LONG-WAVELENGTH INSTABILITIES IN BINARY FLUID LAYERS

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It is known that a fluid layer heated from below,[3] when the boundaries are poorly conducting, gives rise to *long-wavelength* instabilities (see e.g. Refs. 5,7). The same effect appears[8] also in the case of convection in a porous medium.[19] In this work we investigate, analytically and numerically,[6,18] how this effect is influenced by a stabilizing solutal field, both in the case of a fluid layer and in porous media. The solutal field is assigned through fixed concentrations at the boundaries, or more general Robin boundary conditions. The present work concerns a linear stability analysis of the problem and it is part of a larger project including a nonlinear analysis.[9,12,18]

Keywords: Bénard, Porous media, binary fluid, long wavelength, Newton-Robin

1. Introduction

In the simple Bénard problem the instability is driven by a density difference caused by a temperature difference between the lower and upper planes bounding the fluid. When the temperature gradient reaches a critical value the fluid gives rise to a regular pattern of motion (onset of convection).

If the fluid layer additionally has a *solute* dissolved in it, we have a binary fluid mixture and the phenomenon of convection which arises is called *double diffusive convection*. The study of stability and instability of motions of a binary fluid mixture heated and salted from below is relevant in many geophysical applications[1,15,20] (see also Refs. 10,18 and the references therein). It has been studied both in the linear and nonlinear case. Also double diffusive convection in porous layers has many applications.[13,16,19]

Here we consider the problem of a layer heated and salted from below. This means that in the motionless basic state we will have a positive concentration gradient, having a stabilizing effect. The critical linear instability thresholds have been studied in the case of rigid and stress-free boundaries

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and for fixed temperatures and concentrations of mass. Here we consider more general boundary conditions on temperature and solute, in the form of Robin boundary conditions, which are linear expressions in the temperature (or solute concentration) and its normal derivative at a boundary. These boundary conditions are physically more realistic then simply fixing the value of the fields at a boundary, and they have a profound influence on the threshold of stability. A peculiarity of these boundary conditions for the temperature is that in the limit case of fixed heat fluxes the wavelength of the critical periodicity cell tends to infinity. In this work we investigate the influence of the solute field on this long-wavelength phenomenon, with the striking result that, both for the standard Bénard system and for porous media, the critical parameters become *totally independent* from the solute field. This last aspect and non linear stability will require further analysis of the systems.

2. Main equations

Here we derive the equations for the study of stability of a binary fluid mixture heated from below, considering the two cases of a fluid filling a layer (Bénard system), or a fluid saturating a porous medium.

2.1. *Bénard system for a binary mixture*

We denote by $Oxyz$ the cartesian frame of reference, with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and we consider an infinite layer $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$ of thickness $d > 0$ filled with a newtonian fluid $\mathcal{F}$, subject to a vertical gravity field $\mathbf{g} = -g\mathbf{k}$. We suppose that the density of the fluid depends on temperature T and on a solute concentration C according to the linear law $\rho_f = \rho_0[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)]$, where ρ_0, T_0 and C_0 are reference density, temperature and concentration, and α_T, α_C are (positive) density variation coefficients. In the Oberbeck-Boussinesq approximation, the equations governing the motion of the fluid are given by (see Ref. 10)

$$
\begin{cases}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \dfrac{p}{\rho_0} + \dfrac{\rho_f}{\rho_0}\mathbf{g} + \nu \Delta \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0 \\
T_t + \mathbf{v} \cdot \nabla T = \kappa_T \Delta T, \quad C_t + \mathbf{v} \cdot \nabla C = \kappa_C \Delta C,
\end{cases}
\tag{1}
$$

where $\mathbf{v}$ and p are the velocity and pressure fields. Further, ν and κ_T, κ_C are positive constants which represent kinematic viscosity and the thermal and solute diffusivity. ∇ and Δ are the gradient and the Laplacian operators, respectively, and the subscript "$_t$" denotes the partial time derivative.

