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ON BANACH SPACES WITH THE GELFAND-PHILLIPS PROPERTY. II (*)

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We present some result of lifting of the Gelfand Phillips property from Banach spaces E and F to Banach spaces of compact operators and of Bochner integrable functions. Moreover we study $C(K)$ spaces possessing the same property. In the last section we prove some result concerning the so called three space problem for the Gelfand Phillips property too.

1. Introduction.

Let E be a Banach space. A (bounded) subset A of E is called *limited* (in E) [2] if, for every weak* null sequence (x_n^*) in the dual space E^*, we have $x_n^*(x) \to 0$ uniformly for x in A. If all limited subsets of E are relatively (norm) compact, then E is said to have the *Gelfand-Phillips property* [4] or to be a *Gelfand-Phillips space*; in this case we will often write $E \in (GP)$ for short. This property was first considered by Gelfand [13] who «proved» that it is shared by *all* Banach spaces. That it is not so was very soon discovered by Phillips [16] who observed that Gelfand's proof is correct only for separable Banach spaces, and showed that the space l_∞ does not have

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the property in question (inventing his now classic «Phillips' Lemma» for this purpose).

The class of Gelfand-Phillips spaces extends far beyond that of separable spaces, and it has quite good permanence properties (unfortunately, it is not closed with respect to quotients), but some important questions in this area are still open. For instance, no dual characterization of the Gelfand-Phillips property has been found so far, nor do we know for which precisely compact spaces K is $C(K)$ a Gelfand-Phillips space. On the other hand, it would be desirable to know more about those Banach spaces which do not have the Gelfand-Phillips property: There exist $C(K)$ spaces that lack this property and contain no copy of l_∞; however, it seems to be unknown whether or not a Banach space without the Gelfand-Phillips property must have a subspace isomorphic to c_0. The reader is referred to [6], [9] [11] and [18] for more information; some of the above statements will also be clarified below.

In this paper, which we consider to be a continuation of [6], we first prove that if E and F are Gelfand-Phillips spaces, then so is $K_{w^*}(E^*, F)$. Here $K_{w^*}(E^*, F)$ denotes the Banach space of compact and weak*-to-weak continuous linear operators from E^* to F, equipped with the usual operator norm; cf. [17] for more details and references. This enables us to give a unified treatment for the results in [6] concerning the Gelfand-Phillips property of injective tensor products and spaces of compact operators. Next, in Section 3, we show that if E is a Gelfand-Phillips space, so are the Lebesgue-Bochner spaces $L_p(\mu, E)$, for every positive measure μ and $1 \leq p < \infty$. In fact, we also extend this result to a much broader class of Banach spaces of E-valued measurable functions. In Section 4, we consider the class of all compact spaces K such that $C(K) \in (GP)$, and prove that it is closed with respect to arbitrary products. Finally, in Section 5, we seek conditions on a closed subspace $X \subset E$ and/or the quotient space E/X assuring that E is a Gelfand-Phillips space. In general, as shown in a recent work of Schlumprecht [18], the assumption that both X and E/X are Gelfand-Phillips is not enough for E to be Gelfand-Phillips, so at least one of the spaces X and E/X must be required to have a stronger property. Section 5 concludes with some comments on non-Gelfand-Phillips spaces having no subspace isomoprhic to l_∞.

The following facts will be used below, often without explicit reference. Their verification is straightforward except for the «if» part of (B), which is a consequence of a result in [1].

(A) A sequence (x_n) in E is limited (i.e., the set of its terms is limited) iff $x_n^*(x_n) \to 0$ for every weak* null sequence (x_n^*) in E^*.

(B) $E \in (GP)$ iff every limited weakly null sequence in E is norm null.

(C) If for every separable subspace L of E there exists a complemented subspace M in E such that $L \subset M$ and $M \in (GP)$, then, $E \in (GP)$.

