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Let X, Y be two Banach spaces. By $L(X, Y)$ (resp. $K(X, Y)$) we denote the Banach spaces of all bounded, linear (resp. compact, bounded, linear) operators from X into Y. Several papers have been devoted to the question of when c_0 embeds isomorphically into $K(X, Y)$ (see [5, 8, 9] and their references) and its relationship with the following question:

(i) is $K(X, Y)$ always uncomplemented in $L(X, Y)$ when $L(X, Y) \neq K(X, Y)$?

From those papers it turned out that, in special settings, the embeddability of c_0 in $K(X, Y)$ implies that (i) is true. The purpose of this short note is to prove that if c_0 embeds in $K(X, Y)$, then $K(X, Y)$ is always uncomplemented in $L(X, Y)$, regardless of the nature of X and Y.

We need the following result due to Feder.

`Theorem 1 (Feder [6]).` *Assume that $L(X, Y) \neq K(X, Y)$ and that there a noncompact $T \in L(X, Y)$ and a sequence $(T_n) \subset K(X, Y)$ such that the series $\sum T_n(x)$ converges unconditionally to $T(x)$, for all $x \in X$. Then $K(X, Y)$ is uncomplemented in $L(X, Y)$.*

Remark 1. Following the proof of the main theorem in [4] one can prove that if the assumption of Theorem 1 holds then c_0 embeds in $K(X, Y)$. The proof of Theorem 2 below shows that the converse is also true, so that Feder's hypothesis is exactly equivalent to the containment of c_0 in $K(X, Y)$.

`Theorem 2.` *Assume that c_0 embeds in $K(X, Y)$. Then $K(X, Y)$ is uncomplemented in $L(X, Y)$.*

Proof. First suppose that Y contains a copy of c_0. Let (y_n) be a copy of the unit vector basis of c_0 in Y, with coefficient functionals (y_n^*), and (x_n^*) a weak* null sequence in X^* of norm one elements. Putting $T_n = x_n^* \otimes y_n$ for $n \in \mathbb{N}$, it is very easy to see that, for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally and so it defines an element $T(x)$ of Y. The mapping T so defined is linear (obviously) and bounded; indeed, since (y_n) is a copy of the unit vector basis of c_0 and (x_n^*) is a weak* null sequence, there is $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$. If T were in $K(X, Y)$, then T^* would be compact, too; and so the sequence $(T^*(y_n^*)) = (x_n^*)$ would be relatively compact, a contradiction. Hence Theorem 1 applies. Fron now on, we assume that Y contains no copy of c_0. Next, suppose that c_0 embeds in X^*. Let (x_n^*) be a copy of the unit vector basis of c_0 in X^* and (y_n) be a normalized basic sequence in Y with coefficient functionals (y_n^*). Let T_n and T be as above. The series $\sum T_n(x) (x \in X)$ is (absolutely and so) unconditionally convergent, since $\sum x_n^*$ is weakly unconditionally convergent. Again using the fact that $\sum x_n^*$ is weakly unconditionally convergent, we find an $M > 0$ such that $\sum |x_n^*(x)| \leq M\|x\|$ for all $x \in X$. Hence $\|T(x)\| \leq M\|x\|$ for all $x \in X$; thus, T being linear, we have $T \in L(X, Y)$. As above T is not compact and again Theorem 1 yields our result. So we can also assume that c_0 does not embed in X^*. Theorem 4 in [9] gives that ℓ^∞ does not embed in $K(X, Y)$. Now consider a copy (T_n) of the unit vector basis of c_0 inside $K(X, Y)$. For $\xi \in \ell^\infty$ and $x \in X$ consider the series $\sum \xi_n T_n(x)$ that converges unconditionally, because $\sum T_n$ is weakly unconditionally convergent and Y does not contain c_0. It is possible to define an operator $T_\xi(x) = \sum \xi_n T_n(x)$ belonging to $L(X, Y)$. Indeed, T_ξ is linear; let $y^* \in Y^*$, $\|y^*\| \leq 1$ and observe that

$$|T_\xi(x)(y^*)| \leq \sum |\xi_n T_n(x \otimes y^*)| \leq \|\xi\| \sum |T_n(x \otimes y^*)| \qquad (x \in X, \xi \in \ell^\infty). \qquad (1)$$

