COMPETITION AND COOPERATION OF STABILIZING EFFECTS IN THE BÉNARD PROBLEM WITH ROBIN TYPE BOUNDARY CONDITIONS

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Rotation and magnetic field have a stabilizing effect on the Bénard problem if they act separately. However, as is shown in the classical works of Chandrasekhar,[2] when they are both present, these stabilizing effects are often *conflictual*. Instead, other stabilizing effects, such as rotation and concentration field, are cumulative.[8] The previous results were obtained for stress-free boundary conditions, and fixed boundary temperatures and concentrations. In this work, we investigate, analytically and numerically,[3,13] how different boundary conditions on the temperature, such as the Robin and Neumann b.c. used in,[4] influence the competition and cooperation of the aforesaid stabilizing effects. The appearance of long-wavelength perturbations for low thermal conductivity of the boundaries is also investigated. The present work concerns a linear stability analysis of the problem and it is part of a larger project including a nonlinear analysis.[6,13]

Keywords: Bénard problem, competing effects, thermohaline convection, magnetic Bénard problem, long wavelength, Newton-Robin

1. Introduction

The linear stability problem of the motionless state of an infinite layer of homogeneous fluid heated from below has been studied by Chandrasekhar[2] by means of classical normal modes. Moreover, the stabilizing effect of uniform rotation has been predicted by the same author in the *rotating* Bénard problem. A stabilizing effect is obtained even by salting the fluid layer from below,[7] or by immersing it in a normal magnetic field, if the fluid is electrically conducting.[2] Two or more simultaneously acting stabilizing effects allow observation of a very rich variety of phenomena often surprising. As is known, unexpected *conflicting tendencies* among the rotation and magnetic field have been found by Chandrasekhar, instead *cooperative behaviour* has been showed among the rotation and salt concentration field (when the mixture is salted from below).[8] Here we study how the boundary conditions influence interaction of different stabilizing fields. We consider the following two cases: the co-presence of rotation and salt (supplied from below), and the interaction of magnetic field with rotation, both coupled with Robin and Neumann boundary conditions on temperature. Numerical results in this paper are obtained with a Chebyshev-tau method.[3]

2. The Bénard problem of a rotating mixture

Let $Oxyz$ be a cartesian frame of reference with unit vectors **i**, **j**, **k** respectively, rotating at the constant velocity $\bar{\Omega}\mathbf{k}$. Let $d > 0$ and assume that a newtonian fluid is confined in the layer $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$, and subject to a gravity field $\mathbf{g} = -g\mathbf{k}$. We assume also that the density of the fluid depends linearly on temperature T and concentration C of a solute according to $\rho_f = \rho_0[1-\alpha_T(T-T_0)+\alpha_C(C-C_0)]$, with α_T, α_C positive coefficients of volume expansion and T_0, C_0 reference temperature and concentration.[2,7,9] For the temperature field we assume Newton-Robin boundary conditions, which are linear combination of the temperature at a surface and its normal gradient. This boundary conditions describe the physical cases in which the media surrounding the fluid are not thermostatic.[4,7] The limit cases of fixed temperatures or fixed temperature gradients (and hence fixed heat fluxes) are also considered. We use the following general form of the thermal boundary conditions:

$$\begin{aligned} &\alpha_H(T_z + \beta_T)d + (1 - \alpha_H)(T_H - T) = 0, \text{ on } z = -d/2 \\ &\alpha_L(T_z + \beta_T)d + (1 - \alpha_L)(T - T_L) = 0, \quad \text{on } z = d/2, \end{aligned} \tag{1}$$