(D) If A is a limited subset of E, then

$$\lim_{k\to\infty} \sup_{x\in A} ||u_k(x)|| = 0$$

for every sequence (u_k) of continuous linear operator from E into another Banach space F such that $\lim_k ||u_k(x)|| = 0$ for all x in E.

(E) Continuous linear images of limited sets or sequences are limited.

(F) Any Banach space isomorphic to a subspace of a Gelfand-Phillips space is Gelfand-Phillips itself.

2. The Gelfand-Phillips property for $K_{w^*}(E^*, F)$.

THEOREM 2.1. *If E and F are Gelfand-Phillips spaces, then so is $K_{w^*}(E^*, F)$.*

Proof. According to (B), we have to prove that if (h_n) is a limited and weakly null sequence in $K = K_{w^*}(E^*, F)$, then $||h_n|| \to 0$.

For each n choose x_n^* in E^* with $||x_n^*|| = 1$ so that $||h_n|| = ||y_n||$, where $y_n = h_n(x_n^*)$; this is possible because $h_n(B_{E^*})$ is a compact subset of F (but the choice for which $||y_n|| \geq \frac{1}{2}||h_n||$ would work as well).

Claim 1. (y_n) is weakly null.

Fix any y^* in F^*. Then, for each h in K, $h^*(y^*) = y^* \circ h$ is a weak* continuous linear functional on E^* so that $h^*(y^*) \in E \subset E^{**}$. We may therefore consider the operator $h \to h^*(y^*)$ from K into E, and conclude that the sequence $(h_n^*(y^*))$ is limited and weakly null in E. But $E \in (GP)$, so $\|h_n^*(y^*)\| \to 0$. Now, for each y^* in F^*, $y^*(y_n) = \langle h_n^*(y^*), x_n^* \rangle \to 0$, which proves Claim 1.

Claim 2. (y_n) is limited in F.

Let (y_n^*) be a weak* null sequence in F^*. For each n define $\alpha_n \in K^*$ by

$$\alpha_n(h) = y_n^*(h(x_n^*)) = \langle h^*(y_n^*), x_n^* \rangle.$$

If $h \in K$, then h is compact; hence h^* is sequentially weak*-to-norm continuous, so $\|h^*(y_n^*)\| \to 0$. It follows that (α_n) is a weak* null sequence in K^*. Now, $y_n^*(y_n) = \alpha_n(h_n) \to 0$, because (h_n) is limited in K, which proves Claim 2.

Having verified Claims 1 and 2, we may now apply (B) and the assumption that $F \in (GP)$, to conclude that $\|h_n\| = \|y_n\| \to 0$. Therefore, by (B), K is a Gelfand-Phillips space.

COROLLARY 2.2. ([6, Th. 3.1]). *If both E and F are Gelfand-Phillips spaces, so is their injective tensor product $E\check{\otimes}F = E\tilde{\otimes}_\varepsilon F$.*

Proof. $E\check{\otimes}F$ may be identified with a closed subspace of $K_{w^*}(E^*, F)$ via an isometrical isomorphism that sends $x \otimes y$ to the one-dimensional operator $x^* \to x^*(x)y$.

COROLLARY 2.3. ([6, Th. 4.2]). *If the Banach spaces E, F are such that both E^* and F are Gelfand-Phillips spaces, then also $K(E,F)$, the space of compact operators from E to F, is a Gelfand-Phillips space.*

Proof. The map $h \to h^{**}$ is an isometric isomorphism of $K(E,F)$ onto $K_{w^*}(E^{**}, F)$.

COROLLARY 2.4. *If E is a Gelfand-Phillips space, so is $l_1(E)$, the* *space of all unconditionally convergent series* $\sum x_n$ *in* E *equipped with the norm* $||(x_n)|| = \sup\{\sum |x^*(x_n)| : x^* \in B_{E^*}\}$.

Proof. It is easily seen that $l_1(E)$ is isometrically isomorphic to $K(c_0, E)$, and so Corollary 2.3 applies. (Alternatively, the result also follows from [6, Th. 5.1].)

