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Mathematical Proceedings of the Cambridge Philosophical Society
Volume 109, (1991), 161-166

On the reciprocal Dunford-Pettis property in projective tensor products

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(*Received* 19 *December* 1989; *revised* 5 *July* 1990)

Abstract

We prove the following result: if a Banach space E does not contain ℓ^1 and F has the (RDPP), then $E \otimes_\pi F$ has the same property, provided that $L(E, F^*) = K(E, F^*)$. Hence we prove that if $E \otimes_\pi F$ has the (RDPP) then at least one of the spaces E and F must not contain ℓ^1. Some corollaries are then presented as well as results concerning the necessity of the hypothesis $L(E, F^*) = K(E, F^*)$.

In the famous paper [8] A. Grothendieck introduced the following isomorphic property (that he called the *reciprocal Dunford-Pettis property*, in symbols (RDPP)): *a Banach space E has the* (RDPP) *if any Dunford-Pettis operator $T{:}E \to F$ where F is an arbitrary Banach space, is weakly compact.* It is known that a Banach space with the so-called property (V) of Pelczynski [13] has the (RDPP) that is inherited even by quotients; the same happens if a Banach space does not contain ℓ^1: it enjoys the (RDPP). In the case of a Banach lattice, C. Niculescu proved in [12] that E has the (RDPP) if and only if it does not contain complemented copies of ℓ^1. Furthermore, if K is a compact Hausdorff space and E is a Banach space not containing ℓ^1, then $C(K, E)$ has the (RDPP)(see [9]), whereas if K is dispersed, then $C(K, E)$ has the (RDPP) if and only if E has the same property (see [1]). As far as we know, no other results about the (RDPP) are known but the following useful characterization of the (RDPP) obtained in [10]:

`Theorem 1.` *A Banach space E has the* (*RDPP*) *if and only if each (bounded) subset M of E^* such that*

$$\lim_n \sup_M |x_n(x^*)| = 0 \tag{1}$$

for each w-null sequence $(x_n) \subset E$ is relatively weakly compact.

The purpose of this note is to present a result on the construction of new Banach spaces with the (RDPP) from old ones, by taking (suitable) projective tensor products. More precisely, we prove that if E does not contain ℓ^1, F has the (RDPP) and $L(E, F^*) = K(E, F^*)$, then $E \otimes_\pi F$ has the (RDPP). Here L(E,F*) and K(E,F*) denote the Banach spaces of all operators and compact operators from E into F^*, respectivley. Hence we show that if $E \otimes_\pi F$ has the (RDPP), then necessarily at least one of the spaces E and F must not contain ℓ^1. In order to prove our main theorem we need to use Theorem 1 and the following characterization of Banach spaces not containing ℓ^1 proved in [3]:

`Theorem 2.` *A Banach space E does not contain ℓ^1 if and only if each (bounded) subset M of E^* satisfying (1) is relatively compact.*

At the end of the paper we present some results concerning the necessity of the assumption $L(E, F^*) = K(E, F^*)$ considered in the main result, showing that it is not possible to dispense with it completely. We are now ready to prove our main result.

`Theorem 3.` *Let E be a Banach space not containing ℓ^1 and F a Banach space with the (RDPP). If L(E,F*) = K(E,F*), then $E \otimes_\pi F$ has the (RDPP).*

Proof. Let M be a subset of $(E \otimes_\pi F)^* = L(E, F^*) = K(E, F^*)$ satisfying (1) and (h_n) a sequence in M. Observe that the closed subspace

$$H = \overline{span}\{h_n(x) : x \in E, n \in \mathbb{N}\}$$

of F^* is separable and that $(h_n) \subset K(E, H) \subset K(E, F^*)$. Let Y be a countable w$^*-$dense subset of H^*. If $y \in Y$, then the sequence $(h_n^*(y)) \subset E^*$ is a subset satisfying (1). Indeed, let $(x_n) \subset E$ be a w-null sequence. We consider, for $n \in \mathbb{N}$,

$$|h_n^*(y)(x_n)| = |h_n(x_n)(y)| \leq \|y\| \, \|h_n(x_n)\|_H = \|y\| \, \|h_n(x_n)\|_{F^*}$$

and we show that $\|h_n(x_n)\|_{F^*} \to 0$. If this were false, there would exist $r > 0$, $(h_{k(n)})$, $(x_{k(n)})$ and $(z_n) \subset B_F$ such that

