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GELFAND-PHILLIPS PROPERTY IN KÖTHE SPACES OF VECTOR VALUED FUNCTIONS*

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We prove that $E(X)$ is a Gelfand-Phillips space if and only if E and X are.

Let X be a Banach space. A subset M of X is called *limited* if for any weak* null sequence $x_n^* \subset X^*$ one has $\lim_n \sup_M |x_n^*(x)| = 0$. X is called a Gelfand-Phillips space if limited subsets of X are relatively compact. We refer to the paper [1] (and its References) for these definitions, examples and properties of limited sets and Gelfand-Phillips spaces. In particular, L.Drewnowski and the author obtained in [1] the following result

THEOREM 1. ([1], Th. 3.2). *Let F be a Banach space whose elements are (equivalence classes of) strongly measurable functions with respect to a measure space (S, Σ, μ) with values in X. Assume that F satisfies the following conditions:*

(a) for each $A \in \Sigma$, the operator $f \to f\chi_A$ maps F into itself and is continuous.

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(b) if $f \in F$ and (A_n) is a sequence in Σ such that $A_n \downarrow \phi$ then

$$\lim_n \|f\chi_{A_n}\| = 0$$

(c) for every $A \in \Sigma$ with $0 < \mu(A) < \infty$ there exists $B \in \Sigma$ such that $B \subset A$, $0 < \mu(B)$ and

(c') $F(B) = \{f\chi_B : f \in F\} \subset L^1(B, X)$ and the inclusion is continuous

(c") on $F(B) \cap L^\infty(B, X)$ the topology induced from $L^\infty(B, X)$ is stronger than that induced from F.

Then if X is a Gelfand-Phillips space, so is F.

The purpose of this short note is to present an application of Theorem 1 above to Köthe spaces of vector valued functions: the result we prove actually is an extension of Theorem 3.1 in [1] (on which the proof of Theorem 1 is based). Now, let us introduce some definitions and theorems to be utilized in the proof of our result.

Let (S, Σ, μ) be a σ-finite (complete) measure space and $M(S)$ the space of Σ, μ-measurable real valued functions with functions equal μ-almost everywhere identified. A Köthe space E is a Banach subspace of $M(S)$ consisting of locally integrable functions such that

(i) if $|u| \leq |v|$ μ-a.e., with $u \in M(S)$, $v \in E$, then $u \in E$ and $\|u\| \leq \|v\|$

(ii) for each $A \in \Sigma$, $\mu(A) < \infty$ the characteristic function χ_A is an element of E.

Köthe spaces are Banach lattices if we put $u \geq 0$ when $u(s) \geq 0$ μ-a.e. (we refer to [3] for these definitions); furthermore, they are σ-complete Banach lattices.

THEOREM 2. ([2], Th.1).*Given a Köthe space E, there exists an increasing sequence (S_n) in $\Sigma, \mu(S_n) < \infty$ for all $n \in \mathbb{N}$ and $\mu\left(S \setminus \bigcup_{n \in \mathbb{N}} S_n\right) = 0$ for which the following chain of continuous inclusions holds*

$$L^\infty(S_n) \subset E(S_n) \subset L^1(S_n).$$

We recall that a Banach lattice has an *order continuous norm* if, for every downward directed net $\{x_\alpha\}$ with $\inf_\alpha x_\alpha = 0$, then $\lim_\alpha \|x_\alpha\| = 0$.

THEOREM 3. ([3]). *Let E be a σ-complete Banach lattice. If E does not contain l^∞, then it has an order continuous norm.*

In this paper we consider, for a real Banach space X, the family of all strongly measurable functions $f : S \to X$ (identifying μ-a.e. equal functions) such that $\|f(\cdot)\|_X \in E$, E a Köthe space. Such a space, denoted by $E(X)$, is a Banach space under the norm

$$\|f\|_{E(X)} = \| \|f(\cdot)\|_X \|_E.$$

We note that among Köthe spaces defined above one can find Lebesgue-Bochner spaces as well as Orlicz or Musielak-Orlicz spaces of vector valued function. We are now ready to prove our result.

THEOREM 4. *$E(X)$ is a Gelfand-Phillips space if and only if E and X are.*

Proof. It is enough to prove the "if" part only. As we remarked at the beginning we want to show that $E(X)$ verifies all of the assumptions of Theorem 1. We start with (a). Let $f \in E(X)$ and $A \in \Sigma$. It is clear that $f\chi_A$ is strongly measurable; furthermore, $\|f(S)\chi_A(s)\|_X \le \|f(s)\|_X$ μ-a e. on S; since E is a Köthe space, $f\chi_A \in E(X)$ and $\|f\chi_A\|_{E(X)} \le \|f\|_{E(X)}$. Now, let us prove (b) If $f \in E(X)$ and (A_n) is a sequence in Σ with $A_n \downarrow \phi$, we observe that

$$\|f(s)\chi_{A_{n+1}}(s)\|_X \le \|f(s)\chi_{A_n}(s)\|_X \quad \mu\text{-a.e. on } S, n \in N.$$

Now, recall that a Gelfand-Phillips space cannot contain l^∞ and therefore E also has this property; hence E is an order continuous Banach lattice (Theorem 3). Since $\{\|f(\cdot)\chi_{A_n}(\cdot)\|_X\}$ is a downward directed sequence with infimum equal to 0, we can conclude that $\lim_n \|f\chi_{A_n}\|_{E(X)} = 0$. It remains only to show (c). To prove (c) let $A \in \Sigma$ with $0 < \mu(A) < \infty$. Theorem 2 assures the existence of a $n^* \in N$ such that $S_{n^*} \cap A \in \Sigma$, $0 < \mu(S_{n^*} \cap A)$ and the following chain of continuous inclusions holds

$$L^\infty(S_{n^*} \cap A) \subset E(S_{n^*} \cap A) \subset L^1(S_{n^*} \cap A). \tag{1}$$

Let us assume $B = S_{n^*} \cap A$. Thanks to (1) we have that (c') and (c") are true. Indeed, for $f \in E(B, X)$ we have that f is strongly measurable and that $\|f(\cdot)\|_X$ is in ($E(B)$ and hence in) $L^1(B)$. This means that $f \in L^1(B,X)$. By virtue of (1) we also have the existence of a $c > O$ for which

$$\|u\|_{L^1(B)} \leq c\|u\|_{E(B)} \quad \text{for all } u \in E(B); \tag{2}$$

Applying (2) to $u(\cdot) = \|f(\cdot)\|$ we get (c'). To prove (c") let us take f in $E(B,X) \cap L^\infty(B,X)$. The function $u(\cdot) = \|f(\cdot)\|_X$ is in $L^\infty(B)$ and, thanks to (1), there is $c > O$ for which

$$\|u\|_{E(B)} \leq c\|u\|_{L^\infty(B)}. \tag{3}$$

From (3) we easily have

$$\|f\|_{E(B,X)} \leq c\|f\|_{L^\infty(B,X)}. \tag{4}$$

The last inequality gives (c").

Once (a), (b), (c) are proved, Theorem 1 applies to give the assertion.

The above result, already known for the case of Lebesgue-Bochner spaces, is new for Orlicz or Musielak-Orlicz spaces of vector valued functions.

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