Remark 1) Evidently, the converse to each of the above results is also valid.

2) Theorem 2.1 has been recently obtained also by D. Werner [20] (by a slightly different proof).

3. The Gelfand-Phillips property for Banach spaces of measurable vector-valued functions.

Below, $(S, \sum, \mu)$ is an *arbitrary* positive measure space. For $1 \leq p \leq \infty$, $L_p(\mu, E) = L_p(S, \sum, \mu; E)$ are the usual Lebesgue-Bochner spaces of (strongly) measurable functions from S to the Banach space E; see [3], [8].

THEOREM 3.1. *If* E *is a Gelfand-Phillips space, so is* $L_p(\mu, E)$ *for* $1 \leq p < \infty$.

Proof. If f is in $L_p(\mu, E)$, then the support of f is of σ-finite measure. Hence, if L is a separable subspace of $L_p(\mu, E)$, then there exists a set A in $\sum$ of σ-finite measure such that $L \subset L_p(A, \mu, E) = \{f \in L_p(S, \mu, E) : f\chi_{S-A} = 0 \text{ a.e.}\}$, and $L_p(A, \mu, E)$ is a complemented subspace of $L_p(\mu, E)$. On the other hand, since A has σ-finite measure, $L_p(A, \mu, E)$ is isometrically isomorphic to $L_p(\nu, E)$ for some finite measure ν. This combined with (C) implies that we may assume $\mu(S) < \infty$ in what follows.

Thus, let $\mu(S) < \infty$, and let (f_n) be a limited weakly null sequence in $L_p(\mu, E)$. If $A \in \sum$, then applying the integration operator $\int_A \cdot \, d\mu : L_p(\mu, E) \to E$ we see that $\left(\int_A f_n d\mu\right)$ is a limited weakly null sequence in E. Since $E \in (GP)$, we have

$$(*) \qquad \lim_{n\to\infty} \left\| \int_A f_n d\mu \right\| = 0 \quad \text{for all } A \text{ in } \textstyle\sum.$$

Choose an increasing (with respect to refinement) sequence (π_k) of finite measurable partitions of S into disjoint sets of positive measure such that

$$E_{\pi_\infty}(f_n) = f_n \quad \text{for all} \quad n,$$

where E_{π_∞} denotes the conditional expectation operator with respect to the sub-σ-algebra generated by (π_k). Then, if

$$E_{\pi_k}(f) = \sum_{A\in\pi_k} \frac{1}{\mu(A)} \int_A f d\mu \cdot \chi_A \quad \text{for } f \text{ in } \quad L_1(\mu, E),$$

we have

$$\lim_{k\to\infty} ||E_{\pi_k}(f) - E_{\pi_\infty}(f)||_p = 0 \quad \text{for all } f \text{ in } \quad L_p(\mu, E),$$

see [3,Chap. V.2].

Since the sequence (f_n) is limited in $L_p(\mu, E)$, we have

$$\lim_{k\to\infty} \sup_n ||E_{\pi_k}(f_n) - f_n||_p = 0,$$

by (D). From (*) it follows that

$$\lim_{n\to\infty} ||E_{\pi_k}(f_n)||_p = 0 \quad \text{for all} \quad k,$$

and taking the preceding relation into account, we arrive at the desired equality

$$\lim_{n\to\infty} ||f_n||_p = 0.$$

We will use the above theorem in the proof of a much more general result.

