Some stability results in the rotating Bénard problem

P. Falsaperla and G. Mulone
Dipartimento di Matematica e Informatica - Università di Catania, Catania, Italy

September 29, 2008

1 Introduction

The Bénard system is historically the first example of convection to be studied, and for its many geophysical and industrial applications, it is still of great relevance. The essential feature of the phenomenon is as follows. A horizontal layer of fluid in the rest state is heated from below in such a way that an adverse temperature gradient is maintained. The fluid at the bottom expands as it becomes hotter and, when the temperature gradient reaches a critical value (see [19]), the buoyancy overcomes viscosity, and the fluid gives rise to a regular cellular pattern of motion. This phenomenon is called Bénard convection after the experiments of Bénard [2], (see also [12, 13, 16, 21]). The linear stability of this problem has been studied in Chandrasekhar [4], by means of classical normal modes, for perfectly conducting boundaries, and rigid or stress-free boundaries. The nonlinear energy stability has been shown in Joseph [9, 10], (see also [12, 23]).

More general problems include new effects such as a rotation field or a magnetic field. For fixed boundary temperatures, the linear stability theories predict the stabilizing effect of rotation, while the nonlinear L^2-norm stability proves that the rotation about a vertical axis has only a non-destabilizing character (see Rionero [20]). In Mulone and Rionero [14], and Mulone [15], the coincidence of critical linear and nonlinear stability parameters has been proved. In [15] the reduction method has been applied to prove the coincidence of critical linear and nonlinear stability parameters.

In this work we consider mixed, or Newton-Robin, thermal boundary conditions, which involve both the temperature field, and its normal derivative at the boundaries. The mixed boundary conditions have several physical justifications, that originate from a more accurate description of heat transfer phenomena in the media surrounding the fluid [22, 5, 18, 6]. It should be noted also that the usual "thermostatic" boundary conditions are not justified physically, since they imply an infinite conductivity inside the thermostat. An interesting, both physically and mathematically, limit case is that of fixed heat fluxes (also known as "insulating" boundary conditions) [3, 17].

2 The Equations

We consider an infinite layer $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$ of thickness $d > 0$ filled with an incompressible homogeneous newtonian fluid $\mathcal{F}$, subject to the action of a vertical gravity field $\mathbf{g}$. We also assume that the fluid is uniformly rotating about the vertical axis z with an angular velocity $\widehat{\Omega}\mathbf{k}$, and denote by $Oxyz$ the cartesian frame of reference (with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) *rotating* about z with the same angular velocity $\widehat{\Omega}$.

The equations of the fluid in the Bussinesq approximation are given by (see Chandrasekhar [4]):

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\dfrac{p^\star}{\rho_0} + [1-\alpha(T-T_0)]\mathbf{g} - 2\,\widehat{\Omega}\,\mathbf{k}\times\mathbf{v} + \nu\Delta\mathbf{v} \\ \nabla\cdot\mathbf{v} = 0, \qquad T_t + \mathbf{v}\cdot\nabla T = k\Delta T \end{cases} \tag{1}$$

where $\mathbf{v}$, T, $p^\star$ are the velocity, temperature and pressure fields, respectively, and the $p^\star$ field includes the centrifugal force term. Further ρ_0, α, ν and k are positive constants which represent the density of the fluid at some reference temperature T_0, the coefficient of volume expansion, the kinematic viscosity and the thermometric conductivity, ∇ and Δ are the gradient and the Laplacian operators, respectively, and the suffix "t" denotes the partial time derivative. For the velocity field, we assume that the boundaries are either rigid (**R**) or stress free (**F**),

$$\begin{aligned} \mathbf{v} = 0, &\quad \text{on R boundaries,} \\ \mathbf{k}\cdot\mathbf{v} = \partial_z(\mathbf{i}\cdot\mathbf{v}) = \partial_z(\mathbf{j}\cdot\mathbf{v}) = 0, &\quad \text{on F boundaries.} \end{aligned} \tag{2}$$

For the temperature field we assume the following boundary conditions

$$\begin{aligned} \alpha_H(T_z+\beta)d + (1-\alpha_H)(T_H - T) = 0, &\quad \text{on } z = -d/2 \\ \alpha_L(T_z+\beta)d + (1-\alpha_L)(T - T_L) = 0, &\quad \text{on } z = d/2, \end{aligned} \tag{3}$$

where $\alpha_H, \alpha_L \in [0,1]$, $\beta > 0$, and $T_H = T_0 + \beta d/2$, $T_L = T_0 - \beta d/2$ are respectively an higher (T_H) and lower (T_L) temperature.