For the velocity field, we assume that the boundaries are either rigid or stress free, and then

$$\mathbf{v} = 0, \text{ on rigid boundaries,}$$
$$\mathbf{k} \cdot \mathbf{v} = \partial_z(\mathbf{i} \cdot \mathbf{v}) = \partial_z(\mathbf{j} \cdot \mathbf{v}) = 0, \text{ on stress free boundaries.} \tag{2}$$

For temperature we use Newton-Robin boundary conditions, which describe the physically relevant cases in which the media surrounding the fluid is not an ideal thermostat (see e.g. Ref. 10). In the literature, many explicit forms of the Newton-Robin boundary conditions are used, but we find convenient to choose them in such a way that, by varying a single coefficient, different thermal boundary conditions can be obtained, but the basic (motionless) solution is *preserved*:

$$\alpha_H(T_z + \beta_T)d + (1 - \alpha_H)(T_H - T) = 0, \text{ on } z = -d/2$$
$$\alpha_L(T_z + \beta_T)d + (1 - \alpha_L)(T - T_L) = 0, \text{ on } z = d/2, \tag{3}$$

where $\alpha_H, \alpha_L \in [0, 1]$, $\beta_T > 0$, and $T_H = T_0 + \beta_T d/2$, $T_L = T_0 - \beta_T d/2$ are respectively an higher (T_H) and lower (T_L) temperature. For $\alpha_H, \alpha_L = 0$, we obtain the *infinite conductivity* boundary condition, in which we fix the value of the temperature at a boundary. For $\alpha_H, \alpha_L \in (0, 1)$ we get the cases of finite conductivity at the corresponding boundary, or Newton-Robin conditions.[4,14,17] For $\alpha_H, \alpha_L = 1$ we get the *insulating* boundary conditions,[2,4,5] with a fixed heat flux $\mathbf{q}$ directed along the z axis at one or both boundaries, with $q = \beta_T \kappa_T$.

For the solute field, by similar considerations,[10] we use boundary conditions depending on both concentration of solute and its normal derivative at the boundary surfaces. Again, to obtain a range of boundary conditions (from fixed concentrations to fixed fluxes of solute) depending on a single parameter on each boundary, while maintaining the basic solution, we use the following expressions

$$\gamma_H(C_z + \beta_C)d + (1 - \gamma_H)(C_H - C) = 0, \text{ on } z = -d/2$$
$$\gamma_L(C_z + \beta_C)d + (1 - \gamma_L)(C - C_L) = 0, \text{ on } z = d/2, \tag{4}$$

where $\gamma_H, \gamma_L \in [0, 1]$, $\beta_C > 0$, and $C_H = C_0 + \beta_C d/2$, $C_L = C_0 - \beta_C d/2$ are respectively an higher (C_H) and lower (C_L) density.

The steady solution in whose stability we are interested is the motionless state, which, for any choice of the $\alpha_H, \alpha_L, \gamma_H, \gamma_L$ parameters, is

$$\mathbf{v} = \mathbf{0}, \quad \overline{T} = -\beta_T z + T_0, \quad \overline{C} = -\beta_C z + C_0. \tag{5}$$

Note that in (5) β_T, β_C are the temperature and concentration gradients.

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The non-dimensional equations — here we use the non-dimensional form given in 10, §56 — which govern the evolution of a disturbance to (5) are[10]

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p_1 + (\mathcal{R}\vartheta - \mathcal{C}\gamma)\mathbf{k} + \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \\ \mathrm{P}_\theta(\vartheta_t + \mathbf{u} \cdot \nabla \vartheta) = \mathcal{R}w + \Delta\vartheta, \quad \mathrm{P}_c(\gamma_t + \mathbf{u} \cdot \nabla \gamma) = \mathcal{C}w + \Delta\gamma, \end{cases} \tag{6}$$

where $\mathbf{u} \equiv (u, v, w)$, ϑ, γ, p_1 are the perturbations to the velocity, temperature, concentration and pressure fields, respectively. The stability parameters in (6) are the Rayleigh numbers $\mathcal{R}, \mathcal{C}$ for heat and solute, and P_θ and P_c are the Prandtl and Schmidt numbers (as defined in Ref. 10). Note that in (6) we have made use of the transformation $\mathcal{R}\vartheta = \vartheta_1, \mathcal{C}\gamma = \gamma_1$ and we have omitted the subscript "$_1$".