$$r < |h_{k(n)}(x_{k(n)})(z_n)| \qquad \text{for all} \quad n \in \mathbb{N}$$

Observe that $(x_{k(n)} \otimes z_n) \subset E \otimes_\pi F$ and that, for $T \in (E \otimes_\pi F)^*$, one has

$$|T(x_{k(n)} \otimes z_n)| = |T(x_{k(n)})(z_n)| \leq \|T(x_{k(n)})\| \to 0$$

since T is compact and $x_{k(n)} \xrightarrow{w} \theta$. Hence $(x_{k(n)} \otimes z_n)$ is w-null and so $h_{k(n)}(x_{k(n)})(z_n) \to 0$ a contradiction proving our claim. Hence $(h_n^*(y))$ satisfies (1) and by virtue of Theorem 1 we can assume (and we do) that $h_n^*(y))$ is a weak Cauchy sequence of E^* (otherwise we pass to a subsequence) for all $y \in Y$, because Y is countable. Now let $x^{**} \in E^{**}$ and consider $(h_n^{**}(x^{**})) \subset F^*$. We claim that it satisfies (1). Let (z_n) be a w-null sequence in F and consider, for $n \in \mathbb{N}$,

$$|h_n^{**}(x^{**})(z_n)| = |h_n^*(z_n)(x^{**})| \leq \|x^{**}\| \, \|h_n^*(z_n)\|_{E^*}.$$

As above we prove that $\|h_n^*(z_n)\|_{E^*} \to 0$. Thus for all x^{**} in E^{**}, $(h_n^{**}(x^{**}))$ is a relatively weakly compact subset of F^* by virtue of Theorem 1. But $h_n : E \to H$ and h_n is compact, for all $n \in \mathbb{N}$; so $(h_n^{**}(x^{**})) \subset H$. Let z', z'' be two w-sequential cluster points of $(h_n^{**}(x_n^{**}))$. If $y \in Y$ we have

$$\begin{aligned} z'(y) &= \lim_n h_{k(n)}^{**}(x^{**})(y) = \lim_n x^{**}(h_{k(n)}^*(y)) = \lim_n x^{**}(h_n^*(y)) \\ &= \lim_n x^{**}(h_{p(n)}^*(y)) = \lim_n h_{p(n)}^{**}(x^{**})(y) = z''(y)\,, \end{aligned}$$

if $h_{k(n)}^{**}(x^{**}) \xrightarrow{w} z'$ and $h_{p(n)}^{**}(x^{**}) \xrightarrow{w} z''$. Hence $z'(y) = z''(y)$ for all $y \in Y$ and so $z' = z''$ since Y is w$^*-$dense in H^*. This means that, for all $x^{**} \in E^{**}$, there is $\tilde{h}(x^{**}) \in H$ such that

$$\tilde{h}(x^{**}) = w - lim_n h_n^{**}(x^{**}).$$

Of course $\tilde{h} \in L(E^{**}, H) \subset L(E^{**}, F^*)$. Now we show that $\tilde{h}$ is w$^*-$w* continuous from E^{**} into F^*. Let (x_α^{**}) be a w*-null net in E^{**} and $y \in F$. As at the beginning, $(h_n^*(y))$ satisfies (1). Now Theorem 2 comes into play: $(h_n^*(y))$ is a relatively compact subset of E^*. There exist $x^* \in E^*$ and a subsequence $(h_{k(n)}^*(y))$ converging to x^*. This gives that

$$\begin{aligned} \lim_\alpha \tilde{h}(x_\alpha^{**})(y) &= \lim_\alpha(\lim_n h_{k(n)}^{**}(x_\alpha^{**})(y)) = \lim_\alpha(\lim_n x_\alpha^{**}(h_{k(n)}^*(y))) \\ &= \lim_\alpha x_\alpha^{**}(x^*) = 0\,. \end{aligned}$$

Now, consider $h \in L(E, F^*) = K(E, F^*)$ defined by $h = \tilde{h}_{|E}$. We have $h^{**} = \tilde{h}$. Indeed, if $x^{**} \in E^{**}$ there is a (bounded) net $(x_\alpha) \subset E$ w$^*-$converging to x^{**}; so we obtain

$$h^{**}(x^{**}) = w^* - \lim_\alpha h^{**}(x_\alpha) = w^* - \lim_\alpha h(x_\alpha) = w^* - \lim_\alpha \tilde{h}(x_\alpha) = \tilde{h}(x^{**})\,.$$