For $\alpha \in (0,1)$ these conditions are equivalent to the many Newton-Robin boundary conditions used in the literature (eg. [7]), but **(3) have the property to *preserve the basic solution*.** In fact, all the motionless solutions of problem (1), with the boundary conditions (2)-(3) and *any* choice of α_H and α_L, can be expressed as

$$\mathbf{v} = 0, \qquad \widehat{T}(x,y,z) = -\beta z + T_0. \tag{4}$$

Expression (4) implies also $\widehat{T}(x,y,-d/2) = T_H$, $\widehat{T}(x,y,d/2) = T_L$.

Following Chandrasekhar [4], the evolution equations of a nondimensional disturbance of the motionless solution of (1) in the rotating frame of reference (see [11], [23]), are given by

$$\begin{cases} \Delta w_t = \mathcal{R}\Delta^*\theta - \mathcal{T}\zeta_z + \Delta\Delta w + N_1 \\ \zeta_t = \mathcal{T}w_z + \Delta\zeta + N_2 \\ P_r\theta_t = \mathcal{R}w + \Delta\theta + N_3, \end{cases} \tag{5}$$

in $\mathbb{R}^2 \times (-1/2, 1/2) \times (0,\infty)$, where $\mathbf{u} = (u,v,w)$ and θ are the perturbations of the velocity and temperature, $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{u}$ is the z component of the vorticity

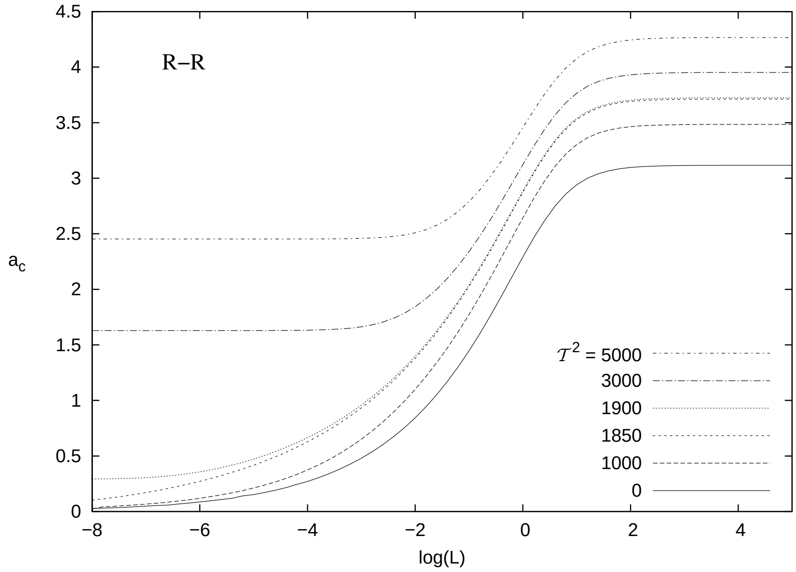

Figure 1: Values of a_c as a function of the parameter L for selected values of $\mathcal{T}^2$ and R-R boundary conditions. Notice that for $L \to 0$ the curves corresponding to $\mathcal{T}^2 \leq 1850$ tend to zero, while the curves for $\mathcal{T}^2 \geq 1900$ tend to finite values.