THEOREM 3.2. *Let F be a Banach space whose elements are (equivalence classes of) strongly measurable functions from S to the Banach space E. Assume that F satisfies the following conditions:*

(a) *For each A in $\sum$, the operator $f \to f_{\chi_A}$ maps F itself and is continuous.*

(b) *If $f \in F$ and (A_n) is a sequence in $\sum$ such that $A_n \downarrow \emptyset$, then*

$$\lim_{n\to\infty} \|f_{\chi_{A_n}}\| = 0.$$

(c) *For every A in $\sum$ with $0 < \mu(A) < \infty$ there exists B in $\sum$ such that $B \subset A$, $\mu(B) > 0$ and*

(c') *$F(B) = \{f_{\chi_B} : f \in F\} \subset L_1(\mu, E)$ and the inclusion map is continuous;*

(c") *on $F(B) \cap L_\infty(\mu, E)$, the topology induced from $L_\infty(\mu, E)$ is stronger than that induced from F.*

Then, if E is a Gelfand-Phillips space, so is F.

Proof. First consider the case when $\mu(S) < \infty$ and conditions (c') and (c") are satisfied for $B = S$. Note that in this case condition (b) is equivalent to the following one:

$$\lim_{\mu(A)\to 0} \|f_{\chi_A}\| = 0 \quad \text{for all } f \text{ in } F. \tag{+}$$

Let (f_n) be a limited weakly null sequence in F. Then (c') implies it is limited and weakly null in $L_1(\mu, E)$ as well. By Theorem 3.1, $L_1(\mu, E)$ is a Gelfand-Phillips space, so $\|f_n\|_1 \to 0$. In particular, $f_n \to 0$ in measure. If $(A_k) \subset \sum$ and $\mu(A_k) \to 0$, then applying (D) to the operators

$$u_k : F \to F; f \to f_{\chi_{A_k}}$$

(which converge pointwise to 0 by (+)) gives

$$\lim_{k\to\infty} \sup_n \|f_n \chi_{A_k}\| = 0.$$

Thus

$$\lim_{\mu(A)\to 0} \sup_n \|f_n \chi_A\| = 0.$$

This, condition (c"), and the fact that $f_n \to 0$ in measure, are easily seen to imply that (f_n) is norm null in F; cf. [5, Prop. 2.1(a)]. Therefore, $F \in (GP)$.

Now consider the general case. First observe that each f in F has a support of σ-finite measure, as can be easily deduced from (b). From this, in view of (C) and (a), it follows that we may assume that μ is a σ-finite measure. Let (B_n) be a maximal (necessarily countable!) collection of pairwise disjoint measurable sets such that $0 < \mu(B_n) < \infty$ and both conditions (c') and (c") are satisfied with $B = B_n$, $n = 1, 2, \ldots$. In view of (c), we may assume that S is the union of the B_n's. Then (b) implies that the subspaces $F(B_n)$ form an unconditional Schauder decomposition of F, and each $F(B_n)$ is in (GP) by the first part of the proof. Hence F is a Gelfand-Phillips space, by [6, Th. 5.1].

COROLLARY 3.3. *Let $E \in (GP)$ and let $\sum$ be a σ-algebra on a set S. Then $cca(\sum, E)$, the Banach space of all countably additive vector measures from $\sum$ to E having a relatively compact range, endowed with the semivariation norm, is a Gelfand-Phillips space.*

Proof. Since $cca(\sum, E)$ can be identified with the injective tensor product $ca(\sum)\check{\otimes}E$ (cf. [15, Th. 3.1]), in view of Corollary 2.2 it is enough to verify that $ca(\sum) \in (GP)$.

Let L be a separable subspace of $ca(\sum)$. Then (cf [8, p. 306]) there exists a finite positive measure μ on $\sum$ such that $L \subset ca(\sum, \mu) = \{\nu \in ca(\sum) : \nu \ll \mu\}$. By the Radon-Nikodym theorem, $ca(\sum, \mu)$ can be identified with $L_1(\mu)$, and the latter space has the Gelfand-Phillips property by Theorem 3.1. [This is also an easy consequence of the fact that $L_1(\mu)$ has the separable complementation property, i.e., the subspace M in Fact (C) can be chosen to be separable, and then of course $M \in (GP)$]. By the Lebesgue decomposition theorem, $ca(\sum, \mu)$ is a complemented subspace in $ca(\sum)$. Appealing to (C), we are done.