By the construction of $\tilde{h}$ we thus have

$$\lim_n h_n^{**}(x^{**})(y^{**}) = h^{**}(x^{**})(y^{**})$$

for all $x^{**} \in E^{**}$ and $y^{**} \in F^{**}$, a fact implying that $h_n \xrightarrow{w} h$ in $K(E, F^*)$ (see [15]). ∎

Theorem 3 has the following interesting corollaries

`Corollary 4.` *Assume E and F have the (RDPP) and E^* and F^* are weakly sequentially complete. If $L(E, F^*) = K(E, F^*)$, then $E \otimes_\pi F$ has the (RDPP).*

Proof. The proof of Theorem 3 (until the definition of $\tilde{h}$) shows that (h_n) is a weak Cauchy sequence (using a result in [11]). But $K(E, F^*)$ is weakly sequentially complete (see [11]).∎

`Corollary 5.` *Let E be a subspace of an order continuous Banach lattice. If E, F have the (RDPP) and L(E,F^*) = K(E,f^*), then $E \otimes_\pi F$ has the (RDPP).*

Proof. If E has the (RDPP) then it cannot contain complemented copies of ℓ^1. hence ℓ^1 cannot lie inside E (see [16]). Now Theorem 3 may be applied. ∎

`Corollary 6.` *Let E^* not contain ℓ^1 and F have the (RDPP). If L(E^*,F^*) = K(E^*,F^*), then the space N_1(E,F) of all nuclear operators from E into F, has the (RDPP).*

Proof. $N_1(E, F)$ is a quotient of $E^* \otimes_\pi F$.∎

`Corollary 7.` *Let E,F be two Banach spaces such that E^{**} and F^* have the (RDPP). Let us assume that ℓ^1 does not embed in either E^{**} or F^* and that either E^{**} or F^* has the Radon-Nikodym property. If L(E^{**},F^{**}) = K(E^{**},F^{**}), then $(K(E, F()^*$ has the (RDPP).*

Proof. $(K(E, F))^*$ is a quotient of $E^{**} \otimes_\pi F^*$ (see [7]).∎

One could also ask to what extent the assumption that E does not contain ℓ^1 is necessary for the validity of Theorem 3. We have the following result

`Theorem 8.` *Let E and F be two Banach spaces with L(E,F*) = K(E,F*). Then the following facts are equivalent:*

(j) E and F have the (RDPP) and ℓ^1 *fails to embed in at least one of them;*

(jj) $E\otimes_\pi F$ *has the (RDPP).*

Proof. (j) $\Rightarrow$ (jj). If E does not contain copies of ℓ^1, then Theorem 3 gives the conclusion. Assume now that ℓ^1 cannot be embedded in F. Since $E \otimes_\pi F$ is isometrically isomorphic with $F \otimes_\pi E$ it is enough to show that $F \otimes_\pi E$ has the (RDPP); this will be done using Theorem 3 again. Hence, we have just to show that $L(F, E^*) = K(F, E^*)$. Let us consider

$$T : F \to E^* \qquad \text{and} \qquad T^* : E^{**} \to F^*.$$

Define $\tilde{T} = T^*_{|E}$ and note that $\tilde{T}$ is compact, by our assumptions. Let $x^{**} \in B_{E^{**}}$; there is a net $(x_\alpha) \subset B_E$ weak* converging to x^{**}. Then the net $(T^*(x_\alpha))$ weak* converges to $T^*(x^{**})$. But $(T^*(x_\alpha))$ is contained in $\tilde{T}(B_E)$, a relatively compact set and so a suitable subnet $(\tilde{T}(x_{\alpha_\beta}))$ must converge strongly; of course its limit is $T^*(x^{**})$. This means that

$$T^*(B_{E^{**}}) \subset \overline{\tilde{T}(B_e)},$$

i.e. T^* and so T is compact.

(jj) $\Rightarrow$ (j). That E and F have the (RDPP) is clear. Assume, now, that ℓ^1 can be embedded in E and F. Hence (L^1 and so) ℓ^2 can be embedded in E^* and F^*. From this we have that $\ell^2 \otimes_\epsilon \ell^2$ is isomorphic to a subspace of $L(E, F^*)$: but c_0 embeds into $\ell^2 \otimes_\epsilon \ell^2$. This is a contradiction.■

From Corollary 4 and Theorem 8 it follows that if E and F are two banach spaces with the (RDPP), with weakly sequentially complete duals and $L(E, F^*) = K(E, F^*)$, then one of them fails to contain ℓ^1. We note that if a Banach space with local unconditional structure (see [2]) has the (RDPP), then its dual space is weakly sequentially complete, as pointed out in the paper [4].

