SOME REMARKS ON THE POSITION OF THE SPACE $K(X,Y)$ INSIDE THE SPACE $W(X,Y)$

New Zealand Journal of Mathematics
Volume 26, (1997), 183-189

SOME REMARKS ON THE POSITION OF THE SPACE $K(X,Y)$ INSIDE THE SPACE $W(X,Y)$[1]

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(Received March, 1996)

Abstract. We present some results showing that sometimes $K(X,Y) = W(X,Y)$ and sometimes $K(X,Y)$ is an u-ideal in $W(X,Y)$ under suitable renormings

In this paper we want to collect several loosely connected statements about the position of the space $K(X,Y)$ (of compact operators from a Banach space X into another Banach space Y) inside the space $W(X,Y)$ of weakly compact operators. We obtained them when trying to investigate the old problem of the complementability of $K(X,Y)$ inside $W(X,Y)$. The first result we present, Proposition 1, gives conditions that imply the equality $K(X,Y) = W(X,Y)$, similarly to recent results in the paper [1]. Next we impose conditions on the Banach space X^* (or on X and X^{**}) to get that if E is any isomorphic predual of X^*, then $W(E)$ can be renormed so that $K(E)$ is a u-ideal or a M-ideal in $W(E)$ in this new norm. In the last part we reformulate the (compact) approximation property in terms of *ideals* in the sense that the subspace $K(X)$ of the space $L(X)$ of all operators is an ideal in $L(X)$ if $K(X)^0$ is complemented inside $L(X)^*$. In fact, this last observation is easy and is implicitly contained in the papers of Lima [7],[8].

Results. Our first result, as announced, concerns with the coincidence of the spaces $K(X,Y)$ and $W(X,Y)$

Proposition 1. *Let us suppose that $K(X,Y)$ is weakly sequentially complete. Moreover, assume that X^* has the bounded compact approximation property and the Radon-Nikodym Property. Then $K(X,Y) = W(X,Y)$.*

Proof. Let us suppose there is a $T \in W(X,Y) \setminus K(X,Y)$; it is possible to find a separable subspace X_0 of X so that $T_0 = T_{|X_0}$ is not compact and, thanks to a result in [4], we may also suppose there is an isometric embedding j of X_0^* into X^*. Moreover, since X^* has the

[1] **1991** *AMS Mathematics Subject Classification:* **46B20, 46B28, 47B07.**
The work of the first author was supported by M.U.R.S.T. of Italy (40%,1993), the work of the second author was supported by the grants of AVČR No.A1019505 and GAČR No.201/94/0069.

Radon-Nikodym Property, even X_0^* is separable. Hence, there is a sequence $A_n \in K(X^*)$ such that

$$A_n(jx_0^*) \to jx_0^*$$

for all $x_0^* \in X_0^*$. Since $jT_0^* : Y^* \to X^*$, we get

$$A_n jT_0^*(y^*) \to jT_0^*(y^*)$$

for all $y^* \in Y^*$. Now, we recall that T_0 is weakly compact and so we may affirm that $A_n jT_0^*$ and jT_0^* are weak*-weak continuous, i.e. they are conjugate operators. So there are $Q_n \in K(X,Y), Q_0 \in W(X,Y)$ such that $Q_n^* = A_n jT_0^*, Q_0^* = jT_0^*$. From the above limit relationships, we get

$$Q_n^*(y^*)(x^{**}) \to Q_0^*(y^*)(x^{**})$$

which means that (Q_n) is a weak Cauchy sequence in $K(X,Y)$ [6]. Since this last space is weakly sequentially complete, the sequence (Q_n) must converge weakly to some element of $K(X,Y)$, that clearly must coincide with Q_0. This gives that T_0 must be compact, a contradiction concluding the proof.

To state the next result we repeat ([2]) that a separable Banach space X has the unconditional metric approximation property if there is a sequence (T_n) of compact operators on X such that $\|I - 2T_n\| \to 1$ and $\|x - T_n x\| \to 0$ for all $x \in X$. Note that then $\limsup_n \|T_n\| \leq 2^{-1} \limsup_n (\|I\| + \|2T_n - I\|) \leq 1$.