Remark 1) The fact that $ca(\sum) \in (GP)$ is actually a direct consequence of Theorem 3.1 because $ca(\sum)$ is an (abstract) L_1-space.

2) The results in [9] on the Gelfand-Phillips property for $K(\mu, E) = \{m \in cca(\sum, E) : m \ll \mu\}$ follows immediately from the above corollary; cf. also [11].

4. Compact spaces K for which $C(K)$ is a Gelfand-Phillips space.

If K is a compact (Hausdorff) space and E is a Banach space, then $C(K,E)$ denotes the Banach space of all continuous functions from K to E, equipped with the usual sup norm; we write $C(K)$ when E is the space of (real or complex) scalars. As noted in [6, Cor. 3.2]), from Corollary 2.2 it follows that $C(K,E)$ is a Gelfand-Phillips space iff both $C(K)$ and E are Gelfand-Phillips spaces. Therefore, as far as the Gelfand-Phillips property is concerned, we may restrict our attention to the standard $C(K)$ spaces.

Let $\mathcal{K}$ the class of all compact spaces K for which $C(K) \in (GP)$. As shown in [6. Th. 2.4], $\mathcal{K}$ contains the class $\mathcal{K}_c$ of compact spaces K satisfying the following condition:

(DCSC) K has a dense subset S which is conditionally sequentially compact, i.e., every sequence in S has a convergent subsequence.

There is also an apparently weaker condition ensuring the Gelfand-Phillips property: Let $\mathcal{K}'$ be the class of compact spaces K such that whenever (G_n) is a decreasing sequence of nonempty open subsets of K, there exists a convergent sequence (t_n) with $t_n \in G_n$ for all n; then we have the following.

THEOREM 4.1. *If* $K \in \mathcal{K}'$, *then* $C(K) \in (GP)$.

Proof. Let (f_n) be a limited weakly null sequence in $C(K)$; thus $f_n \to 0$ pointwise. Suppose that (f_n) is not norm null. Then, by passing to a subsequence, we may assume that for some $r > 0$ and all n, $A_n = \{t \in K : |f_n(t)| > r\} \neq \emptyset$; Let $G_n = \bigcup_{m \geq n} A_m$. Since $K \in \mathcal{K}'$, there exists a sequence (t_n) converging to some point $t \in K$ and such that $t_n \in G_n$ for all n. By the definitions of A_n's and G_n's, and passing again to a subsequence of (f_n) if necessary, we may arrange things to have $|f_n(t_n)| > r$ for all n. Continuing similarly as in the proof of Theorem 2.4 in [6], we arrive at a contradiction.

From the above it follows that $\mathcal{K}_c \subset \mathcal{K}' \subset \mathcal{K}$; unfortunately, we have been unable so far to distinguish between these three classes.

As the following two results indicate, these classes have similar permanence properties.

PROPOSITION 4.2. *Each of the classes $\mathcal{K}$, $\mathcal{K}'$, and $\mathcal{K}$ is closed with respect to continuous images: If K and L are compact spaces, ϕ is a continuous map from K onto L, and K is in one of these classes, then so is, respectively, L.*

Proof. The assertion is obvious in the case of classes $\mathcal{K}$ and $\mathcal{K}'$. If $K \in K$, then the induced composition operator $C_\phi : C(L) \to C(K)$ is a linear isometric embedding, hence $C(K) \in (GP)$ implies $C(L) \in (GP)$.

THEOREM 4.3. *Each of the classes K', K'', and K is closed with respect to arbitrary products.*

Proof. The productivity of $\mathcal{K}$ has been already noted in [6, p. 406]; since it is rather straightforward in the case of $\mathcal{K}'$, we will prove it only for the class $\mathcal{K}$.