where $\alpha_H, \alpha_L \in [0, 1]$, $\beta_T > 0$, and $T_H = T_0 + \beta_T d/2$, $T_L = T_0 - \beta_T d/2$ are respectively an higher (T_H) and lower (T_L) temperature. Note that, from (1), we obtain fixed temperature, fixed heat flux, or a Newton-Robin boundary condition[1,11,12] at $z = d/2$, when α_L is equal to $0, 1$ or $0 < \alpha_L < 1$, respectively. The same observations apply to α_H and the boundary $z = -d/2$. For the velocity field, we assume that the boundaries are either rigid ($\mathbf{v} = 0$) or stress free ($\mathbf{k} \cdot \mathbf{v} = \partial_z(\mathbf{i} \cdot \mathbf{v}) = \partial_z(\mathbf{j} \cdot \mathbf{v}) = 0$).[2] Concentrations at the boundaries are $C(x, y, -d/2) = C_0 + \beta_C d/2$, $C(x, y, d/2) = C_0 - \beta_C d/2$, where β_C is an assigned concentration gradient. The form of (1) ensures that for any choice of α_H, α_L, (and rigid or stress-free boundaries) the basic solution m_0 is the same, simplifying further analysis

$$\mathbf{v} = 0, \quad T(x, y, z) = -\beta_T z + T_0, \quad C(x, y, z) = -\beta_C z + C_0. \tag{2}$$

The non-dimensional evolution equations of a perturbation to the basic motionless state m_0 are[8]

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p^* + (\mathcal{R}\vartheta - \mathcal{C}\gamma)\mathbf{k} + \Delta\mathbf{u} + \mathcal{T}\mathbf{u}\times\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0,\ P_T(\vartheta_t + \mathbf{u}\cdot\nabla\vartheta) = \mathcal{R}w + \Delta\vartheta,\ P_C(\gamma_t + \mathbf{u}\cdot\nabla\gamma) = \mathcal{C}w + \Delta\gamma \end{cases} \tag{3}$$

in $\Omega_1 \times (0,\infty)$ where $\Omega_1 = \mathbb{R}^2 \times (-1/2, 1/2)$. In this system $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, ϑ, γ and p^* are functions of (x, y, z, t) which represent the perturbations of the velocity, temperature, concentration and pressure fields, respectively; ∇ is the gradient operator and Δ is the Laplacian. The parameters $\mathcal{R}^2, \mathcal{C}^2, \mathcal{T}^2, P_T, P_C$ are the standard Rayleigh number for heat and solute, Taylor number, Prandtl and Schmidt numbers, respectively.[2,7] We study the linear instability of the basic motion, following Chandrasekhar.[2] We assume for the perturbation fields the general form, periodic in x, y, $f = F(z)\exp\{\,i\,(a_x x + a_y y) + p\,t\}$, where f denotes any of the fields $w, \zeta(= k\cdot\nabla\times\mathbf{u}), \theta, \gamma$ and $p = \sigma + i\tau$. Then, following standard calculations,[2,8] and adopting a suitable rescaling of fields, we can derive the equations

$$\begin{cases} p\,(D^2 - a^2)W = (D^2 - a^2)^2 W - \mathcal{T}DZ + \mathcal{C}a^2\,\Gamma - \mathcal{R}a^2\,\Theta \\ pZ = \mathcal{T}DW + (D^2 - a^2)Z \\ p\,P_T\,\Theta = (D^2 - a^2)\Theta + \mathcal{R}\,W, \quad p\,P_C\,\Gamma = (D^2 - a^2)\Gamma + \mathcal{C}\,W, \end{cases} \tag{4}$$

where $a = (a_x^2 + a_y^2)^{1/2}$ is the wave number, and D^n denotes the n-th partial derivative respect to z. The boundary conditions for system (4) are $W = \Gamma = 0$ on $z = \pm 1/2$, $D^2W = DZ = 0$ on stress-free boundaries, $DW = Z = 0$ on rigid boundaries, $\alpha_H D\Theta - (1 - \alpha_H)\Theta = 0$ on $z = -1/2$ and $\alpha_L D\Theta + (1 - \alpha_L)\Theta = 0$ on $z = 1/2$. When the Principle of Exchange of Stabilities (PES) holds ($\sigma = 0 \Rightarrow \tau = 0$), for stress-free and thermostatic boundaries, it is possible to find[8] for the critical Rayleigh number