Proposition 2. *Let X be a Banach space such that X^* is separable and has an equivalent (not necessarily dual) norm $\|\cdot\|$ such that $(X^*, \|\cdot\|)$ has the unconditional metric compact approximation property. Let Y be an arbitrary Banach space. Then there is an equivalent norm $\|\cdot\|$ on $W(X,Y)$ such that $K(X,Y)$ is a u-ideal in $(W(X,Y), \|\cdot\|)$.*

Proof. We define, for $f \in W(X,Y)$,

$$\|f\| \stackrel{def}{=} \|f^*\| = \sup_{|y^*| \leq 1} \|f^*(y^*)\|$$

Let $(T_n) \subset K(X^*)$ be the unconditional compact approximating sequence existing by assumptions. Let $\Phi \in K(X,Y)^*$ and $f \in W(X,Y)$. Then $u_n(f) \stackrel{def}{=} T_n f^* \in K_{w^*}(Y^*, X^*)$, so that $u_n(f) = f_n^*$, for suitable $f_n \in K(X,Y)$. Let us denote by A the index set formed by the unit ball of $W(X,Y)$; then u_n maps A into some multiple B of the closed unit ball of $K(X,Y)^{**}$. Thus (u_n) is a sequence in the product space B^A and the latter space is compact when considering the weak* topology on B. Let (u_{n_α}) be a converging subnet; this means that for any $f \in W(X,Y)$ and any $\Phi \in K(X,Y)^*$ we may define

$$J(\Phi, f) = \lim_\alpha \Phi(f_{n_\alpha})$$

It is clear that J is a bilinear mapping and

$$|J(\Phi, f)| \leq \limsup_\alpha \Phi(f_{n_\alpha}) \leq \|\Phi\| \|f\| \limsup_n \|T_n\| \leq \|\Phi\| \cdot \|f\| \tag{1}$$

We also observe that for $f \in K(X,Y)$ we have

$$J(\Phi, f) = \Phi(f) \qquad \forall \Phi \in K(X,Y)^* \tag{2}$$

Indeed, if $x^{**} \in X^{**}, y^* \in Y^*$ and $f \in K(X,Y)$, we have

$$x^{**} f_n^*(y^*) = x^{**} T_n f^* y^* \to x^{**} f^* y^*$$

so that by Kalton ([6]) we have $f_n \stackrel{w}{\to} f$; and (2) follows. If $Re : W(X,Y)^* \to K(X,Y)^*$ is the restriction mapping, we may define a projection P in $W(X,Y)^*$ by putting

$$P\Phi(f) = J(Re\Phi, f) \qquad \forall \Phi \in W(X,Y)^*$$

P is evidently a projection because (2) implies that $ReP\Phi = Re\Phi$ and thus

$$(P^2\Phi)f = J(ReP\Phi, f) = J(Re\Phi, f) = P\Phi(f)$$

By (2) it is also clear that $P^{-1}(0) = K(X,Y)^0$ where $K(X,Y)^0 \stackrel{def}{=} \{\Phi \in W(X,Y)^*,$ $\Phi_{|K(X,Y)} = 0\}$. Finally we prove that

$$\|\Phi - 2P\Phi\| \leq \|\Phi\| \qquad \forall \Phi \in W(X,Y)^* \tag{3}$$

so showing, by definition, that $K(X,Y)$ is a u-ideal in $(W(X,Y), \|\cdot\|)$. Indeed, we have

$$\|\Phi - 2P\Phi\| = \sup_{\substack{\|f\| \leq 1 \\ f \in W(X,Y)}} \limsup_\alpha |\Phi(f) - 2\Phi(f_{n_\alpha})| \leq$$
$$\|\Phi\| \sup_{\substack{\|f\| \leq 1 \\ f \in W(X,Y)}} \limsup_\alpha \|f^* - 2T_{n_\alpha} f^*\| \leq \|\Phi\| \ .$$

This completes the proof.

We say that (K_n) is a countable compact approximation of the identity in a Banach space, if the sequence (K_n) converges to the identity in the strong operator topology. Now we may state

Proposition 3. *Let X be a Banach space such that X^* has an equivalent (not necessarily dual) norm $\|\cdot\|$ so that $(X^*, \|\cdot\|)$ has a countable compact approximation (K_n) of the identity for which*

$$\limsup_n \|K_n S + (Id - K_n)T\| \leq \max(\|S\|, \|T\|) \tag{4}$$

for all $S, T \in W(Y^*, X^*)$ with Y an arbitrary Banach space. Then there is an equivalent norm on $W(X, Y)$ so that $K(X, Y)$ is an M-ideal in $W(X, Y)$.