2.2. *Binary mixture in a layer of porous medium*

We assume that the layer has the same geometry used in Sec. 2.1 and that the flow in the porous medium is governed by Darcy's law. Moreover we assume that the Oberbeck-Boussinesq approximation is valid with the same formal dependency of density on the temperature and the concentration of solute.

Under these assumptions, we follow the derivation of Ref. 13 (see also Refs. 16,19). For the velocity, temperature and concentration fields, we use the same boundary conditions (2), (3), (4) described previously, and we consider the same motionless basic state (5). In this way we obtain the following non-dimensional perturbation equations for a disturbance $(\mathbf{u}, \vartheta, \gamma, p_1)$ to (5)

$$\begin{cases} \nabla p_1 = (\mathcal{R}\,\vartheta - \mathrm{Le}\,\mathcal{C}\,\gamma)\mathbf{k} - \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0 \\ \vartheta_t + \mathbf{u} \cdot \nabla \vartheta = \mathcal{R}w + \Delta\vartheta, \quad \epsilon\,\mathrm{Le}\,\gamma_t + \mathrm{Le}\,\mathbf{u} \cdot \nabla \gamma = \mathcal{C}w + \Delta\gamma, \end{cases} \tag{7}$$

(quantities $\mathcal{R}$ and $\mathcal{C}$ appearing in Eq. 11 of Ref. 13 are respectively equal to quantities $\mathcal{R}^2$ and $\mathcal{C}^2$ used in the previous equation, and the changes of variables $\mathcal{R}\vartheta \to \vartheta, \mathcal{C}\gamma \to \gamma$ where performed). The quantities appearing in (7) have the same meaning of those used in the previous paragraph, $\mathrm{Le} = \kappa_T/\kappa_C$ is the Lewis number and ϵ is the normalized porosity.[13]

2.3. *Linear instability equations*

We will perform a linear instability analysis of systems (6) and (7). The procedure to obtain the final system is similar in the two cases, so we will execute the same calculations on both systems.

We follow the standard analysis of Chandrasekhar,[3] applying twice the curl operator to the first equation of both systems. We then consider only

the linear terms of the resulting systems and obtain

$$
\begin{cases}
\Delta w_t = \mathcal{R}\Delta^*\vartheta - \mathcal{C}\Delta^*\gamma + \Delta\Delta w \\
\mathrm{P}_\theta\vartheta_t = \mathcal{R}w + \Delta\vartheta, \quad \mathrm{P_c}\gamma_t = \mathcal{C}w + \Delta\gamma,
\end{cases}
\tag{8}
$$

$$
\begin{cases}
0 = \mathcal{R}\Delta^*\vartheta - \mathrm{Le}\mathcal{C}\Delta^*\gamma - \Delta w \\
\vartheta_t = \mathcal{R}w + \Delta\vartheta, \quad \epsilon\mathrm{Le}\gamma_t = \mathcal{C}w + \Delta\gamma.
\end{cases}
\tag{9}
$$

where $\Delta^\star = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We assume, as usual, that the perturbation fields are sufficiently smooth, and that they are periodic in the x and y directions (this is not a restriction, see 18). We denote by $a = (a_x^2 + a_y^2)^{1/2}$ the wave number. We search then solutions of both systems in the form

$$
f = F(z)\exp\{\,i\,(a_x x + a_y y) + p\,t\}
\tag{10}
$$

for fields w, ϑ, γ, where $p = \sigma + i\tau$ is a complex constant. By substituting expressions (10) in (8) and (9) we obtain the following two ODE systems for the perturbation fields W, Θ, Γ