$$\mathcal{R}_1^2 = \frac{(1+x)^3}{x} + \frac{\mathcal{T}_1^2}{x} + \mathcal{C}_1^2, \tag{5}$$

where $\mathcal{R}_1 = \mathcal{R}/\pi^2$, $x = a^2/\pi^2$, $\mathcal{C}_1 = \mathcal{C}/\pi^2$, and $\mathcal{T}_1 = \mathcal{T}/\pi^2$. In (5), the second and third term are exactly the stabilizing contributes appearing when only one of the two fields is present,[2,7] moreover the critical wave number appears independent of $\mathcal{C}$.

We consider the case of fixed heat fluxes, stress-free boundaries and $P_T = P_C = 1$. At a difference from the case $\mathcal{T} = 0$ (Ref. 5, in this proceedings collection), where $\mathcal{R}^2$ is constant ($\mathcal{R}^2 = 120$, dashed line in Fig. 1a), now the Rayleigh number is an increasing function of $\mathcal{C}$ and $\mathcal{T}$, and so the stabilizing effect of the solute is restored. Angular points correspond to a transition to a region of vanishing values of a_c. Moreover, we observe (Fig. 1b) a "competition" on the wave number between rotation and concentration gradient, in the sense that the wave number, for any fixed $\mathcal{T}$, becomes zero for sufficiently large values of $\mathcal{C}$; on the other hand, for fixed $\mathcal{C}$ and sufficiently large $\mathcal{T}$, it is $a_c > 0$. The same figure shows that, for large values of $\mathcal{T}, \mathcal{C}$, the region $a_c = 0$ is very closely defined by $\mathcal{C}^2 > \mathcal{T}^2$.

It is possible to check that the concentration and rotation fields remain cooperative in their stabilizing effects, for any $\alpha_H, \alpha_L \in [0, 1]$, and different values of P_T, P_C. For $P_C > P_T$ overstability effects appear. At least for stationary convection, thermal boundary conditions are more destabilizing as the parameters α_H, α_L increase, i.e. in the transition from fixed temperatures to fixed heat fluxes.

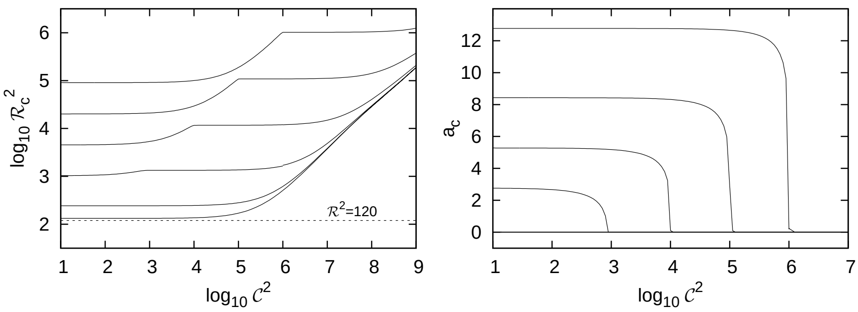

Fig. 1. $\mathcal{R}_c^2$ and a_c as a function of $\mathcal{C}^2$ for fixed heat fluxes. Taylor number is equal to $10^6, 10^5, \dots, 10$ from top to bottom in both graphics. For $\mathcal{T}^2 = 10, 100$, a_c is identically equal to zero for any $\mathcal{C}$.