Proof. The proof is the same as that of the previous Proposition, with the only change that, instead of (3), we have to prove that

$$\|P\Phi\| + \|\Phi - P\Phi\| \leq \|\Phi\| \qquad \forall \Phi \in (W(X, Y), \|\cdot\|)^* \tag{5}$$

But this is the case, because, given $\epsilon > 0$ and Φ, we find $f \in W(X, Y)$ and $g \in W(X, Y)$ with $\|f\| = \|g\| = 1$ and $\|P\Phi\| \leq P\Phi(f) + \epsilon$ and $\|\Phi - P\Phi\| \leq \Phi(g) - P\Phi(g) + \epsilon$. Furthermore, we find an $n \in N$ such that $P\Phi(f) \leq \Phi(f_n) + \epsilon$, $P\Phi(g) \geq \Phi(g_n) - \epsilon$ and $\|K_n f^* + (Id - K_n)g^*\| \leq 1 + \epsilon$. Then

$$\|P\Phi\| + \|\Phi - P\Phi\| \leq 4\epsilon + \Phi(f_n + g - g_n) \leq 4\epsilon + \|\Phi\|\|K_n f^* + g^* - K_n g^*\| \leq 4\epsilon + \|\Phi\|(1 + \epsilon)$$

Because ϵ is arbitrary, (5) follows.

Remark 4. The assumption (4) is, for instance, verified if $X = Y$ and $K(X^*, \|\cdot\|)$ is an M-ideal in $L(X^*, \|\cdot\|)$ [6]. This assumption (4) is also satisfied if X^* is isomorphic to some Z^* with $K(Z)$ an M-ideal in $L(Z)$ and Z^{**} has the M^*-property. Indeed, let $(K_n) \subset K(Z)$ be the shrinking approximation of the identity in Z, the existence of which is guaranteed by the Theorem VI.4.17 in [3], due to Kalton. Following the proof of the implication $(vi) \Rightarrow (ii)$ in this theorem, we have for any $m \in N$

$$\limsup_n \|K_m + Id - K_n\| \leq \|Id - 2K_m\|.$$

If we show that for any $S \in K(X^*), T \in W(X^*), \|S\|, \|T\| \leq 1$ we have

$$\limsup_n \|S + (Id - K_n^*)T\| \leq 1 \tag{6}$$

the result will follow. To prove (6) we again follow the Kalton's result quoted above: we fix $m \in N$ and $n \in N$ and we pick $x_n^{**} \in X^{**}$ with norm 1 so that

$$\limsup_n \|K_m^* S + (Id - K_n^*)T\| = \limsup_n \|S^* K_m^{**} x_n^{**} + T^*(Id - K_n^{**})x_n^{**}\|$$

We observe that $(Id - K_n^{**})(x_n^{**}) \xrightarrow{w^*} 0$ and thus also $[T^*(Id - K_n^*)](x_n^{**}) \xrightarrow{w^*} 0$; indeed, let $x^* \in X^*$; we have $x^*[(Id - K_n^{**})x_n^{**}] = x_n^{**}(x^* - K_n^* x^*) \to 0$ since $K_n^* x^* \to x^*$ in the norm topology. Also observe that $(K_m^{**} x_n^*)_n$ are contained in a relatively norm compact set; thus we get

$$\limsup_n \|S^* K_m^{**} x_n^{**} + T^*(Id - K_n^{**} x_n^{**})\| \leq \limsup_n \|K_m^{**} x_n^{**} + T^*(Id - K_n^{**} x_n^{**})\| \qquad (7)$$

Now we observe that also the M^* analogue of Lemma 4.14 [3] holds: Suppose that X^* has property M^* and that $T \in L(X)$ has norm less or equal to 1. If $(u_\alpha), (v_\alpha)$ are relatively norm compact with $\|u_\alpha\| \leq \|v_\alpha\|$ and a (x_α) is a bounded net weak* converging to zero, then

$$\limsup_\alpha \|u_\alpha + T x_\alpha\| \leq \limsup_\alpha \|v_\alpha + x_\alpha\|.$$

According to this Lemma we may continue the estimation (7):

$$\limsup_n \|K_m^{**} x_n^{**} + T^*(Id - K_n^{**} x_n^{**})\| \leq \limsup_n \|(K_m^{**} + Id - K_n^{**})x_n^{**}\| \leq \|Id - 2K_m\|.$$

Thus

$$\limsup_n \|S + (Id - K_n^*)T\| \leq \|S - K_m^* S\| + \|Id - 2K_m\|$$

This last inequality gives (6) for m large enough.

The last result we present relates the following generic notion of "ideal" (cf.[2]) to the approximation properties (cf.[7],[8]).