If K, $L \in K$, then $C(K \times L) \simeq C(K, C(L))$ (cf. [19]), and $C(K, C(L)) \simeq C(K)\check{\otimes}C(L)$; hence $K \times L \in \mathcal{K}$ by Corollary 2.2. Thus $\mathcal{K}$ is closed with respect to finite products.

Now let $K_j \in \mathcal{K}$ for $j \in J$ (an index set), and let $K = \Pi_{j\in J} K_j$.

First consider the case when J is infinite countable, say $J = N$. Fix a point $(S_j)_{j\in N}$ in K and, for each j, let $L_j = K_1 \times \ldots \times K_j$. Let (f_n) be a limited weakly null sequence in $C(K)$. Then, for each j, the sequence $(g_{j,n})$ in $C(L_j)$ defined by

$$g_{j,n}(t_1, \ldots, t_j) = f_n(t_1, \ldots, t_j, s_{j+1}, s_{j+2}, \ldots)$$

is limited and weakly null. Since $L_j \in K$ (by the first part of the proof), it follows that

$$(*) \qquad ||g_{j,n}|| = \sup\{|f_n(t_1, \ldots, t_j, s_{j+1}, \ldots)| : (t_i)_{i=1}^{j} \in L_j\} \to \text{ as } n \to \infty.$$

Suppose (f_n) is not norm null; then we may assume that there exists $r > 0$ such that $||f_n|| > 2r$ for all n.

Claim Given j and m, there exists $t = (t_i)_{i \in N} \in K$ and $n > m$ such that

$$|f_n(t) - f_n(t_1, \ldots, t_j, s_{j+1}, s_{j+2}, \ldots)| > r.$$

Indeed, using (*) we can find $n > m$ so that $||g_{j,n}|| < r$; next, as $||f_n|| > 2r$, we can choose t in K such that $|f_n(t)| > 2r$, and then the above inequality is immediate.

An easy induction based on the above claim produces now a strictly increasing sequence (n_j) in N, and a sequence (t^j) of points in K such that, denoting $u^j = (t_1^j, \ldots, t_j^j, s_{j+1}, s_{j+2}, \ldots)$ and $h_j = f_{n_j}$, we have

$$(\ddagger) \qquad |h_j(t^j) - h_j(u^j)| > r \text{ for all } j.$$

Now observe that if we define functionals $\rho_j \in C(K)^*$ by

$$\rho_j(f) = f(t^j) - f(u^j),$$

then, by the uniform continuity of $f, \rho_j(f) \to 0$ for all f in $C(K)$. Thus (ρ_j) is a weak* null sequence in $C(K)^*$. Therefore, since (h_j) is limited, $\rho_j(h_j) \to 0$, contradicting (‡). Thus $||f_n|| \to 0$, which concludes the proof in the case of countable products.

Now let J be an arbitrary (uncountable) index set, and let, as before, (f_n) be a limited weakly null sequence in $C(K)$. Since each $f \in C(K)$ depends only on a contable set of coordinates (cf. [12, 3.2.H]), there exists a countable subset I of J such that $f_n((t_j)) = f_n((t'_j))$ whenever $t_j = t'_j$ for $j \in I$. Let (s_j) be a fixed element of K, and let $g_n \in C(K')$, where $K' = \Pi_{j \in I} K_j$, be defined by

$$g_n((t'_j)_{j \in I}) = f_n((t_j)_{j \in J}),$$

where $t_j = t'_j$ for $j \in I$, and $t_j = s_j$ for $j \in J - I$. Then (g_n) is a limited and weakly null sequence in $C(K')$; hence, by the preceding part of the proof,

$$||f_n|| = ||g_n|| \to 0 \text{ as } n \to \infty.$$

Thus $K \in \mathcal{K}$

Remark Since $C(\beta N) \simeq l_\infty \notin (GP)$ although βN is homeomorphic to a subspace of the product space $[0,1]^{[0,1]} \in \mathcal{K}$, none of the classes $\mathcal{K}, \mathcal{K}$, or $\mathcal{K}'$ is closed with respect to closed subspaces.

