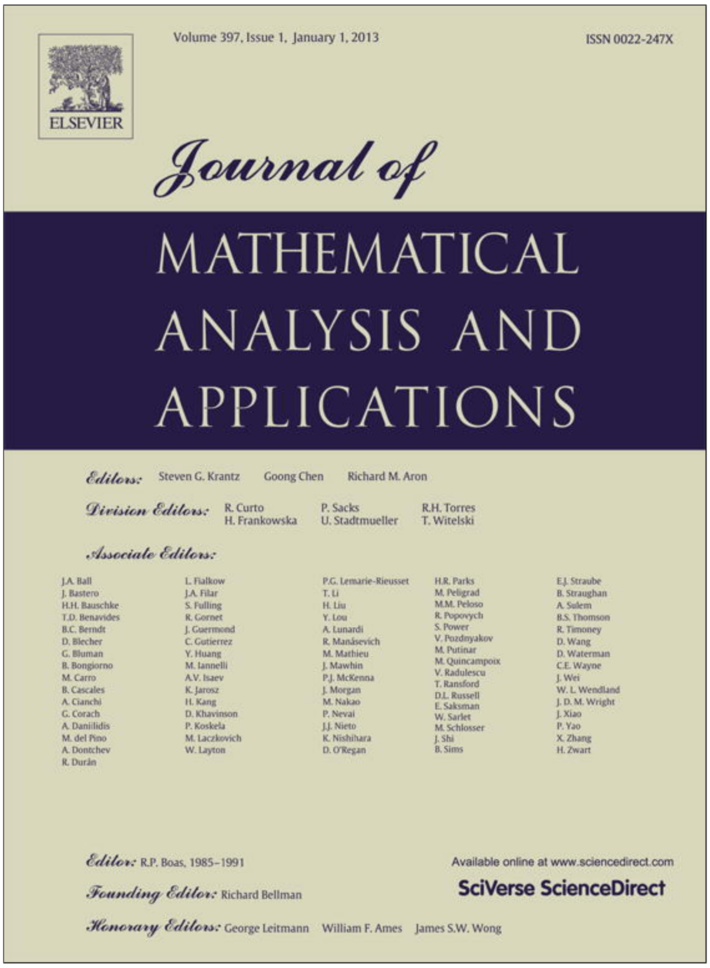
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On relative compactness in $K(X, Y)$

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I wish to dedicate this note to Francesca.

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ABSTRACT

We give a sufficient condition for the relative compactness of subsets of $K(X, Y)$ that is similar to an 11-year-old result obtained by Mayoral, but actually extends Mayoral's result.

In this note, X, Y will always denote Banach spaces and $K(X, Y)$ the usual space of compact operators from X to Y endowed with the operator norm.

In 2001, Mayoral [1] proved a sufficient condition for relative compactness in $K(X, Y)$, after introducing the following notion of sequential weak-norm equicontinuity.

Definition 1 (*[1]*)**.** A subset $H \subset K(X, Y)$ is said to be sequentially weak-norm equicontinuous if for each weak null sequence $(x_n) \subset X$ the sequence $(T(x_n))$ converges uniformly on $T \in H$.

He used this in order to prove the following result.

Theorem 2 (*Mayoral [1]*)**.** *Assume that X is a Banach space not containing copies of l_1 and that Y is an arbitrary Banach space. Let H be a subset of $K(X, Y)$ satisfying the following two conditions:*

(i) *for each $x \in X$ the set $\{T(x) : T \in H\}$ is relatively compact in Y,*
(ii) *H is sequentially weak-norm equicontinuous.*

Then, H is relatively compact.

We observe that conditions (i) and (ii) are also necessary conditions for a subset of $K(X, Y)$ to be relative compact, with no assumption on X. So, it appears quite natural to ask if (i) and (ii), without any other assumptions, are sufficient conditions for relative compactness in $K(X, Y)$. This is not the case, as has been proved in [2]. Indeed, using an old result of ours [3], which characterizes Banach spaces not containing l_1 via the relative compactness of suitable subsets of X^*, the authors of [2] showed that the implication "if $H \subset K(X, Y)$ satisfies (i) and (ii), thus it is relatively compact" in Mayoral's theorem is true only if X does not contain copies of l_1. So there is also no hope of obtaining a generalization of Mayoral's theorem just enlarging the class of Banach spaces X to which it applies, but maintaining (i) and (ii) as the only conditions equivalent to relative compactness. In order to get an improvement of Mayoral's theorem, as we shall do here, we have considered a similar but more restrictive condition (in the general case) that, jointly with (i), allows us to present a new sufficient condition for relative compactness in $K(X, Y)$ just supposing that X^* satisfies the Gelfand–Phillips property, according to the following.

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Definition 3 (*[4,5]*). A bounded subset M of a Banach space X is called a limited set if

$$\sup_{x\in M} |x_n^*(x)| \to 0$$

for each weak* null sequence $(x_n^*) \subseteq X^*$. A Banach space X is said to have the Gelfand–Phillips property if any of its limited subsets is relatively compact.