Because $\sum T_n$ is weakly unconditionally convergent and because $x \otimes y^* \in (K(X, Y))^*$, there is $M > 0$ such that

$$\sum |T_n(x)(y^*)| \leq M\|x \otimes y^*\| = M\|x\|\|y^*\| \qquad (x \in X). \qquad (2)$$

The inequalities (1) and (2) imply

$$\|T_\xi(x)\| \leq M\|\xi\|\|x\| \qquad (x \in X, \xi \in \ell^\infty), \qquad (3)$$

so giving the continuity of each T_ξ. Furthermore, (3) implies

$$\|T_\xi\| \leq M\|\xi\| \qquad (\xi \in \ell^\infty).$$

Hence it is also possible to define an operator $\phi : \ell^\infty \to L(X,Y)$ by putting $\phi(\xi) = T_\xi$. If $T_\xi \in K(X,Y)$ for all $\xi \in \ell^\infty$, then ϕ is weakly compact, since we are supposing that ℓ^∞ does not embed in $K(X,Y)$ (see [1], chapter 6); hence, ϕ maps weakly null sequences into norm null ones ([1], chapter 6) and so we have

$$\|\phi(e_n)\| = \|T_n\| \to 0,$$

where (e_n) denotes the unit vector basis of c_0. This is a contradiction. Hence for some $\xi_0 \in \ell^\infty$ the operator T_{ξ_0} is not compact. This T_{ξ_0} and the series $\sum \xi_{0n} T_n$ must satisfy Feder's assumption in Theorem 1, and our proof is complete.

As a consequence of proposition 1 in [8] and the present Theorem 2 we have the following.

`Corollary.` *Let Y have the bounded approximation property. If c_0 embeds in $K(X,Y)$, then $K(X,Y)$ is not isomorphic to a complemented subspace of a dual space.*

In [6] Feder put the following question.

Problem 2 ([6]). Do Banach spaces X and Y exist such that $L(X,Y) \neq K(X,Y)$ and such that for each element in $L(X,Y) \setminus K(X,Y)$ there does not exist a series as in Theorem 1?

In the light of Remark 1 this problem by Feder can be reformulated as it follows:

Problem 2 (reformulated). Do Banach spaces X and Y exist such that $L(X,Y) \neq K(X,Y)$ and such that c_0 does not embed in $K(X,Y)$?

Hence conditions assuring that c_0 embeds into $K(X,Y)$ are very useful; a number are contained in the papers [4, 5, 8, 9] as already quoted. Here we present a further method to construct isomorphic copies of c_0 in spaces of compact operators which is different from those already cited. It is an extension of techniques used by Holub in [7]; in passing we observe that we also present a sufficient condition for $K(X,Y)$ to contain a complemented copy of the sequence space c_0. We shall use the definition of a Gelfand-Phillips space and a limited subset from [2].

Definition. A Banach space E has the Gelfand-Phillips property if any bounded subset M of E such that

$$\lim_n \sup_M |x_n^*(x)| = 0 \qquad \text{for each weak}^* \text{ null sequence} \qquad (X_n^*) \subset E^* \tag{4}$$

is relatively compact. A set satisfying (4) is called 'limited'.

Theorem 3. *Let X, Y be two Banach spaces satisfying the following assumptions:*
(ii) there exist a Banach space G with an unconditional basis (g_n) and a biorthogonal coefficients (g_n^*) and two operators $R : G \to Y, S : G^* \to Y^*$ mapping (g_n) and (g_n^*) into normalized basic sequences.
Then c_0 embeds in $K(X,Y)$ (indeed in any subspace H of $L(X,Y)$ containing $X^* \otimes_\epsilon Y$). If moreover X^* (or Y) has the Gelfand-Phillips property, then $K(X,Y)$ contains a complemented copy of c_0.