Definition 5. *A closed subspace X of a Banach space Y is called an ideal with constant λ and projection P in Y if X^0 is the kernel of a projection P in Y^* with $\|P\| \leq \lambda$.*

In the next result, by $x \otimes x^*$ where $x \in X$ and $x^* \in X^*$ we denote an element of $L^*(X)$ and also its restriction to any subspace of $L(X)$ such that for $f \in L(X)$ we have $(x \otimes x^*)f = x^*(fx)$.

Proposition 6. *The following are equivalent for any Banach space X*

(1) X has the λ-bounded compact approximation property

(2) $K(X)$ is an ideal in $L(X)$ with constant λ such that

$$x \otimes x^* \in \text{range } P \qquad \forall \quad x \otimes x^* \qquad (8)$$

(3) there is a bilinear form $J : K^*(X) \times L(X) \to R$ such that $\|J\| \leq \lambda$, J is equal to the canonical pairing $\langle K^*(X), K(X)\rangle$ on $K^*(X) \times K(X)$ and $J(x \otimes x^*, f) = (x \otimes x^*)f$, for all $x \otimes x^*$ and $f \in L(X)$
(4) there is an extension operator $A : K^*(X) \to L^*(X)$ such that $\|A\| \leq \lambda$ and $A(x \otimes x^*) = x \otimes x^*$, for all $x \otimes x^*$
(5) there exists an operator $B : L(X) \to K^{**}(X)$ such that $B_{|K(X)}$ is the canonical injection of $K(X)$ into $K^{**}(X)$, $\|B\| \leq \lambda$ and $Bf(x \otimes x^*) = (x \otimes x^*)f$, for all $x \otimes x^*$
(6) for every $\epsilon > 0$, for any finite dimensional subspace F of $L(X)$ and for any finite number of $x_i \otimes x_i^*, i = 1, 2 \dots n$, there is an operator $T : F \to K(X)$ such that $\|T\| \leq \lambda + \epsilon$, $Tf = f$ for any compact operator $f \in F$ and $(x_i \otimes x_i^*)Tf = (x_i \otimes x_i^*)f$, for all $i = 1, 2 \dots n$, $x \otimes x^*$.
Proof. (1) $\Rightarrow$ (2) follows from a result by J.Johnson ([5]) and noting that in fact $P\Phi(f) = \lim_\alpha \Phi(f_\alpha f)$ where $\{f_\alpha\} \subset K(X)$ is suitably chosen with $f_\alpha(x) \to f(x)$ for all $x \in X$ and $\|f_\alpha\| \leq \lambda$.
(2) $\Rightarrow$ (3). If P is the projection in the condition (2) we define $J(\Phi, f) = (P\tilde{\Phi})f$, where $\tilde{\Phi}$ is any Hahn-Banach extension of Φ to $L(X)$. J is well defined and linear also in Φ because $P(K(X)^0) = 0$; indeed,let $\tilde{\Phi}_1, \tilde{\Phi}_2, \widetilde{\Phi_1 + \Phi_2}$ be Hahn-Banach extensions of $\Phi_1, \Phi_2, \Phi_1 + \Phi_2 \in K(X)^*$; we have

$$J(\Phi_1 + \Phi_2, f) - J(\Phi_1, f) - J(\Phi_2, f) = P(\widetilde{\Phi_1 + \Phi_2} - \tilde{\Phi}_1 - \tilde{\Phi}_2)f = 0$$

because $\widetilde{\Phi_1 + \Phi_2} - \tilde{\Phi}_1 - \tilde{\Phi}_2 \in K(X)^0$. Similarly for the uniqueness

$$J(x \otimes x^*, f) = P(x \otimes x^*)f = (x \otimes x^*)f$$

(3) $\Leftrightarrow$ (4). The relation between the operator A and the bilinear form J is given by

$$A\Phi(f) = J(\Phi, f)$$

The rest is trivial.
(3) $\Leftrightarrow$ (5). Again the relation of J to B is

$$B(f)\Phi = J(\Phi, f)$$

(5) $\Rightarrow$ (6). Let $F \subset L(X)$ be a finite dimensional subspace and let $x_i \otimes x_i^* \in X \otimes X^*$ for $i = 1, 2...n$. The principle of local reflexivity now offers for any $\epsilon > 0$ an operator

$R : B(F) \to K(X)$ so that $R(f) = f$ if $f \in K(X) \cap B(F)$, $\|R\| \leq 1 + \epsilon$ and taking each $x_i \otimes x_i^*$ onto itself, for $i = 1, 2...n$. Then $T = RB_{L(X)}$ is the desired operator from $L(X)$ into $K(X)$.