The family of Banach spaces with the Gelfand–Phillips property is strictly larger than the one considered by Mayoral; it is indeed well known that Banach spaces not containing l_1 have a dual space with the Gelfand–Phillips property, but there exist spaces X with copies of l_1 inside such that X^* has the Gelfand–Phillips property (see [4,6,5], for instance).

The proof of Mayoral's theorem uses the idea of isomorphically embedding the considered set H in some space of continuous functions where it is possible to use an Ascoli–Arzelá-type theorem to get the relative compactness of H. Our proof, instead, uses the following old result by Palmer [7], which is a necessary and sufficient condition for a subset of $K(X, Y)$ being relatively compact.

Theorem 4 (*Palmer, [7]*). *A subset $H \subset K(X, Y)$ is relatively compact if and only if*

(h) $H^*(B_{Y^*}) = \{T^*(y^*) : T \in H, y^* \in Y^*, \|y^*\| \le 1\}$ *is relatively compact (in such a case we shall say that H^* is* collectively compact*), and*
(hh) $H(x) = \{T(x) : T \in H\}$ *is relatively compact, for all $x \in X$.*

Now we are ready to present our main theorem, proof of which is very much simpler than that of Mayoral's theorem (and could be used to simplify the proof of Mayoral's theorem, too, thanks to the quoted characterization of Banach spaces not containing l_1 from [3]).

Theorem 5. *Assume that X is a Banach space such that X^* satisfies the Gelfand–Phillips property and Y is an arbitrary Banach space. Let $H \subset K(X, Y)$ be a subset of $K(X, Y)$ satisfying the following two conditions:*

(j) *for each $x \in X$ the set $\{T(x) : T \in H\}$ is relatively compact in Y,*
(jj) *for each weak * null sequence $(x_n^{**}) \subset X^{**}$ the sequence $(T^{**}(x_n^{**}))$ converges uniformly on $T \in H$.*

Then, H is relatively compact.

Proof. According to (h) and (hh) in Palmer's theorem, it is enough to show that under our assumptions the set $\{T(x) : T \in H\}$ is relatively compact in Y (this is just our hypothesis (j)) and the set $\{T^*(y^*) : T \in H, y^* \in Y^*, \|y^*\| \le 1\}$ is relatively compact in X^*. To get this second condition it is enough to prove that the set $\{T^*(y^*) : T \in H, y^* \in Y^*, \|y^*\| \le 1\}$ is limited in X^*, thanks to our assumption on X^*. So let us consider a weak* null sequence $(x_n^{**}) \subset X^{**}$. Because of (jj) we have that $\sup_{T\in H} \|T^{**}(x_n^{**})\| \longrightarrow 0$. Hence, $\sup_{T\in H, y^*\in Y^*, \|y^*\|\le 1} |x_n^{**}[T^*(y^*)]| = \sup_{T\in H, y^*\in Y^*, \|y^*\|\le 1} |T^{**}(x_n^{**})(y^*)| \longrightarrow 0$, which means that the set $\{T^*(y^*) : T \in H, y^* \in Y^*, \|y^*\| \le 1\}$ is limited in X^*. We are done. □

It is clear that our assumption (jj) is stronger than Mayoral sequential weak-norm continuity, so one could think that Mayoral's result and our Theorem 5 (working in a larger class of Banach spaces) are not comparable. Actually Theorem 5 improves Mayoral's result. This happens because again our old characterization of Banach spaces not containing l_1 [3] can be used to show that (i) and (ii) when X does not contain copies of l_1 imply the assumptions we have considered in Theorem 5 above, as we shall see immediately. Indeed we have the following result, for the proof of which we need to use the following notion very recently introduced by Serrano et al. [8].

Definition 6. A subset $H \subset K(X, Y)$ is said to be equicompact if there is a norm null sequence $(x_n^*) \subset X^*$ such that

$$\|T(x)\| \le \sup_{n\in\mathbb{N}} |x_n^*(x)| \quad \forall x \in X,\ T \in H,$$

or, equivalently, if there is a norm null sequence $(x_n^*) \subset X^*$ such that

$$\|T^{**}(x^{**})\| \le \sup_{n\in\mathbb{N}} |x^{**}(x_n^*)| \quad \forall x^{**} \in X^{**},\ T \in H.$$

Theorem 7 (*Mayoral Theorem, [1]*). *Assume that X is a Banach space not containing copies of l_1 and that Y is an arbitrary Banach space. Let $H \subset K(X, Y)$ be a subset of $K(X, Y)$ satisfying the following two conditions:*

(i) *for each $x \in X$ the set $\{T(x) : T \in H\}$ is relatively compact in Y,*
(ii) *H is sequentially weak-norm equicontinuous.*

Then, H is relatively compact.

