Remarks on Hammerstein integral operators. [1]

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Summary. We present a result showing that the usual Hammerstein integral operator maps suitable bounded subsets of L^1 onto relatively compact subsets; then we apply it to get an existence result for the Hammerstein integral equation.

In this note we consider the following Hammerstein Integral Operator

$$(\mathcal{K}x)(\cdot) = \int_0^1 k(\cdot, s) f(s, x(s)) ds \qquad (HIO)$$

from $L^1([0,1], E), E$ a Banach space, into itself and we show that it acts as a compact operator on suitable bounded subsets under quite general assumptions on the kernel k and the space E involved; even if our result is not a compactness one, but just a weaker form, neverthless it may be utilized to derive an existence theorem for the Hammerstein integral equation

$$x = g + \mathcal{K}x \qquad (HIE)$$

(where $g \in L^1([0,1], E)$) in the space $L^1([0,1], E)$, as we shall do in Theorem 2.

We refer the reader to [1], [2], [5], [6], [8], [9], [10], [11], [12], [13] and References therein for further results about (HIO) and (HIE).

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In order to get our main result we need the following facts

Proposition 1 ([7]). *Let M be a bounded subset of a separable Banach space E. M is relatively compact if and only if for any weak*-null sequence $(x_n^*) \subset E^*$ one has* $\lim_n \sup\{|x_n^*(x)| : x \in M\} = 0$.

Proposition 2 ([3]). *Let F be a reflexive Banach space. If $\phi \in L^1([0,1])$ the set*

$$X = \{x \in L^1([0,1], F) : \|x(s)\|_F \leq \phi(s), a.e. \quad in \quad [0,1]\}$$

is relatively weakly compact.

Proposition 3 ([3],[4]). *Let $(g_n) \subset L^1([0,1], Y), Y$ a Banach space. Then there is a separable Banach space $Z \subset Y$ such that $(g_n) \subset L^1([0,1], Z)$.*

Proposition 4 ([2]). *Let (S, d) a complete metric space and $T : S \to S$ an application such that there are $L \in [0,1[, n_0 \in N$, for which*

$$d(T^{n_0}(x), T^{n_0}(y)) \leq Ld(x,y) \qquad \forall x, y \in S.$$

Then T has a (unique) fixed point.

We present now the announced compactness result on the (HIO)

Theorem 1. *Let E, F be two Banach spaces, with F reflexive. Let $k : [0,1]^2 \to \mathbb{K}(F,E) = \{$compact operators from F into $E\}$ be a measurable function; furthermore, for each $t \in [0,1]$, let the functions $s \to k(t,s)$ belong to $L^\infty([0,1], \mathbb{K}(F,E))$. Also suppose that the linear operator $\tilde{K} : z \to \int_0^1 \|k(\cdot,s)\|_{\mathbb{K}(F,E)} z(s)ds$ is continuous from $L^1([0,1], E)$ into itself. Let $f : [0,1] \times E \to F$ be a Caratheodory function for which there are $a \in L^1([0,1]), b \in R^+$, such that*

$$\|f(s, x(s))\|_F \leq a(s) + b\|x(s)\|_E \qquad \forall x \in L^1([0,1], E), s \quad a.e. \ in \quad [0,1].$$

If $\phi \in L^1([0,1])$, we consider the bounded, closed and convex set

$$Q = \{x : x \in L^1([0,1], E), \|x(s)\|_E \leq \phi(s), s \quad a.e. \ in \quad [0,1]\} \subset L^1([0,1], E).$$

Then $\mathcal{K}(Q)$ is relatively compact in $L^1([0,1], E)$

Proof. First of all, observe that

$$\|f(s, x(s))\|_F \leq a(s) + b\phi(s) \qquad \forall x \in Q, s \quad \text{a.e. in} \quad [0,1] \tag{1}$$

so that the set $f(\cdot, Q)$ is relatively weakly compact in $L^1([0,1], F)$, by Proposition 2; hence, if $(x_h) \subset Q$ there is (x_{h_p}) such that

$$f(\cdot, x_{h_p}(\cdot)) \xrightarrow{w} \psi$$

for a suitable $\psi \in L^1([0,1], F)$. For any $\bar{t} \in [0,1]$, we have that $k(\bar{t}, \cdot) \in L^\infty([0,1], \mathbb{K}(F,E))$ and so

$$\int_0^1 k(\bar{t}, s) f(s, x_{h_p}(s)) ds \xrightarrow{w} \int_0^1 k(\bar{t}, s)\psi(s) ds \tag{2}$$

in E, since it is easily seen that, for each $x^* \in E^*$, one has $x^* k(\bar{t}, \cdot) \in L^\infty([0,1], F) \subset (L^1([0,1], F))^*$. We wish to show that the sequence $\left(\int_0^1 k(t,s) f(s, x_{h_p}(s)) ds\right)$ is relatively compact in E, for each $t \in [0,1]$; thanks to Proposition 3 we may assume that E is separable, so that Proposition 1 may be utilized, and so we shall do it. Let $(x_n^*) \subset E^*$ be a weak*-null sequence; (1) implies that $\{f(\bar{s}, x_{h_p}(\bar{s})) : k \in N\}$ is bounded in E, for almost all $\bar{s} \in [0,1]$, so that $\{k(\bar{t}, \bar{s}) f(\bar{s}, x_{h_p}(\bar{s})) : k \in N\}$ is relatively compact; hence