5. The three-space problem for the Gelfand-Phillips property.

T. Schlumprecht [18] has recently constructed a closed subspace E in l_∞ containing c_0 such that $E/c_0 \in (GP)$ but $E \notin (GP)$. Thus, in general, it is not true that a Banach space E is Gelfand-Phillips provided it has a closed subspace X such that both X and E/X are Gelfand-Phillips. Such a conclusion is nevertheless valid when the assumptions about X and-/or E/X are appropriately strengthened.

PROPOSITION 5.1. *Suppose X is a closed subspace of a Banach space E such that $E/X \in (GP)$. Then $E \in (GP)$ if (and only if) every weakly null sequence $(x_n) \subset X$ which is limited in E is norm null.*

Proof. Let $Q : E \to E/X$ be the quotient map, and let (z_n) be a limited weakly null sequence in E. Then the sequence (Qz_n) is limited and weakly null in $E/X \in (GP)$, hence $||Qz_n|| \to 0$. It follows that there is a sequence $(x_n) \subset X$ such that $||z_n - x_n|| \to 0$. Clearly, (x_n) is weakly null and limited in E so, by the assumption, $||x_n|| \to 0$. In consequence, $||z_n|| \to 0$, which proves that $E \in (GP)$.

COROLLARY 5.2. *If a closed subspace X of a Banach space E has the Schur property (i.e., weakly null sequences in X are norm null), and $E/X \in (GP)$, then $E \in (GP)$.*

THEOREM 5.3. *Suppose X is a Gelfand-Phillips subspace in a Banach space E, and suppose there exists a closed subspace Y in E such that its dual closed unit ball B_{Y^*} is w^*-sequentially compact and $X + Y$ is dense in E. Then $E \in (GP)$.*

Proof. First of all observe that the quotient map maps Y onto a dense subspace of E/X, hence also $B_{(E/X)^*}$ is w^*-sequentially compact (cf. [2]) which implies that $E/X \in (GP)$ (see e.g. [6, Cor. 2.3]). Since also $X \in (GP)$, the condition in Proposition 5.1 will be satisfied (and we will conclude that $E \in (GP)$) if we prove that every sequence $(x_n) \subset X$ that is limited in E is limited in X as well. For this to hold, it suffices to know that every w^*-null sequence $(x_n^*) \subset X^*$ can be extended (term by term) to a sequence $(z_n^*) \subset E^*$ having a w^*-null subsequence. We are now going to show that it is indeed so.

Let a sequence $(x_n^*) \subset X^*$ be w^*-null (hence norm bounded). For each n apply the Hahn-Banach theorem to find a norm-preserving extension $w_n^* \in E^*$ of x_n^*. By the assumption about Y, there is a subsequence $(w_{k_n}^*)$ such that its restriction to Y is pointwise convergent. Since, moreover, the sequence $w_{k_n}^*|X = x_{k_n}^*$ converges pointwise to zero, we see that $(w_{k_n}^*)$ is pointwise convergent on the dense subspace $X+Y$ of E. But this sequence is also bounded, hence it must converge pointwise on E to a functional $w^* \in E^*$. Now let $z_{k_n}^* = w_{k_n}^* - w^*$; then the sequence $(z_{k_n}^*) \subset E^*$ is weak* null and $z_{k_n}^*|X = x_{k_n}^*$ for every n. This concludes the proof.

COROLLARY 5.4. *Let X be a closed subspace of a Banach space E, and assume that $X \in (GP)$ and E/X is separable. Then $E \in (GP)$.*

Remark As was observed above, the assumptions about Y in Theorem 5.3 imply that E/X has a weak* sequentially compact dual ball. It should be however pointed out that this last property alone would not permit us to prove that $E \in (GP)$. In fact, the non-Gelfand-Phillips space E in Schlumprecht's counterexample is such that $B_{(E/c_0)^*}$ is weak* sequentially compact!