Proof. In [8] it has been shown that Mayoral's assumption (ii) and "X does not contain copies of l^1" together give that H is equicompact. And so there is a null sequence $(x_n^*) \subset X^*$ such that

$$\|T^{**}(x^{**})\| \le \sup_n |x^{**}(x_n^*)| \quad \forall x^{**} \in X^{**},\ T \in H.$$

Assuming now that $(x_p^{**}) \subset X^{**}$ is weak*-null, we have

$$\|T^{**}(x_p^{**})\| \le \sup_n |x_p^{**}(x_n^*)| = L_p \quad \forall\, p \in \mathbb{N},\ T \in H.$$

Fixing $p \in \mathbb{N}$, there is $n_p \in \mathbb{N}$, so $L_p < |x_p^{**}(x_{n_p}^*)| + \frac{1}{p}$. Since $(x_p^{**}) \subset X^{**}$ is weak*-null we may define an operator $P : X^* \to c_0$ by putting $P(x^*) = (x_p^{**}(x^*)), x^* \in X^*$, for which $(P(x_n^*))$ goes to zero. Hence, for $\epsilon > 0$ there is $\nu \in \mathbb{N}$, so $\|P(x_n^*)\| = \sup_{p\in\mathbb{N}} |x_p^{**}(x_n^*)| < \epsilon$ for all $n > \nu$. Since

$$\begin{aligned} L_p &= \max(|x_p^{**}(x_1^*)|, |x_p^{**}(x_2^*)|, \ldots .|x_p^{**}(x_\nu^*)|, \sup_{n>\nu} |x_p^{**}(x_n^*)|) \\ &\le \max(|x_p^{**}(x_1^*)|, |x_p^{**}(x_2^*)|, \ldots .|x_p^{**}(x_\nu^*)|, \sup_p \sup_{n>\nu} |x_p^{**}(x_n^*)|) \\ &< \max(|x_p^{**}(x_1^*)|, |x_p^{**}(x_2^*)|, \ldots .|x_p^{**}(x_\nu^*)|, \epsilon). \end{aligned}$$

But $x_p^{**}(x_i^*) \to 0$ as $p \to \infty$ for all $i = 1, 2, \ldots, \nu$. So there is $p_0 \in \mathbb{N}$ such that $p > p_0$ gives $|x_p^{**}(x_i^*)| < \epsilon$, $i = 1, 2, \ldots, \nu$. It follows that

$$\|T^{**}(x_p^{**})\| < \epsilon \quad p > p_0$$

with p_0 not depending on the choice of $T \in H$. We are done. □

Even in the case of Theorem 5, as for Mayoral's theorem, assumptions (j) and (jj) are easily shown to be also necessary for the relative compactness of a subset $H \subset K(X, Y)$, but again the result cannot be extended to a larger class of Banach spaces because if X, Y are such that any subset $H \subset K(X, Y)$ satisfying (j) and (jj) is relatively compact, then X^* must have the Gelfand–Phillips property. Indeed, let us consider a bounded limited subset $M \subset X^*$ and a $y_0 \in Y$, $y_0 \neq 0$ and define $T_{x^*} \in K(X, Y)$ by $T_{x^*}(x) = x^*(x)y_0 : X \to Y$ for each $x^* \in M$. It is clear that the set $\{T_{x^*} : x^* \in M\}$ satisfies (j). Moreover, if $(x_n^{**}) \in X^{**}$ is a weak* null sequence, we have $\sup_{x^*\in M} \|T_{x^*}^{**}(x_n^{**})\| = \sup_{x^*\in M} \|x_n^{**}(x^*)y_0\| \longrightarrow 0$ since M is limited in X^*. (jj) is thus verified. Hence the set $\{T_{x^*} : x^* \in M\}$ is relatively compact in $K(X, Y)$; but this easily gives the relative compactness of M. We are done.

Our assumption (jj) is verified if $H \subset K(X, Y)$ is a limited set. Indeed, suppose H is limited but for some weak* null sequence $(x_n^{**}) \subset X^{**}$ there is $\epsilon > 0$ such that $\sup_{T\in H} \|T^{**}(x_n^{**})\| > \epsilon$ for each $n \in \mathbb{N}$. Now choose $(T_n) \subset H$ so that $\|T_n^{**}(x_n^{**})\| > \epsilon$ and then $(y_n^*) \subset Y^*$, $\|y_n^*\| \le 1$, so that $|T_n^{**}(x_n^{**})(y_n^*)| > \epsilon$. The sequence $(x_n^{**} \otimes y_n^*)$ is in $[K(X, Y)]^*$ and converges weak* to zero, because for an arbitrary $Q \in K(X, Y)$ we have

$$(x_n^{**} \otimes y_n^*)(Q) = y_n^*[Q^{**}(x_n^{**})] \le \|Q^{**}(x_n^{**})\| \to 0$$

since Q (and hence Q^{**}) is compact and x_n^{**} is weak* null. But H is limited and so

$$\sup_{T\in H} (x_n^{**} \otimes y_n^*)(T) \to 0,$$

which is a contradiction with our previous assumption. We are done.

As an easy consequence we get, again, the following old result.

Corollary 8 ([6]). *Assume that X^*, Y have the Gelfand–Phillips property. Then also $K(X, Y)$ has the Gelfand–Phillips property.*

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