(6) $\Rightarrow$ (5). Let us choose, for any finite dimensional subspace $F \subset L(X)$ the operator $T_F : F \to K(X)$ according to the condition (6). Let us extend this operator by zero to all of $L(X)$ and denote this nonlinear operator again by $T_F : L(X) \to K(X)$. Considering $K(X)$ embedded into $K(X)^{**}$, an usual compactness argument reveals that there is a subnet $\{T_{F_\alpha}\}$ of the net $\{T_F\}$ such that for every $\Phi \in K(X)^*$, the net $\{T_F(\Phi)\}$ is converging to some $B(\Phi) \in K(X)^{**}$. It is again trivial to check the properties of B .

(5) $\Rightarrow$ (1). Let $\tilde{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_n)$ be an n-tuple of elements of $K(X)^*$ and $f \in L(X)$. By the bipolar theorem we may now find an element $f_{\tilde{\Phi}} \in K(X)$, with $\|f_{\tilde{\Phi}}\| \leq \lambda$, such that

$$|\Phi_i(f_{\tilde{\Phi}}) - B(f)\Phi_i| < \frac{1}{n}$$

for all $i = 1, 2, \ldots n$. Evidently $\{f_{\tilde{\Phi}}\}$ is a net in $K(X)$ in a usual way.If now $x \otimes x^* \in X \otimes X^*$ then evidently

$$x^*(f_{\tilde{\Phi}}(x)) \to B(f)(x \otimes x^*) = f(x \otimes x^*)$$

showing that $\{f_{\tilde{\Phi}}\}$ tends to f in the weak operator topology. Now the result follows, for instance, by VI.4.9 in [3].

Remark 7. The proof above actually shows that $f_{\tilde{\Phi}} \xrightarrow{w^*} B(f)$. But evidently $B = A^*/L$ so that for every $\Phi \in K^*$ we have $\Phi(f_{\tilde{\Phi}}) \to \Phi(Bf) = (A\phi)f$, i.e. $f_{\tilde{\Phi}} \to f$ in the weak topology $w(L, AK^*) = w(L, PL^*)$. Here and below we have set $L = L(X)$ and $K = K(X)$.

Remark 8. It may easily be checked that the operators A and B are one to one and that $\|A^{-1}\| \leq 1$, $\|B^{-1}\| \leq 1$. We also note that the "local reflexivity" mapping R in the proof of (5) $\Rightarrow$ (6) may be chosen so that we have also $\|R\| \cdot \|R^{-1}\| \leq 1 + \epsilon$. This yields that (6) may be rephrased:

The space $L(X)$ is finitely representable in $K(X)$ in such a way that the representations are identical for the elements of $K(X)$ and the representations keep the duality with any finite subset of elements of the type $x \otimes x^*$.

Remark 9. The condition (8) is evidently equivalent to $A(x \otimes x^*) = x \otimes x^*$ in (4), or to $Bf(x \otimes x^*) = (x \otimes x^*)f$ and similarly for T in (6). Evidently all the conditions (2) to (6) are again equivalent also without (8). From the Remark 4 it follows that they imply

1') For any $f \in L(X)$ there exists a net $\{f_\alpha\} \subset K(X)$ converging to f in the topology $w(L, PL^*) = w(L, AK^*)$.

Thus the property that $K(X)$ forms an ideal with the constant λ in $L(X)$ may formally be considered as a generalization of the concept of the λ-bounded compact approximation property.

Note that the condition (8) implies that PL^* is total on L and thus the topology $w(L, PL^*)$ is Hausdorff.

Remark 10. The Proposition 6 and the remarks afterwards hold also for λ-bounded approximation property. In this case we just have to take for example the projection P in L^* to have the kernel equal to the polar $F^0 \subset L^*$, where F denotes the set if all finite dimensional operators in L.

Remark 11. Lima [7] proved that if X is an Asplund space and if $\lambda = 1$ then the condition (i) is automatically implied by the rest of the conditions in the Proposition 6.

Remark 12. If X has the λ-bounded compact approximation property then $K(X)$ cannot be reflexive. Indeed, the Proposition 6 yields then that $K \subset L \subset K^{**} = K$, identifying L with its image $B(L)$ in K^{**}. Thus $K = L$ which is not the case.

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