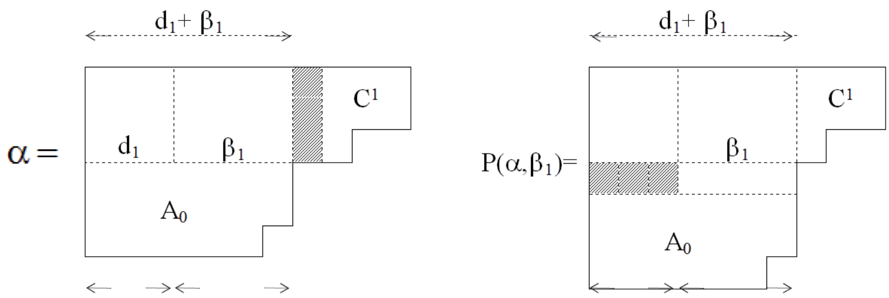


$$\beta'_k = \beta_k + k - 1, \text{ where } \beta = (\beta_1, \ldots, \beta_q), \ k = 1, 2, \ldots, q$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{r-d_1}, \alpha_{r-d_1+1} \ldots \alpha_{r-d_2}, \ldots, \alpha_{r-d_q+1}, \ldots, \alpha_r).$$

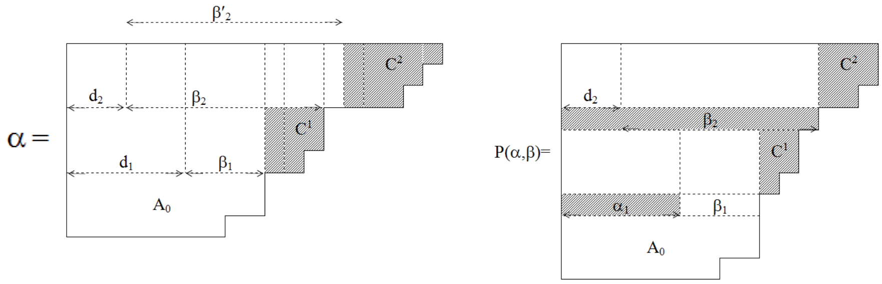
Example 4.2. The diagrams below describe pictorially the calculation of $P(\alpha, \beta)$, for $s = 1, 2, 3$, where $s = l(D(\alpha, \beta))$.

a) $s = 1$, $\beta = (\beta_1)$, $D(\alpha, \beta) = (d_1)$, $\quad d_1 > 0$.

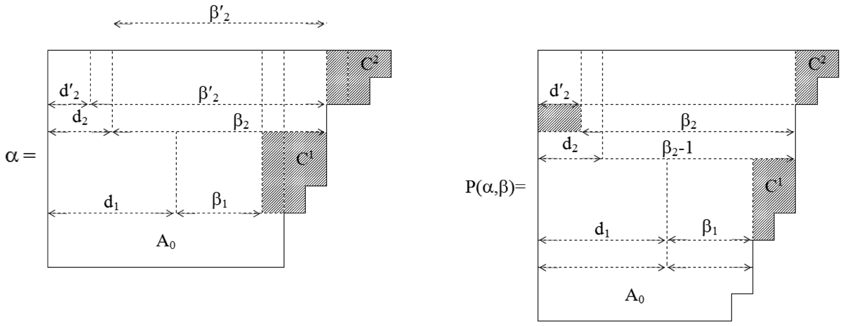


For example for $\alpha = (9, 11, 11)$, $\beta = (4)$, $\quad D(\alpha, \beta) = (3)$, $P(\alpha, \beta) = (7, 8, 10, 10)$;

b) $s = 2$, $\beta = (\beta_1, \beta_2)$, $\quad D(\alpha, \beta) = (d_2, d_1)$, $\quad d_2 \neq 0$:

i) 

$$d_1 + \beta_1 < d_2 + \beta_2, \quad d_2' = d_2.$$

ii) 

$$d_1 + \beta_1 < d_2 + \beta_2, \quad d_2' = d_2 - 1.$$

For example:

i) $\alpha = (12,13,13,17,18,19)$, $\beta = (4,10)$, $D(\alpha,\beta) = (3,6)$, $P(\alpha,\beta) = (10,11,12,12,13,15,16,16)$.

ii) $\alpha = (12,13,13,13,15,17)$, $\beta = (4,10)$, $D(\alpha,\beta) = (3,6)$.

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Dipartimento di Matematica
Seconda Università degli Studi di Roma Tor Vergata
via della Ricerca Scientifica
00173 - Roma
e-mail: artale@mat.uniroma2.it