field, $\Delta^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The quantities $\mathcal{R}^2$, $\mathcal{T}^2$, and P_r are the standard Rayleigh, Taylor and Prandtl numbers, respectively [4]. N_1, N_2, N_3 are nonlinear terms. Following the standard analysis in normal modes of the system, we search then solutions of the linearized system (5) in the form

$$\left\{ \begin{array}{l} w = W(z) \exp\{\, i\,(a_x x + a_y y) + p\, t\} \\ \zeta = Z(z) \exp\{\, i\,(a_x x + a_y y) + p\, t\} \\ \theta = \Theta(z) \exp\{\, i\,(a_x x + a_y y) + p\, t\} \end{array} \right. \tag{6}$$

where $p = \sigma + i\tau$ is a complex constant, with $\mathrm{Re}(p) = \sigma$ and $\mathrm{Im}(p) = \tau$. Substituting expressions (6) into the linearized system (5) we obtain

$$\left\{ \begin{array}{l} (D^2 - a^2)^2 W - \mathcal{T}\, DZ - \mathcal{R} a^2\, \Theta = p\,(D^2 - a^2) W \\ (D^2 - a^2) Z + \mathcal{T}\, DW = p\, Z \\ (D^2 - a^2)\Theta + \mathcal{R}\, W = p\, Pr\, \Theta, \end{array} \right. \tag{7}$$

where "D" represents the operator of derivation along the z axis. System (7), with its boundary conditions,

$$\begin{array}{ll} \text{on a rigid surface} & W = DW = Z = 0, \\ \text{on a stress-free surface} & W = D^2 W = DZ = 0, \\ \text{on } z = -1/2 & \alpha_H D\Theta - (1 - \alpha_H)\Theta = 0, \\ \text{on } z = 1/2 & \alpha_L D\Theta + (1 - \alpha_L)\Theta = 0, \end{array} \tag{8}$$

is an eigenvalue problem with eigenvalue p, depending on the parameters $\mathcal{T}$, $\mathcal{R}$, a, Pr, α_H, α_L. From expression (6) we see that the perturbations decay, and then (linear) stability is achieved, if $\sigma < 0$ for all eigenvalues obtained for

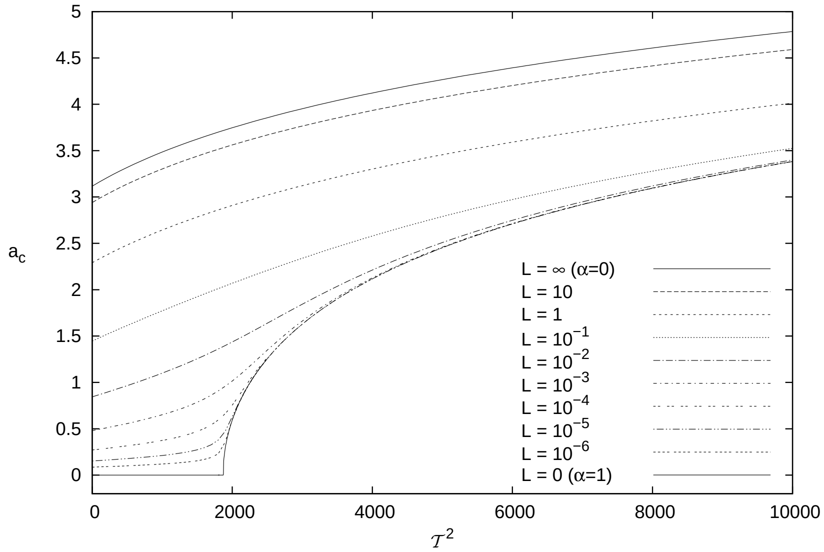

Figure 2: Values of a_c as a function of $\mathcal{T}^2$ for selected values of L ranging from $L = \infty$ (fixed temperatures) to $L = 0$ (fixed heat fluxes), and R-R boundary conditions.

a certain set of values of the parameters. Criticality is then obtained when the eigenvalue with the largest real part has $\sigma = 0$. If at the criticality it is $\tau = 0$, we say that the principle of exchange of stabilities (PES) holds, and a simplified form of (7) can be studied [4]. By fixing all parameters except $\mathcal{R}$ and a we get a locus of critical states $\mathcal{R}(a)$. The onset of instability is then expected at the minimum value $\mathcal{R}_c$ of $\mathcal{R}$ along the curve, corresponding to a cellular motion of wave number a_c.