$$
\begin{cases}
p\,(D^2 - a^2)W = (D^2 - a^2)^2 W + \mathcal{C}a^2\,\Gamma - \mathcal{R}a^2\,\Theta \\
p\,\mathrm{P}_\theta\,\Theta = (D^2 - a^2)\Theta + \mathcal{R}\,W, \quad p\,\mathrm{P_c}\,\Gamma = (D^2 - a^2)\Gamma + \mathcal{C}\,W,
\end{cases}
\tag{11}
$$

$$
\begin{cases}
0 = (D^2 - a^2)W + \mathcal{R}a^2\,\Theta - \mathrm{Le}\mathcal{C}a^2\Gamma \\
p\,\Theta = (D^2 - a^2)\Theta + \mathcal{R}\,W, \quad p\,\epsilon\mathrm{Le}\,\Gamma = (D^2 - a^2)\Gamma + \mathcal{C}W.
\end{cases}
\tag{12}
$$

where "D" represents the derivation along z. In this new variables, the hydrodynamic, thermal and solute boundary conditions become

on a rigid surface	$DW = W = 0,$
on a stress-free surface	$D^2 W = W = 0,$
on $z = -{}^1\!/_2$	$\alpha_H D\Theta - (1 - \alpha_H)\Theta = 0,\ \gamma_H D\Gamma - (1 - \gamma_H)\Gamma = 0,$
on $z = {}^1\!/_2$	$\alpha_L D\Theta + (1 - \alpha_L)\Theta = 0,\ \gamma_L D\Gamma + (1 - \gamma_L)\Gamma = 0.$

$$
\tag{13}
$$

The conditions on rigid and stress-free boundaries apply only to system (11), while the conditions on Θ, Γ and the condition $W = 0$, are common to both systems. When the principle of exchange of stabilities (PES) holds, a simplified form of both systems is obtained.[3]

3. Some results

3.1. *Note on fixed heat fluxes*

It is well known[2,4] that in the Bénard system, for Newton-Robin BCs approaching fixed heat fluxes on both boundaries, the critical wave number of

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the perturbations goes to zero (and so the wavelength goes to infinity), and the critical Rayleigh numbers $\mathcal{R}_c^2$ tend to the integer values $720, 320, 120$, respectively for RR, RF, and FF boundary conditions. This has been verified[8] also for a Darcy flow in a porous medium, with $\mathcal{R}_c^2 = 12$. We check here how a solute field affects the stability of a fluid, free or in a porous medium, under Newton-Robin or fixed heat flux BCs.

3.2. *Numerical methods*

In general, the eigenvalue problems obtained for this kind of boundary conditions must be solved numerically. We solved our eigenvalue problems with a Chebyshev Tau method (see Refs. 18 and 6). The accuracy of the method has been checked by evaluation of the tau coefficients, by comparison with known or analytical results, and, when PES holds, comparing the solutions of PES and non-PES problems.

3.3. *Results for Bénard system*

We performed a series of computations for different choices of Prandtl and Schmidt numbers, and thermal and hydrodynamic BCs. In the following we present some results for stress free boundaries. In this case the analytic solution is known (see e.g. Ref. 11) for thermostatic boundaries, and over-stability phenomena are present for $P_c/P_\theta > 1$. We show graphics obtained for $P_c = 3$ and $P_\theta = 1$. The same qualitative behavior appears also for more physically meaningful values, such as $P_c = 670$ and $P_\theta = 6.7$ (for sea water).