We conclude the paper with a few remarks on the assumption $L(E, F^*) = K(E, F^*)$ used in Theorem 3. If E^* has the Schur property and F the (RDPP), then it always holds true. Indeed, the unit ball B_E of E is a Dunford-Pettis set (i.e. a set such that $\lim_n \sup_{B_E} |x_n^*(x)| = 0$ for each w-null sequence $(x_n^*) \subset E^*$). Hence, $T(B_E)$ satisfies (1) in

F^*, for any $T \in L(E, F^*)$. Such a T must be weakly compact and hence compact, thanks to our assumption on E^*. We remark that if E^* merely has the Schur property, then E cannot contain ℓ^1. Hence we have

`Corollary 9.` *Let E^* have the Schur property and F the (RDPP). Then $E \otimes_\pi F$ has the (RDPP).*

`Corollary 10.` *Let $E = \ell^p$, where $1 < p \leq \infty$, and $F = c_0$. Then $E \otimes_\pi F$ has the (RDPP).*

Observe, now, that if E is an $\mathcal{L}_\infty$-space and F^* a subspace of an $\mathcal{L}_1$-space, then any $T \in L(E, F^*)$ is (2-absolutely summing and hence) Dunford-Pettis. So we obtain

`Corollary 11.` *Let E be an $\mathcal{L}_\infty$-space not containing ℓ^1 and F have the (RDPP). If F^* is a subspace of an $\mathcal{L}_1$-space, then $E \otimes_\pi F$ has the (RDPP).*

The last results show that the assumption $L(E, F^*) = K(E, F^*)$ is even necessary, in special cases, for the validity of Theorem 3 and Theorem 8.

`Corollary 12.` *Let E not contain ℓ^1 and F have the (RDPP). If F^* is complemented in a Banach space Z with an unconditional Schauder decomposition (Z_n), with Z_n satisfying the Schur property for all $n \in \mathbb{N}$, then the following facts are equivalent:*

(h) $E \otimes_\pi F$ has the (RDPP);

(hh) $L(E,F^*) = K(E,F^*)$.

Proof. (h) $\Rightarrow$ (hh). If (hh) were false, a result in [5] would give the existence of a copy of c_0 into $K(E, F^*)$, a contradiction. The implication (hh) $\Rightarrow$ (h) follows from Theorem 3.∎

Corollary 12 even proves that the hypotheses that E does not contain ℓ^1 and F has the (RDPP) alone are not strong enough to guarentee that $E \otimes_\pi F$ has the (RDPP): $\ell^p \otimes_\pi \ell^q$, where $1 < p < q' < \infty$ and q, q' are dual numbers, does not have the (RDPP) since $L(E, F^*) \neq K(E, F^*)$ in this setting. We conclude with the following

`Theorem 13.` *Assume one of the following hypotheses holds:*

(k) E is an $\mathcal{L}_\infty$-space and F^* a subspace of an $\mathcal{L}_1$-space;

(kk) $E = C(K)$, K Hausdorff compact space, F^* is a space with co-type 2;

(kkk) E has the Dunford-Pettis property and F contains a copy of ℓ^1.

If $E \otimes_\pi F$ has the (RDPP), then L(E,F^*) = K(E,F^*).

Proof. Assume that $E \otimes_\pi F$ has the (RDPP) and (k) or (kk) holds. It is known that any operator from E into F^* is 2-absolutely summing, from [14], and so it factorizes through a Hilbert space; in [6] we remarked that this fact implies that $L(E, F^*)$ contains a copy of c_0, if $L(E, F^*) \neq K(E, F^*)$. This is a contradiction. If (kkk) is true, we can proceed as follows. Assume $E \otimes_\pi F$ has the (RDPP); from Theorem 8, E is not allowed to contain copies of ℓ^1. Hence the unit ball of E is a Dunford-Pettis set as well as $T(B_E)$ for each T in $L(E, F^*)$. Since F has the (RDPP) and a Dunford-Pettis set in F^* satisfies (1), $T(B_E)$ is weakly compact. Since E has the Dunford-Pettis property, T is a Dunford-Pettis operator and hence a compact operator, because ℓ^1 does not embed in E, as stated at the beginning. Since T is arbitrary, the proof is complete.■

This work was done under the auspices of GNAFA of CNR and partially supported by MURST of Italy (40%).

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