Nonlinear stability

By using the classical energy $E(t) = 1/2(\|\mathbf{u}\|^2 + Pr\,\|\theta\|^2)$ we obtain global nonlinear stability whenever $\mathcal{R}^2 < \mathcal{R}_c^2(0)$, where $\mathcal{R}_c^2(0)$ is the critical Rayleigh value for $\mathcal{T} = 0$. In order to prove the stabilizing effect of rotation in the nonlinear context we can use the reduction method [15].

3 Results

We solved problem (7)-(8), (and that obtained when PES holds) with a Chebyshev Tau method, using up to 35 polynomials per unknown function. The accuracy of the method has been checked by evaluation of the tau coefficients, by comparison with known or analytical results, and, when PES holds, comparing the solutions of PES and non-PES problems. All of the results presented here satisfy PES, at least for sufficiently large values of Pr.

In figures (1)-(2) we use symmetric boundary conditions, with $\alpha = \alpha_H = \alpha_L$, and denote by L the quantity $(1-\alpha)/\alpha$ (for a direct comparison with [22, 17]).

The approximate threshold values of $\mathcal{T}_a^2$ at which a_c becomes positive are given in the following table. The numerical computation of these values presents

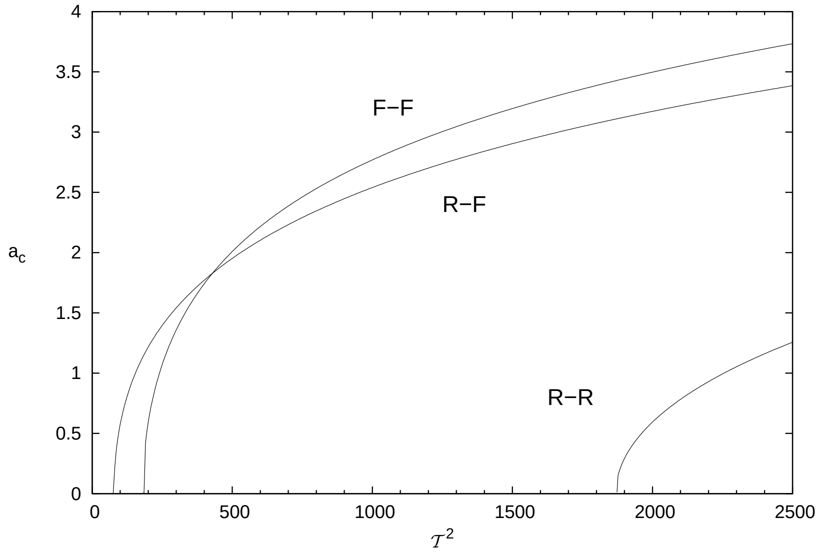

Figure 3: Critical wave number as a function of $\mathcal{T}^2$ for fixed heat fluxes at both boundaries. The critical wave number is equal to zero up to a certain threshold in all three cases.

some difficulties since the derivative $da_c/d\mathcal{T}^2$ is singular for $\mathcal{T}^2 \approx \mathcal{T}_a^2$.

	R-R	F-F	R-F
$\mathcal{T}_a^2$	≈ 1880	≈ 181	≈ 75

The integer values of $\mathcal{R}_c^2$ for $\mathcal{T} = 0$, remarked in fig. 4 $(120, 320, 720)$, are know in the literature [8, 3], and can be computed analytically.

4 Conclusions

For fixed heat fluxes, we find that the critical wave number is asymptotically equal to zero up to a given threshold of rotation speed, dependent on the boundary conditions on the velocity field. This appears to be a new result. This behavior is consistent with the results obtained for Newton-Robin boundary conditions. From an physical point of view, this implies that up to some threshold of rotation speed, the convection cells would probably be the largest allowed by the experimental setup. **Preliminary results show the same qualitative dependency of the wave number (for fixed heat fluxes) on concentrations, magnetic field and diffusivity**, in mixed fluids, electrically conducting fluids, and porous media, respectively. The study of these systems, detailed overstability analysis, the nonlinear stability, and study of analogous problems for compressible fluids (see eg. [1]), will be the object of future works.