We see in Fig. 1 that the overstability region disappears during the transition from fixed temperatures to fixed heat fluxes, (at a smaller value of α for sea water). The solute, as expected, has a stabilizing effect, but we note that in the limit case of fixed heat fluxes the stabilizing effect is **totally lost**, since the critical Rayleigh number becomes *independent of the concentration gradient*. This result is somehow surprising, and will require further investigation. (In the case of the rotating Bénard system we observed a stabilizing effect of rotation *even for fixed heat fluxes*). The result is nevertheless correct: an asymptotic analysis of the system for $a \to 0$ confirms that $\mathcal{R}_c^2 = 120$ (for FF boundaries), independently of $\mathcal{C}$. A point to note is that, since the critical Rayleigh number is independent of the concentration for fixed heat fluxes, we can use the classical non-linear energy stability analysis in the absence of a solute. The critical value $\mathcal{R}_c^2$ ensures then *global stability* (w.r.t. the classical energy norm) for any solute gradient. In Fig. 2 we show the critical wavenumber corresponding to the critical
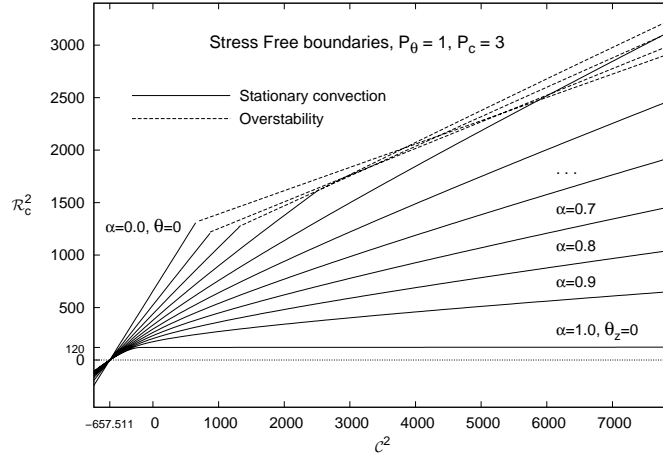
Fig. 1. $\mathcal{R}_c^2$ as a function of $\mathcal{C}^2$ for thermal BCs going from fixed temperatures ($\alpha = 0$) to fixed heat fluxes ($\alpha = 1$), FF boundaries.
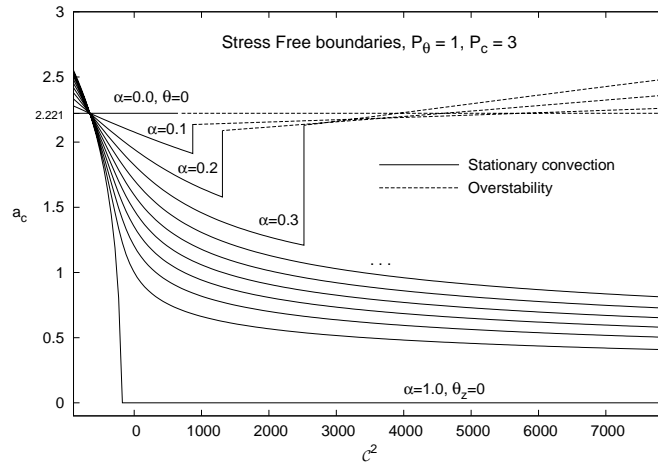

Fig. 2.   Critical wavenumber a_c as a function of $\mathcal{C}^2$ for thermal boundary conditions going from fixed temperatures ($\alpha = 0$) to fixed heat fluxes ($\alpha = 1$), FF boundaries. The vertical segments are added as a guide for the eye.

Rayleigh numbers of Fig. 1. For fixed heat fluxes, and for any stabilizing solute gradient ($\mathcal{C}^2 > 0$) the wave number is equal to zero. For fixed temperatures the critical wavenumber a_c is the constant $\pi/\sqrt{2}$. At the transi-

tion between stationary convection and overstability, for non-thermostatic boundaries, the wavenumber has a discontinuity. Figure 3 shows the influ-
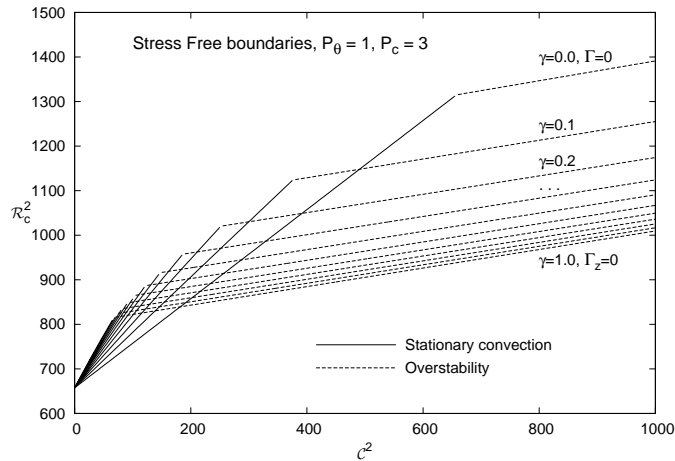