Proof. First note that $(S(g_n^*) \otimes_\epsilon R(g_n))$ is contained in $X^* \otimes_\epsilon Y$. As $(S(g_n^*))$ and $(R(g_n))$ are basic sequences, there is a constant $C > 0$ such that for any real numbers $a_1, a_2, ..., a_n$ we have

$$C \max_{1 \le i \le n} |a_i| \le \left\| \sum_{i=1}^n a_i \left[S(g_n^*) \otimes_\epsilon R(g_n) \right] \right\|_\epsilon$$

thanks to the definition of $\epsilon-$tensor norm. On the other hand, S and R induce an operator $S \otimes_\epsilon R$ from $G^* \otimes_\epsilon G$ into $X^* \otimes_\epsilon Y$ (see [1], p. 229) mapping $(g_n^* \otimes_\epsilon g_n)$ to $(S(g_n^*) \otimes_\epsilon R(g_n))$. So we have, for $a_1, a_2, ..., a_n \in \mathbb{R}$

$$\left\| \sum_{i=1}^n a_i \left[S(g_i^*) \otimes_\epsilon R(g_i) \right] \right\|_\epsilon \le \|S \otimes_\epsilon R\| \left\| \sum_{i=1}^n a_i \left[g_i^* \otimes_\epsilon g_i \right] \right\|_\epsilon \le K \|S \otimes_\epsilon R\| \max_{1 \le i \le n} |a_i|,$$

where the last inequality is due to the unconditionality of (g_n). Hence $(S(g_i^*) \otimes_\epsilon R(g_i))$ is a copy pf the unit vector basis of c_0. Assume now that X^* has the Gelfand-Phillips property (we shall proceed similarly if Y has the Gelfand-Phillips property). Then $(S(g_n^*))$ cannot be a limited subset of X^*. Hence, by passing to a subsequence if necessary, we can find a weak* null sequence $(x_n^{**}) \subset X^{**}$ such that $x_n^{**}(S(g_m^*)) = \delta_{nm}$ (see [11]). If (z_n^*) is a (bounded) sequence of biorthogonal coefficients for $(R(g_n))$, the sequence $(x_n^{**} \otimes_\pi z_n^*)$ belongs to $K(X,Y)^*$. Furthermore, if $T \in K(X,Y)$, we have

$$| < T, x_n^{**} \otimes_\pi z_n^* > | = |T^*(z_n^*)(x_n^{**}))| \to 0$$

because $(T^*(z_n^*))$ is a relatively compact sequence and (x_n^{**}) is weak* null. Hence $(x_n^{**} \otimes_\pi z_n^*)$ is a weak* null sequence in $(K(X,Y))^*$ and, by our choices above, one has $(x_n^{**} \otimes_\pi z_n^*)(S(g_m^*) \otimes_\epsilon R(g_m)) = \delta_{nm}$. It is now easy to see that the operator P from $K(X,Y)$ to $\overline{span}(S(g_n^*) \otimes_\epsilon R(g_n))$ defined by $P(T) = \sum < T, x_n^{**} \otimes_\pi z_n^* > (S(g_n^*) \otimes_\epsilon R(g_n))$ is a projection. This completes the proof.

Remark 2. Observe that the hypothesis that X^* has the Gelfand-Phillips property was used just to provide the sequence (x_n^{**}) used in the proof of Theorem 3. It could be replaced by the hypothesis that $(S(g_n^*))$ (resp. $(R(g_n))$) spans a complemented subspace of X^* (resp. Y) because even in this case $(S(g_n^*))$ cannot be limited: otherwise it would be limited in its closed linear span, a separable space, i.e. a space with the Gelfand-Phillips property. Hence $(S(g_n^*))$ would be relatively compact.

We note that if X is a Banach space containing ℓ^1 and Y a Banach space containing ℓ^p, for some $p \geq 2$, then c_0 embeds in $K(X,Y)$ which cannot be complemented in $L(X,Y)$. Indeed, by a famous result in [10], X^* must contain a copy of (L^1 and hence of) ℓ^2. Furthermore, there is an operator as required in Theorem 3 from ℓ^2 into ℓ^p, since $p \geq 2$. Theorem 3 can be applied to this situation. This way we generalize the following result due to Feder ([6]): $K(C(S), L^1)$ is not complemented $L(C(S), L^1)$, when S is not dispersed (and other results from [6]), when answering a question put in [8].