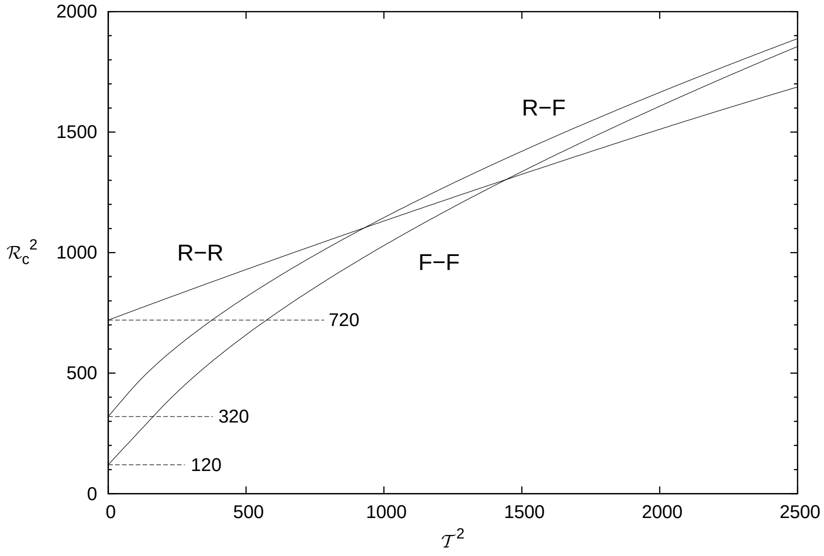

Figure 4: Critical Rayleigh number as a function of the Taylor number $\mathcal{T}^2$ for fixed heat fluxes at both boundaries.

References

[1] Barbera E., Müller I. (2006), Acta Mechanica **184**, 205-216

[2] Bénard H (1900), Rev. generale Sciences Pur. Appl. **11**, 1261-1271 and 1309-1328

[3] F. H. Busse, N. Riahi (1980), J. Fluid Mech. **96-2**, 243-256

[4] S. Chandrasekhar (1961), *Hydrodynamic and Hydromagnetic stability*, Oxford University Press

[5] C. J. Chapman, M. R. E. Proctor (1980), J. Fluid Mech. **101-4**, 759-782

[6] R. M. Clever, F. H. Busse (1998), Phys. Rev. E **57-4**, 4198-4205

[7] S. M. Cox (1998), SIAM J. Appl. Math. **58-4**, 1338-1364

[8] D. T. J. Hurle, E. Jakeman and E. R. Pike (1966), Proc. Roy. Soc. Lond. A **296** n. 1447, 469-475

[9] D. D. Joseph (1965), Arch. Rational Mech. Anal. **20**, 59-71

[10] D. D. Joseph (1966), Arch. Rational Mech. Anal. **22**, 163-184

[11] D. D. Joseph (1976), *Stability of fluid motions*, Springer Tracts in Nat. Phil., **27, 28** Berlin, Springer

[12] E.L. Koschmieder (1967), Beit. zur Phys. der Atmos. **40**, 216-225

[13] E.L. Koschmieder (1993), *Bénard cells and Taylor vortices*, Cambridge Univ. Press

[14] G. Mulone and S. Rionero (1997), Continuum Mech. Termodyn **9**, 347-363

[15] G. Mulone (2004), Far East J. Appl. Math. **15** n.2, 117-134

[16] Nakagawa and P. Frenzen (1955), Tellus **7**, 1-21

[17] D. A. Nield (1964), J. Fluid Mech. **19**, 341-352

[18] M. R. E. Proctor (1981), J. Fluid Mech. **113**, 469-485

[19] Lord Rayleigh (1916), Phil. Mag. **32**, 529-546

[20] S. Rionero S (1968), Ricerche di Mat. **17**, 64-78

[21] H. T. Rossby (1969), J. Fluid Mech. **36**, 309-335

[22] E. M. Sparrow, R. J. Goldstein, V. K. Jonsson (1963), Fluid Mech. **18**, 513-527

[23] B. Straughan (2004), *The Energy Method, Stability, and Nonlinear Convection*, 2nd ed., Ser. in Appl. Math. Sci., **91**, Springer New-York