Fig. 3. Critical Rayleigh number $\mathcal{R}_c^2$ as a function of $\mathcal{C}^2$ for fixed temperatures, solute boundary conditions going from fixed solute ($\gamma = 0$) to fixed solute fluxes ($\gamma = 1$) on both boundaries, hydrodynamic FF boundaries.

ence on stability of the boundary conditions on the solute. The results for fixed solute concentrations are analytically known (see e.g. Refs. 10 and 11), and are linear both in the convective and overstable regime. From the numerical results shown in the figure, the dependence on $\mathcal{C}^2$ seems linear also for Robin boundary conditions on the solute field, so we can suspect that an analytical solution exists even in this case. We observe also that in the convective regime the most stable condition is always obtained to fixed solute fields, while the situation is reversed in the case of overstability. The solute field remains in any case a stabilizing field. Contrary to the case of fixed heat fluxes, for fixed solute fluxes the critical Rayleigh number does not become independent on the solute gradient. Moreover, the corresponding critical wave number is not zero, so the long wavelength phenomenon seems linked only to a fixed flux of the *destabilizing* field.

3.4. *Results for porous media*

Results for porous media are qualitatively very similar, with respect to the dependence on thermal BCs, and also in the limit case of fixed heat fluxes.

We present only a graphic for the critical Rayleigh number, for a choice of the Lewis number Le = 1 and of the normalized porosity $\epsilon = 2$ such that overstability is present for fixed temperatures. The same comments made on Fig. 1 apply to Fig. 4. The critical Rayleigh number becomes *independent of the concentration gradient* for fixed heat fluxes. An asymptotic analysis for $a \to 0$ confirms, even in this case, that $\mathcal{R}$ is constant, $\mathcal{R}_c^2 = 12$, independently of $\mathcal{C}$. Also in this case we obtain *global stability* (w.r.t. the classical energy norm) for fixed heat fluxes and any solute gradient.
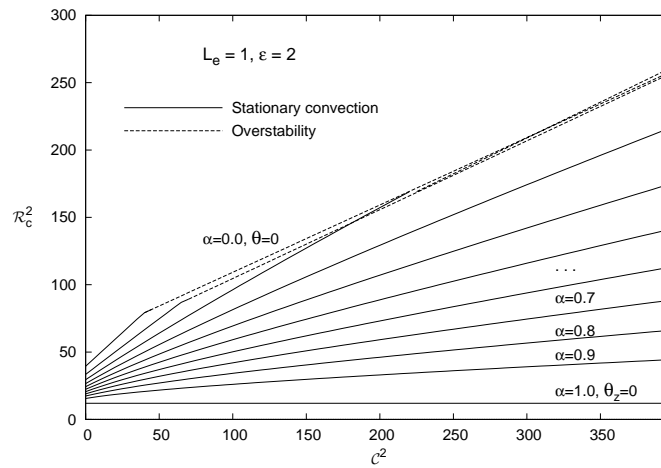

Fig. 4. $\mathcal{R}_c^2$ as a function of $\mathcal{C}^2$ for thermal BCs going from fixed temperatures ($\alpha = 0$) to fixed heat fluxes ($\alpha = 1$).

4. Conclusions

The stability of a binary fluid layer, or a binary fluid saturating a porous medium, subject to Neumann boundary conditions on the temperature, i.e. subject to fixed heat fluxes, results totally independent on the solute field. A clear physical interpretation of the phenomenon is yet to be found. The most stabilizing thermal boundary conditions, at least for stationary convection, are thermostatic boundaries. A better analysis of the transition between stationary convection and overstability is required. A nonlinear stability analysis for the general Newton-Robin case is in progress.

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