As another consequence of Theorem 3 (second part) we note that the space $\ell^p \otimes_\epsilon \ell^q$ (a closed subspace of $K(\ell^{p'}, \ell^q)$, with $1/p + 1/p' = 1$) contains a complemented copy of c_0, provided that $1 < p' \leq q < \infty$; indeed, in this case there is an operator as required in Theorem 3 from $\ell^{q'}$, with $1/q + 1/q' = 1$, into ℓ^p, mapping the unit vector basis of $\ell^{q'}$ onto the unit vector basis of ℓ^p. Hence the considered space is neither a Grothendieck space (i.e. a space such that any operator from it to c_0 is weakly compact) nor a dual space. We recall that in the remaining case, (i.e. when $1 < q < p' < \infty$), it is well known that $\ell^p \otimes \ell^q$ is reflexive (and so both Grothendieck and dual). Further, let us consider $\ell^\infty \otimes_\pi \ell^p$ where $1 < p \leq 2$. Its dual space is $L(\ell^\infty, \ell^{p'})$, where $1/p + 1/p' = 1$, and this latter space contains a copy of c_0 as a consequence of Theorem 3. Therefore $\ell^\infty \otimes_\pi \ell^p$, where $1 < p \leq 2$, is not a Grothendieck space because its dual is not weakly sequentially complete.

Finally, we present a result similar to Theorem 6 in [9], but for different classes of Banach spaces; it makes use of Theorem 2.

`Theorem 4.` *Assume that X and Y satisfy one of the following two assumptions:*
(i) X is an $\mathcal{L}_\infty-$space and Y is a closed subspace of an $\mathcal{L}_1-$space, or
(ii) $X = C[0,1]$ and Y is a space with cotype 2.
Then the following four assertions are equivalent:
(1) $K(X,Y) \neq L(X,Y)$
(2) c_0 embeds in $K(X,Y)$
(3) ℓ^∞ embeds in $L(X,Y)$
(4) $K(X,Y)$ is uncomplemented in $L(X,Y)$

Proof. We start by showing that (1) implies (2). We observe that any element in $L(X,Y)$ factorizes through a suitable $\ell^2(\Gamma)$ space, as any $2-$absolutely summing operator must do (see [3]). Hence, if $T \in L(X,Y) \setminus K(X,Y)$, there are two non-compact operators $R : X \to \ell^2(\Gamma)$ and $S : \ell^2(\Gamma) \to Y$ so that $T = SR$. If P_j is the projection from $\ell^2(\Gamma)$ onto the closed span generated by the element e_j (belonging to a basis $(e_i)_{i\in I}$), then $T_j = SP_jR \in K(X,Y)$. Furthermore, for all $x \in X$, we have $T(x) = \sum_{j\in\Gamma} T_j(x)$ unconditionally. We choose a sequence $(x_n) \subset B_X$ such that $(T(x_n))$ is not relatively compact. Let $n \in N$; it is clear that the set $\Gamma_n = \{j : j \in \Gamma, T_j(x_n) \neq 0\}$ is at most countable and so the set $\Gamma_0 = \bigcup_n \Gamma_n$ is at most countable, too; we can order it as a sequence (j_k) and consider, for $x \in X$, the series $\sum T_{j_k}(x)$ that converges unconditionally to an element $B(x)$ of Y. The mapping $x \to B(x)$ is clearly linear. It is quite easy to show that the same mapping is s-w sequentially continuous and so, by virtue of the Closed Graph Theorem, it is bounded, i.e. $B \in L(X,Y)$. Moreover, $B(x_n) = T(x_n)$ for all $n \in \mathbb{N}$, and so B is not compact. The proof of the main result in [4] now shows that (2) is true. Since the implication (2) $\Rightarrow$ (4) is just Theorem 2 and (4) $\Rightarrow$ (1) is obvious, we see that (1), (2) and (4) are equivalent. That (2) implies (3) is a general fact (see [5]); so we have to show that (3) gives (2). Under (3) we can have $K(X,Y) \neq L(X,Y)$; in this case the above proof works again to give (2). otherwise, $K(X,Y) = L(X,Y)$ from which we get that ℓ^∞ embeds into $K(X,Y)$. This completes the proof of Theorem 4.

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Note added in proof. The reformulation of Problem 2 above shows that it has a positive solution, because of the existence of a $\mathcal{L}_\infty$−space X with Schur property (see [12]). Indeed, it is clear that $L(X,X) \neq K(X,X)$ and yet c_0 does not embed into $K(X,X)$ which is weakly sequentially complete, because of the following facts: X and X^* are weakly sequentially complete and X has the Schur property, so that each weakly compact operator from X into Y is compact. Though the paper [12] appeared before Feder raised Problem 2, the condition in Problem 2 is hard to study. Our reformulation makes it easier to study and leads to the solution above.

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