We conclude with the following consequence of a result in [7].

PROPOSITION 5.5. *If a Banach space E has a closed subspace X such that both X and E/X are Gelfand-Phillips spaces, then E contains no isomorphic copy of l_∞.*

Remark Thus, in particular, the non-Gelfand-Phillips space in Schlumprecht's counterexample contains no copy of l_∞. Also the l_∞-free closed subspaces $E(= X_{\mathcal{A}})$ of l_∞ constructed earlier, for different purposes, by Haydon [14], are easily checked to lack the Gelfand-Phillips property. In fact, that this is so for the first of his two examples (the one given in Section 1 of [14]) is readily seen by inspecting the proof of his Proposition 1B. (Alternatively, this space contains c_0 so, by a result in [10], if it were a Gelfand-Phillips space, it would contain a complemented copy of c_0, which is impossible because it is a Grothendieck space.) As for the second of Haydon's spaces $E = X_{\mathcal{A}}$, the algebra $\mathcal{A}$ defining this space is such that every infinite subset of $\omega = \{0, 1, \ldots\}$ contains an infinite set from $\mathcal{A}$ which, again by inspecting the proof of Proposition 1B (see also [18, Proof of Th. 8]), implies that E is not a Gelfand-Phillips space.

REFERENCES

[1] Bourgain J., Diestel J., *Limited operators and strict cosingularity,* Math. Nachrichten **119** (1984), 55-58.

[2] Diestel J., *Sequences and Series in Banach Spaces*, Graduate Texts in Math. **92**, Springer, 1984.

[3] Diestel J., Uhl J.J., *Vector Measures,* Math. Surveys No. **15**, American Math. Soc., 1977.

[4] Diestel J., Uhl J.J., Jr., *Progress in vector measures - 1977-83,* in: Lecture Notes Math. vol. 1033, pp. 144-192, Springer, 1983.

[5] Drewnowski L., *On subseries convergence in some function spaces,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. **22** (1974), 797-803.

[6] Drewnowski L., *On Banach spaces with the Gelfand-Phillips property,* Math. Z. **193** (1986), 405-411.

[7] Drewnowski L., Roberts J.W., in preparation.

[8] Dunford N., Schwartz J.T., *Linear Operators, Part I,* Interscience, 1958.

[9] Emmanuele G., *Gelfand-Phillips property in a Banach space of vector valued measures,* Math. Nachrichten **127** (1986), 21-23.

[10] Emmanuele G., *A note on Banach spaces containing complemented copies of c_0*, Preprint.

[11] Emmanuele G., Räbiger F., *On the Gelfand-Phillips property in ε-tensor products,* Math. Z. **191** (1986), 485-488.

[12] Engelking R., *General Topology,* Polish Scientific Publishers, 1977.

[13] Gelfand I., *Abstrakte Funktionen und lineare Operatoren,* Mat. Sbornik (Recueil Math.) **4** (1938), 235-284.

[14] Haydon R., *A non-reflexive Grothendieck space that does not contain l_∞,* Israel J. Math. **40** (1981), 65-73.

[15] Lewis D.R., *Conditional weak compactness in certain inductive tensor products,*

Math. Ann. **201** (1973), 201-209.

[16] Phillips R.S., *On linear transformations,* Trans. Amer. Math. Soc. **48** (1940), 516-541.

[17] Ruess W., *Duality and geometry of spaces of compact operators,* in: *Functional Analysis: Surveys and Recent Results III,* pp. 59-78. North-Holland, 1984.

[18] Schlumprecht T., *Liftings in the theory of limited sets,* Preprint.

[19] Semadeni Z., *Banach Spaces of Continuous Functions,* Polish Scientific Publishers, 1971.

[20] Werner D., *A note on the paper «On the Gelfand-Phillips property in* ε*-tensor products»,* Preprint.

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