$$\sup_p |x_n^* k(\bar{t}, \bar{s}) f(\bar{s}, x_{h_p}(\bar{s}))| \to 0 \quad \text{as} \quad n \to +\infty;$$

since

$$\sup_p |x_n^* k(\bar{t}, \bar{s}) f(\bar{s}, x_{h_p}(\bar{s}))| \leq (\sup_n \|x_n^*\|) \|k(\bar{t}, \cdot)\|_{L^\infty([0,1], \mathbb{K}(F,E))} (a(s) + b\phi(s))$$

we easily get

$$\int_0^1 \sup_p |x_n^* k(\bar{t}, s) f(s, x_{h_p}(s))| \to 0$$

from which it follows that

$$\sup x_n^* \int_0^1 k(\bar{t}, s) f(s, x_{h_p}(s)) ds \to 0$$

so that the set

$$\left\{ \int_0^1 k(\bar{t}, s) f(s, x_{h_p}(s)) ds : p \in N \right\} \subset E$$

is relatively compact (see Proposition 1). Hence, from (2) we get

$$\lim_p \int_0^1 k(\bar{t}, s) f(s, x_{h_p}(s)) ds = \int_0^1 k(\bar{t}, s) \psi(s) ds.$$

What we have got can be obtained for all $t \in [0,1]$; since, also,

$$\left\| \int_0^1 k(t, s) f(s, x_{h_p}(s)) ds \right\|_E \leq \int_0^1 \|k(t, s)\|_{\mathbb{K}(F,E)} (a(s) + b\phi(s)) ds \in L^1([0,1])$$

we may apply the Dominated Convergence Theorem to derive that $\left(\int_0^1 k(\cdot, s) f(s, x_{h_p}(s)) ds\right)$ is a sequence strongly converging in $L^1([0,1], E)$. The proof is over.■

As announced in the introduction, Theorem 1 may be applied to prove the following existence result

Theorem 2. *Suppose that all of the assumptions of Theorem 1 are verified. Also assume that g is an element of $L^1([0,1], E)$ and that there is $n_0 \in N$ such that $\|(b\tilde{K})^{n_0}\| < 1$. Then the following Hammerstein Integral Equation (HIE)*

$$x = g + \mathcal{K}x$$

has a solution in $L^1([0,1], E)$.

Proof. Define $A : L^1([0,1]) \cap \{x \in L^1([0,1]), x(s) \geq 0 \quad a.e.\} \to L^1([0,1]) \cap \{x \in L^1([0,1]), x(s) \geq 0 \quad a.e.\}$ by putting

$$Ax(\cdot) = \|g(\cdot)\|_{L^1([0,1],E)} + [(b\tilde{K})x](\cdot) + \int_0^1 \|k(\cdot, s)\|_{\mathbb{K}(F,E)} a(s) ds.$$

It is easily seen that

$$\|A^{n_0}(x) - A^{n_0}(y)\| \leq \|(b\tilde{K})^{n_0}\| \|x - y\| \qquad \forall x, y \in L^1([0,1]) \cap \{x \in L^1([0,1]), x(s) \geq 0 \quad a.e.\}$$

so that A has a fixed point $\phi_0 \in L^1([0,1]) \cap \{x \in L^1([0,1]), x(s) \geq 0 \quad a.e.\}$ (Proposition 4), since $L^1([0,1]) \cap \{x \in L^1([0,1]), x(s) \geq 0 \quad a.e.\}$ is a complete metric space. Now we consider the following bounded, closed and convex subset of $L^1([0,1], E)$

$$Q = \{x \in L^1([0,1], E), \|x(s)\|_E \leq \phi_0(s), \quad \text{s a.e. in} \quad [0,1]\}$$

and we observe that the operator $x \to g + \mathcal{K}x$ maps Q into itself. If $\phi_0 = \theta_{L^1([0,1])}$ it is easy to show that

$$\theta_{L^1([0,1],E)} = g + \mathcal{K}\theta_{L^1([0,1],E)};$$

whereas if $\phi_0 \neq \theta_{L^1([0,1])}$, it is not difficult to show that $x \to g + \mathcal{K}x$ maps Q into a relatively compact set, thanks to Theorem 1; hence the Schauder Fixed Point Theorem applies to get our thesis. We are done.■

Remark 1. The assumption of the existence of a $n_0 \in N$ such that $\|(b\tilde{K})^{n_0}\| < 1$ was used just to guarentee the existence of a solution of the equation $x = Ax$; hence any other assumption implying the existence of such a solution may be used to reach our target.

Remark 2. If one wants to avoid the reflexivity of F, he could, but, as far as we know, assuming that k is a Caratheodory function, too; in such a case, indeed, the same technique used in [6] works to reach our goal (we observe that in [6] an assumption of separability was made on F; but actually it can be avoided as in the present Theorem 1, because we may work just with sequences). It would be interesting to see if the reflexivity of F may be dropped with k only measurable.

Remark 3. Similar compactness and then existence results may be obtained for the following *functional-integral equation*

$$x(\cdot) = f\left(t, r\int_0^1 k(\cdot, s)g(s, x(s))ds\right).$$

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