3. The rotating magnetic Bénard problem

The magnetic Bénard problem deals with the onset of convection of a horizontal layer of a homogeneous, viscous, and electrically conducting fluid, permeated by an imposed uniform magnetic field normal to the layer, and heated from below.[2,14]

We suppose here that the system has the same geometry considered in the previous section, and the corresponding fields are subject to the same boundary conditions. Following the procedure of Chandrasekhar,[2] Chapter

V, we arrive to the linear instability analysis system:

$$\begin{cases} (D^2-a^2)(D^2-a^2-p)W + D(D^2-a^2)K - DZ - a^2\,\Theta = 0, \\ (D^2-a^2-p)Z + \mathcal{T}^2 DW + DX = 0,\ (D^2-a^2-P_m p)X + \mathcal{Q}^2\, DZ = 0, \\ (D^2-a^2-P_m p)K + \mathcal{Q}^2\, DW = 0,\ (D^2-a^2-P_T p)\Theta + \mathcal{R}^2\, W = 0, \end{cases} \tag{6}$$

where X and K are the perturbations to the third component of current density and H (the imposed magnetic field), $\mathcal{Q}^2$ and P_m are the Chandrasekhar and magnetic Prandtl numbers.[2] When PES holds, it is possible to set $p = 0$ and eliminate field K from (6). The same elimination is possible for $P_m = 0$ (this happens to a good approximation[2] for liquid metals, e.g. mercury), and our stability analysis is performed under this hypothesis. On X, we impose $DX = 0$ or $X = 0$ for electrically conducting or non-conducting boundaries, respectively.[2]

In the analytically solvable case of stress-free, thermostatic and non-conducting boundaries, Chandrasekhar finds (see Ref. 2, Chap. V, eq. 59) for stationary convection

$$\mathcal{R}_1^2 = \frac{(1+x)^3}{x} + \frac{\mathcal{T}_1^2}{x} + \frac{(1+x)\mathcal{Q}_1^2}{x} - \frac{\mathcal{Q}_1^2\mathcal{T}_1^2}{x((1+x)^2+\mathcal{Q}_1^2)},$$

where $\mathcal{Q}_1 = \mathcal{Q}/\pi$. The second and third terms in the previous expression can be found when the system is subject only to rotation or a magnetic field (see Chandrasekhar[2] Chap. III eq. 130, Chap. IV eq. 165). The presence of the last term shows that the two effects are not simply additive.

For stress free boundaries and $P_T = 0.025$ (mercury), competition of

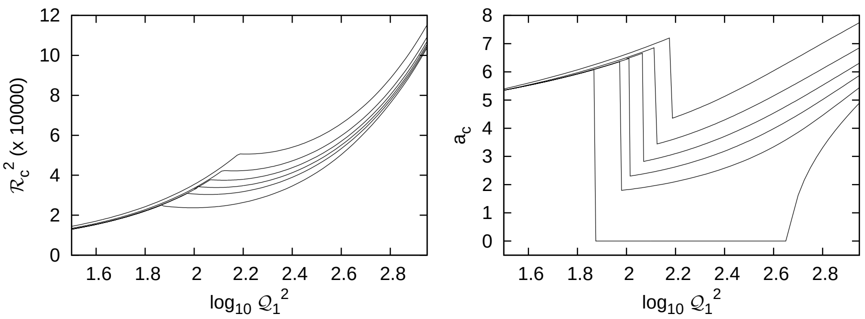

Fig. 2. $\mathcal{R}_c^2$ and a_c as a function of $\mathcal{Q}_1^2$ for $\mathcal{T}_1^2 = 10^4$. Thermal boundary conditions vary in both graphics from fixed temperatures (top curves) to fixed heat fluxes (bottom curves), with $\alpha_H = \alpha_L = 0, 0.2, 0.4, 0.6, 0.8, 1$. The vertical segments in the a_c graphs correspond to transitions from overstability to stationary convection. The same transition appears as a discontinuity in the slope of the $\mathcal{R}_c^2$ graphs.

magnetic field and rotation is enhanced by the new thermal boundary conditions, and appears clearly in Fig. 2a where the slope becomes negative. We can observe also in Fig. 2b a dramatic reduction of the critical parameter a_c when heat flux is prevalent. We note that $a_c = 0$ in a finite range of values of $\mathcal{Q}$ for Neumann conditions on temperature.

A more extensive study of the system, for different values of the Prandtl numbers and other hydrodynamic and magnetic boundary conditions will be the subject of a future work.

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